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Classification of residuated lattices by filters

東京電機大学・情報環境学部　近藤　通朗 (Michiro KONDO)
School of Information Environment
Tokyo Denki University

概要

In this short note we prove that for any residuated lattice X and filter F of X,

(a) F is an implicative filter if and only if X/F is a Heyting algebra;

(b) F is a positive implicative filter if and only if X/F is a Boolean algebra;

(c) F is a fantastic filter if and only if X/F is an MV-algebra.

It follows from this characterization that, for any filter F of a residuated lattice, it is a positive implicative filter if and only if it is an implicative and fantastic filter.

1 はじめに

BL-algebras were invented by Hájek ([2]) in order to prove the completeness theorem of basic fuzzy logic, BL-logic in short. Soon after Cignoli et al. ([1]) proved that Hájek's logic really is the logic of continuous t-norms as conjectured by Hájek. At the same time started a systematic study of BL-algebras, too. Indeed, Turunen ([6]) published where BL-algebras were studied by deductive systems. Deductive systems correspond to subsets closed with respect to Modus Ponens and they are called filters, too. In [5], Boolean deductive systems and implicative deductive systems were introduced. Moreover, it was proved that these deductive systems coincide. In [3], M.Haveshki, A.B.Saeid and E.Eslami continued an algebraic analysis of BL-algebras and they introduced e.g. implicative filters of BL-algebras. Notice that implicative deductive systems and implicative filters are not, in general, the same subsets.

In this short note, we give classification of residuated lattices by filters. We show that for any residuated lattice X and filter F of X,

(a) F is an implicative filter if and only if X/F is a Heyting algebra;

(b) F is a positive implicative filter if and only if X/F is a Boolean algebra;

(c) F is a fantastic filter if and only if X/F is a MV-algebra.

From the above, we see that, for any filter F of a residuated lattice, it is a positive implicative filter if and only if it is an implicative and fantastic filter.

2 residuated lattice と filter

We recall a definition of residuated lattices and filters. An algebra $(X, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a *residuated lattice* if it satisfies

(C1) $(X, \wedge, \vee, 0, 1)$ is a bounded lattice;

(C2) $(X, \odot, 1)$ is a commutative monoid with a unit 1;

(C3) For all $x, y, z \in X$, we have

$$x \odot y \leq z \iff x \leq y \rightarrow z.$$

By $\mathcal{RL}$, we mean the class of all residuated lattices. The following result is familiar ([4]).

命題 1. *For $X \in \mathcal{RL}$ and all $x, y, z \in X$, we have*
(1) $x \leq y \iff x \rightarrow y = 1$
(2) $x \odot (x \rightarrow y) \leq y$
(3) $x \leq y \implies x \odot z \leq y \odot z,\ z \rightarrow x \leq z \rightarrow y,\ y \rightarrow z \leq x \rightarrow z$
(4) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$

A subset F of X is called a *filter* ([4]), for all $x, y \in X$, if it satisfies

(F0) $1 \in F$;

(F1) $x, y \in F$ implies $x \odot y \in F$;

(F2) $x \in F$ and $x \leq y$ imply $y \in F$.

It is easy to prove that for a subset $F \subseteq X$, F is a filter if and only it is a *deductive system* defined in [5], that is, it satisfies

(DS1) $1 \in F$ and

(DS2) If $x \in F$ and $x \to y \in F$ then $y \in F$

For any filter F of X, a relation $\equiv_F$ on X defined by

$$x \equiv_F y \iff x \to y, y \to x \in F,$$

is a congruence on X and, since $\mathcal{RL}$ is a variety, a quotient structure $X/F = \{x/F \mid x \in X\}$ by $\equiv_F$ is also a residuated lattice by the following definition : For all $x/F, y/F \in X/F$,

$$x/F \wedge y/F = (x \wedge y)/F, \quad x/F \vee y/F = (x \vee y)/F$$
$$x/F \to y/F = (x \to y)/F, \quad x/F \odot y/F = (x \odot y)/F.$$

Since the class $\mathcal{BL}$ of all BL-algebras is a subvariety of $\mathcal{RL}$, we use the same terminology about definitions of some types of filters according to [3]. Let X be a residuated lattice. A subset $F \subseteq X$ is called an *implicative filter* if it satisfies

(F0) $1 \in F$;

(I) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$ for all $x, y, z, \in X$.

Also it is called a *positive implicative filter* if

(F0) $1 \in F$;

(PI) $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$ for all $x, y, z, \in X$.

Lastly, a subset F is said to be a *fantastic filter* if

(F0) $1 \in F$;

(FF) $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$ for all $x, y, z, \in X$.

3 filterの特徴付け

In this section we give characterizations of filters defined in the previous section. First of all we treat implicative filters of residuated lattices.

We note that if F is an implicative filter then it is also a filter. We have an identity characterization of implicative filters, which can be proved easily.

命題 2. *For a filter F of $X \in \mathcal{RL}$, the following conditions are equivalent.*

(i) F is an implicative filter.

(ii) $x \to x^2 \in F$ for every $x \in X$, where $x^2 = x \odot x$.

From the above we can show the next result without difficulty.

定理 1. *For any filter F, F is an implicative filter if and only if X/F is a Heyting algebra.*

Concerning to the case of positive implicative filters, we note that if F is a positive implicative filter then it is also a filter.

命題 3. *Every positive implicative filter is an implicative filter.*

Now we have an identity characterization of positive implicative filters.

命題 4. *Let F be a filter of X. Then the following conditions are equivalent:*

(i) F is a positive implicative filter

(ii) $(x \to y) \to x \in F$ implies $x \in F$ for all $x, y \in X$

(iii) $((x \to y) \to x) \to x \in F$ for all $x, y \in X$.

We note from the above that, for a filter F of X, it is a positive implicative filter if and only if X/F satisfies the condition $(a \to b) \to a = a$ for all $a, b \in X/F$. This implies a following result.

定理 2. *Let X be a residuated lattice and F a filter of X. F is a positive implicative filter if and only if X/F is a Heyting algebra with $a \vee b = (a \to b) \to b = (b \to a) \to a$, that is, a Boolean algebra.*

Remark : In [4], an implicative deductive system is defined for Rℓ-monoids which mean residuated lattices with $x \wedge y = x \odot (x \to y)$. A subset F of a residuated lattice X is called an *implicative deductive system* of X if it satisfies the condition that $x \to (z' \to y) \in F$ and $y \to z \in F$ imply $x \to z \in F$, where $z' = z \to 0$. We can verify that implicative deductive systems are exactly same as positive implicative filters in residuated lattices. Indeed, let F be an implicative deductive system of X. Since $(x' \to x) \to (x' \to x) = 1 \in F$ and $x \to x = 1 \in F$, we have $(x' \to x) \to x \in F$. Namely, F is the positive implicative filter.

Conversely, suppose that F is a positive implicative filter. Assume that $x \to (z' \to y) \in F$ and $y \to z \in F$. Since $x \odot (x \to (z' \to y)) \odot (y \to z) \leq (z' \to y) \odot (y \to z) \leq z' \to z$, we have $(x \to (z' \to y)) \odot (y \to z) \in F$ and $(x \to (z' \to y)) \odot (y \to z) \leq x \to (z' \to z)$. This implies $x \to (z' \to z) \in F$. Since F is the positive implicative filter, we also have $(z' \to z) \to z \in F$. Thus $(x \to (z' \to z)) \odot ((z' \to z) \to z) \in F$ and $(x \to (z' \to z)) \odot ((z' \to z) \to z) \leq x \to z$. This means that $x \to z \in F$ and hence that F is the implicative deductive system.

At last we consider property of fantastic filters. It is clear that every fantastic filter is a filter. Moreover we have

補題 1. *Let F be a filter. The following conditions are equivalent.*

(i) F is a fantastic filter;

(ii) If $y \to x \in F$, then $((x \to y) \to y) \to x \in F$ for all $x, y \in X$;

(iii) $((x \to y) \to y) \to ((y \to x) \to x) \in F$ for all $x, y \in X$.

Proof. • $(i) \Longrightarrow (ii)$. Suppose that F is a fantastic filter and $y \to x \in F$. Since $y \to x = 1 \to (y \to x) \in F$ and $1 \in F$, we have $((x \to y) \to y) \to x \in F$.

• $(ii) \Longrightarrow (i)$. Conversely, we assume that $z \to (y \to x) \in F$ and $z \in F$. Since F is the filter, we have $y \to x \in F$ and thus $((x \to y) \to y) \to x \in F$. This means that F is the fantastic filter.

• $(ii) \Longrightarrow (iii)$. Since $y \to ((y \to x) \to x) = 1 \in F$, it follows from assumption (ii) that $((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x) \in F$. From $x \leq (y \to x) \to x$, we have $(x \to y) \to y \leq (((y \to x) \to x) \to y) \to y$ and $((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x) \leq ((x \to y) \to y) \to ((y \to x) \to x)$. This implies $((x \to y) \to y) \to ((y \to x) \to x) \in F$.

• $(iii) \Longrightarrow (ii)$. Suppose that $y \to x \in F$. Since $(y \to x) \to (((x \to y) \to y) \to x) = ((x \to y) \to y) \to ((y \to x) \to x) \in F$ and F is the filter, we get $((x \to y) \to y) \to x \in F$ and thus F is the fantastic filter. □

It follows from the above that

定理 3. *Let X be a residuated lattice and F a filter of X. F is a fantastic filter if and only if X/F is a residuated lattice with meeting the condition $(a \to b) \to b = (b \to a) \to a$, that is, an MV-algebra.*

Concerning to three types of filters in residuated lattices, we see the following characterization of these filters.

系 1. *Let X be a residuated lattice and F a filter of X. F is a positive implicative filter if and only if it is an implicative and fantastic filter.*

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