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On anti-structurable algebras[1]

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1 Definitions and preamble

1.1 (ε, δ)-Freudenthal Kantor triple systems

We are concerned in this paper with triple systems which have finite dimension over a field Φ of characteristic $\neq 2$ or 3, unless otherwise specified.

In order to render this paper as self-contained as possible, we recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order).

A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto (xyz)$ is said to be a *GJTS of 2nd order* if the following conditions are fulfilled:

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \tag{1.1}$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0, \tag{1.2}$$

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where $L(a,b)c := (abc)$ and $K(a,b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (1.1) and the following condition

$$(abc) = (cba). \tag{1.3}$$

We can generalize the concept of GJTS of 2nd order as follows (see [13], [14], [17]-[21], [52] and the earlier references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \tag{1.4}$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) + \varepsilon K(a,b)L(x,y) = 0, \tag{1.5}$$

where

$$L(a,b)c := (abc), \quad K(a,b)c := (acb) - \delta(bca), \tag{1.6}$$

is called an (ε,δ)-*Freudenthal Kantor triple system* (for short (ε,δ)-FKTS).

Remark. We note that

$$K(b,a) = -\delta K(a,b). \tag{1.7}$$

Remark. The concept of GJTS of 2nd order coincides with that of $(-1,1)$-FKTS. Thus we can construct the simple Lie algebras by means of the standard embedding method ([6], [13]-[17], [21], [24], [26], [35], [52]).

For an (ε,δ)-FKTS U we denote

$$A(a,b) := L(a,b) - \varepsilon L(b,a), \tag{1.8}$$

where $L(a,b)$ is defined by (1.6). Then $A(a,b)$ is an anti-derivation of U since we notice that

$$[A(a,b), L(c,d)] = L(A(a,b)c,d) - L(c,A(a,b)d). \tag{1.9}$$

An (ε,δ)-FKTS U is called *unitary* if the identity map Id is contained in $\kappa := K(U,U)$ i.e., if there exist $a_i, b_i \in U$, such that

$$\Sigma_i K(a_i,b_i) = Id. \tag{1.10}$$

If U is an (ε,δ)-FKTS and $a,b \in U$ then (a,b) is called a *left neutral pair* if $L(a,b) = Id$.

For $\delta = \pm 1$, a triple system $(a, b, c) \mapsto [abc], a, b, c \in V$ is called a *δ-Lie triple system* (for short δ-LTS) if the following three identities are fulfilled

$$\begin{gathered}
[abc] = -\delta[bac], \\
[abc] + [bca] + [cab] = 0, \\
[ab[xyz]] = [[abx]yz] + [x[aby]z] + [xy[abz]],
\end{gathered} \tag{1.11}$$

where $a, b, x, y, z \in V$. An 1-LTS is a *LTS* while a -1-LTS is an *anti-LTS*, by [14].

1.2 δ-structurable algebras

The motivation for the study of such nonassociative algebras is as follows. The existence of the class of nonassociative algebras called structurable algebras is an important generalization of Jordan algebras giving a construction of Lie algebras. Hence from our concept, by means of triple products, we define a generalization of such class to construct Lie superalgebras as well as Lie algebras.

Our start point briefly described in a historical setting is the construction of Lie (super)algebras starting from a class of nonassociative algebras. Hence within the general framework of (ε, δ)-FKTSs ($\varepsilon, \delta = \pm 1$) and the standard embedding Lie (super)algebra construction studied in [6],[7],[13]-[15], [26] (see also references therein) we define δ-structurable algebras as a class of nonassociative algebras with involution which coincides with the class of structurable algebras for $\delta = 1$ as introduced and studied in [1], [2]. Structurable algebras are a class of nonassociative algebras with involution that include Jordan algebras (with trivial involution), associative algebras with involution, and alternative algebras with involution. They are related to GJTSs 2nd order (or $(-1, 1)$-FKTSs) as introduced and studied in [33], [34] and further studied in [3], [4], [32], [41]-[44], [49] (see also references therein). Their importance lies with constructions of five graded Lie algebras

$$\mathcal{L}(U) := L(\varepsilon, \delta) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2, \quad [L_i, L_j] \subseteq L_{i+j}. \tag{1.12}$$

For $\delta = -1$ the anti-structurable algebras defined here are a new class of nonassociative algebras that may similarly shed light on the notion of $(-1, -1)$-FKTSs hence (by [6], [7]) on the construction of Lie superalgebras and Jordan algebras as it will be shown.

Let $(\mathcal{A}, ^-)$ be a finite dimensional nonassociative unital algebra with involution (involutive anti-automorphism, i.e. $\overline{\overline{x}} = x, \overline{xy} = \overline{y}\,\overline{x}, x, y \in \mathcal{A}$) over Φ. The identity element of $\mathcal{A}$ is denoted by 1. Since char$\Phi \neq 2$, by [1] we have $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$, where $\mathcal{H} = \{a \in \mathcal{A} | \overline{a} = a\}$ and $\mathcal{S} = \{a \in \mathcal{A} | \overline{a} = -a\}$.

Suppose $x, y, z \in \mathcal{A}$. Put $[x, y] := xy - yx$ and $[x, y, z] := (xy)z - x(yz)$. Note that

$$\overline{[x, y, z]} = -[\overline{z}, \overline{y}, \overline{x}]. \tag{1.13}$$

The operators L_x and R_x are defined by $L_x(y) := xy, R_x(y) := yx$.

For $\delta = \pm 1$ and $x, y \in \mathcal{A}$ define

$${}^{\delta}V_{x,y} := L_{L_x(\overline{y})} + \delta(R_x R_{\overline{y}} - R_y R_{\overline{x}}), \tag{1.14}$$

$${}^{\delta}B_{\mathcal{A}}(x, y, z) := {}^{\delta}V_{x,y}(z) = (x\overline{y})z + \delta[(z\overline{y})x - (z\overline{x})y], x, y, z \in \mathcal{A}. \tag{1.15}$$

${}^{+}B_{\mathcal{A}}(x, y, z)$ is called the *triple system obtained from the algebra* $(\mathcal{A}, ^-)$. We will call ${}^{-}B_{\mathcal{A}}(x, y, z)$ the *anti-triple system obtained from the algebra* $(\mathcal{A}, ^-)$. We write for short

$$V_{x,y} := {}^{\delta}V_{x,y}, \quad B_{\mathcal{A}} := ({}^{\delta}B_{\mathcal{A}}, \mathcal{A}). \tag{1.16}$$

Remark. The upper left index notation is chosen in order not to be mixed with the upper right index notation of [1] which has a different meaning.

A unital non-associative algebra with involution $(\mathcal{A}, ^-)$ is called a *structurable algebra* if the following identity is fulfilled

$$[V_{u,v}, V_{x,y}] = V_{V_{u,v}(x),y} - V_{x,V_{v,u}(y)}, \tag{1.17}$$

for $V_{u,v} = {}^{+}V_{u,v}, V_{x,y} = {}^{+}V_{x,y}, u, v, x, y \in \mathcal{A}$, and we will call $(\mathcal{A}, ^-)$ an *anti-structurable algebra* if the identity (1.17) is fulfilled for $V_{u,v} = {}^{-}V_{u,v}, V_{x,y} = {}^{-}V_{x,y}$.

If $(\mathcal{A}, ^-)$ is structurable then, by [34], the triple system $B_{\mathcal{A}}$ is called a *generalized Jordan triple system* (abbreviated GJTS) and by [8], $B_{\mathcal{A}}$ is a GJTS of 2nd order, i.e. satisfies the identities (1.4) and (1.5). If $(\mathcal{A}, ^-)$ is anti-structurable then we call $B_{\mathcal{A}}$ an *anti-GJTS*.

2 Several properties

2.1 Properties satisfying the second order condition

From now on we assume $\delta = -1$ and let $(\mathcal{A}, ^-)$ be an anti-structurable algebra. Define $C(a, b, c) \in End\,\mathcal{A}$ by

$$C(a, b, c)d := [a\overline{b}, \overline{d}, c] - [a, \overline{b}, \overline{d}]c, \quad a, b, c, d \in \mathcal{A}. \tag{2.18}$$

We say that $\mathcal{A}$ satisfies *condition* $\mathcal{C}$ if

$$C(x,y,w) - C(w,y,x) = C(w,x,y) - C(y,x,w), \quad x,y,w \in \mathcal{A}. \tag{2.19}$$

Theorem 2.1 *Let $(\mathcal{A},^-)$ be an anti-structurable algebra. Then the second order condition (1.5) and condition $\mathcal{C}$ are equivalent.*

Remark. An anti-structurable algebra satisfying the condition $\mathcal{C}$ is a $(-1,-1)$-FKTS.

2.2 Lie admissible structures

Theorem 2.2 *Let $(\mathcal{A},^-)$ be an anti-structurable algebra such that $^- = Id$. Then $\mathcal{A}$ is a LTS with respect to the new product $[x,y,z] = B_{\mathcal{A}}(x,y,z) - B_{\mathcal{A}}(y,x,z)$, $x,y,z \in \mathcal{A}$.*

Theorem 2.3 *Let $(\mathcal{A},^-)$ be an anti-structurable algebra satisfying the second order condition (1.5). Then*

i) $\mathcal{A}$ is a Lie admissible, i.e. the Jacobi identity is fulfilled:

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0, \; x,y,z \in \mathcal{A},$$

ii) $[x,y,z] + [z,y,x]$ is totaly symmetric in any exchanges of $x,y,z \in \mathcal{A}$,

iii) $[h,x,y] = [x,h,y] = [x,y,h] = 0$, for all $h \in \mathcal{H}, x,y \in \mathcal{A}$.

Theorem 2.4 *Let $(\mathcal{A},^-)$ be an anti-structurable algebra satisfying the second order condition (1.5) and let $F(x,y,z) \in End\,\mathcal{A}$ be defined by*

$$F(x,y,z)w := [x\overline{y},w,z] + [z,x\overline{y},w] + ([x,y,w] - [y,x,w])z, \quad x,y,z,w \in \mathcal{A}. \tag{2.20}$$

Then it satisfies

i) $F(x,y,z) = -F(y,x,z), \quad x,y,z \in \mathcal{A}$,

ii) $F(x,y,z) + F(y,z,x) + F(z,x,y) = 0, \quad x,y,z \in \mathcal{A}$.

Remark. We have also $K(u,v)K(x,y) + K(x,y)K(u,v) = K(K(u,v)x,y) + K(x,K(u,v)y$, for $x,y,u,v \in \mathcal{A}$ so the set of $K(x,y), x,y \in \mathcal{A}$ form a Jordan algebra (see [30] for details).

3 Examples of anti-structurable algebras with left neutral pairs

We give examples of anti-structurable algebras with left neutral pairs and invertible elements.

Let $U := \mathcal{M}_{k,k}(\Phi)$ denote the space of square matrices of order k over Φ. Then, by [29], U with the product $(xyz) = xy^\top z - zy^\top x + zx^\top y$, where $x^\top$ denotes the transposed matrix of x is an anti-structurable algebra satisfying the second order condition (1.5).

Let $(u, v), u, v \in U$ be a left neutral pair, i.e. $L(u, v) = Id$, and denote

$$GL_k(\Phi) := \{A \in \mathcal{M}_{k,k}(\Phi) | \det A \neq 0\}.$$

If $u \in GL_k(\Phi)$ then set $v = (u^\top)^{-1}$, where the involution is transposition and so $L(u, v)z = uu^{-1}z - zu^{-1}u + zu^\top(u^\top)^{-1} = z$. Thus there exists a left neutral pair $(u, (u^\top)^{-1})$. Also we have

$$U_u z = u^\top zu - u^\top zu + u^\top uz, \quad U_{(u^\top)^{-1}} = (u^\top)^{-1}((u^\top)^{-1})(u^\top)^\top z = (u^\top)^{-1}u^{-1}z$$

thus by straightforward calculation follows $U_u U_{(u^\top)^{-1}} z = z$. Then the map U_u is invertible. This implies that with any element $u \in GL_k(\Phi)$ there can be constructed a left neutral pair $(u, (u^\top)^{-1})$.

Set $O(\Phi) := \{A \in \mathcal{M}_{k,k}(\Phi) | AA^\top = Id\}$. Then in the example above, if any element $u \in O(\Phi)$ it follows that (u, u) is a left neutral pair, i.e. u is a left unit element.

Theorem 3.1 *Let U be a $(-1, -1)$-FKTS. Then, the following are equivalent*
i) (u, v) is a left neutral pair,
ii) (v, u) is a left neutral pair.

Proof. We shall prove that $L(u, v) = Id$ if and only if $L(v, u) = Id$.

If $L(u, v) = Id$ then $[L(u, v), L(v, x)] = 0$ so $L((uvv), x) - L(v, (vux)) = 0$, by (1.4), hence $L(v, x - (vux))v = 0$, since $L(u, v) = Id$. Now, since U_v is invertible follows from the last identity that $(vux) = x$, hence $L(v, u) = Id$.

Conversely, if $L(v, u) = Id$ follows then that $L(u, v) = Id$, by an analogous proof. $\square$

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