Some Distortion Theorems for Starlike Log-Harmonic Functions

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Abstract

In this paper, we consider univalent log-harmonic mappings of the form $f(z) = zh(z)\overline{g(z)}$ defined on the unit disk $D$ which are starlike. Some distortion theorems are obtained.

1 Introduction

Let $\mathcal{H}(D)$ be the linear space of all analytic functions defined on the open unit disc $D = \{z \in \mathbb{C}||z|<1\}$. A log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = wf_z(\overline{\frac{f}{f}})$$

(1.1)

where the second dilatation function $w \in \mathcal{H}(D)$ is such that $|w(z)| < 1$ for all $z \in D$. It has been shown that if $f$ is non-vanishing log-harmonic mapping in $D$, then $f$ can be expressed as

$$f(z) = h(z)\overline{g(z)},$$

(1.2)

where $h(z)$ and $g(z)$ are analytic in $D$ with the normalization $h(0) \neq 0$, $g(0) = 1$. On the other hand if $f$ vanishes at $z = 0$, but not identically zero then $f$ admits the following representation

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)},$$

(1.3)

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where $\text{Re}\beta > -1/2$, $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with the normalization $h(0) \neq 0$, $g(0) = 1$ ([4]). We also note that univalent log-harmonic mappings have been studied extensively in [1], [2], [3], [4], [5], [6] and the class of all univalent log-harmonic mappings is denoted by $S_{LH}$.

The Jacobian of a logharmonic function of the form $f(z) = zh(z)\overline{g(z)}$ is defined by

$$J_f(z) = |f(z)|^2 \left( \left| \frac{1}{z} + \frac{h'(z)}{h(z)} \right|^2 - \left| \frac{g'(z)}{g(z)} \right|^2 \right) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2$$

for all $z$ in $\mathbb{D}$.

Let $f(z) = zh(z)\overline{g(z)}$ be a univalent log-harmonic mapping. We say that $f$ is a starlike log-harmonic mapping if

$$\frac{\partial}{\partial \theta} (\text{arg} f(re^{i\theta})) = \text{Re} \left( \frac{zf_z - \overline{z}f_{\overline{z}}}{f} \right) > 0 \quad (1.4)$$

for every $z \in \mathbb{D}$. The class of all starlike log-harmonic mappings is denoted by $S_{LH}^*$ ([3]).

Let $\Omega$ be the family of functions $\phi(z)$ which are analytic in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$, and let $s_1(z) = z + a_2z^2 + \cdots$, $s_2(z) = z + b_2z^2 + \cdots$ be analytic functions in $\mathbb{D}$. We say that $s_1(z)$ is subordinate to $s_2(z)$ if there exist $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ and it is denoted by $s_1(z) \prec s_2(z)$.

Let $\varphi(z)$ be analytic function in $\mathbb{D}$ with the normalization $\varphi(0) = 0$, $\varphi'(0) = 1$. If $\varphi(z)$ satisfies the condition

$$\text{Re} \left( z \frac{\varphi'(z)}{\varphi(z)} \right) > 0 \quad (1.5)$$

for every $z \in \mathbb{D}$, then $\varphi(z)$ is called starlike function. The class of all starlike functions is denoted by $S^*$.

In our proofs we need following theorems.

**Theorem 1.1.** [7] Let $\varphi(z)$ be an element of $S^*$, then

$$\frac{1 - r}{1 + r} \leq \left| z \frac{\varphi'(z)}{\varphi(z)} \right| \leq \frac{1 + r}{1 - r} \quad (|z| = r < 1). \quad (1.6)$$

**Theorem 1.2.** [3] $f(z) = zh(z)\overline{g(z)}$ be a log-harmonic function on $\mathbb{D}$, $0 \notin hg(\mathbb{D})$. Then $f \in S_{LH}^*$ if and only if $\varphi(z) = \left( z \frac{h(z)}{g(z)} \right) \in S^*$. 
Theorem 1.3. [3] Let \( f(z) = zh(z)g(z) \in S_{LH}^{*} \), with \( w(0) = 0 \). Then we have

\[
re^{-\frac{4r}{1+r}} \leq |f(z)| \leq re^{\frac{4r}{1-r}}
\]

(1.7)

for all \( |z| = r < 1 \). The equalities occur if and only if \( f(z) = \zeta f_0(\zeta z), |\zeta| = 1 \), where

\[
f_0(z) = z \left( \frac{1 - \bar{z}}{1 - z} \right) e^{Re\left( \frac{4z}{1-z} \right)}.
\]

2 Main Results

Lemma 2.1. Let \( f(z) = zh(z)g(z) \) be an element of \( S_{LH}^{*} \), then

\[
\frac{\varphi'(z)/\varphi(z)}{f_z/f} \prec 1 - z \quad \text{and} \quad \frac{\overline{f_z}/\overline{f}}{\varphi'(z)/\varphi(z)} \prec \frac{z}{1 - z}
\]

(2.1)

where \( \varphi(z) = z \frac{h(z)}{g(z)} \in S^{*} \) for all \( z \in \mathbb{D} \).

Proof. Since \( f(z) = zh(z)g(z) \) is the solution of the non-linear elliptic partial differential equation

\[
\overline{f_z} = w(z) f_z \left( \frac{\overline{f}}{\overline{f}} \right),
\]

then we have

\[
w(z) = \frac{f_z/f}{\overline{f_z}/\overline{f}} = \frac{z \frac{g'(z)}{g(z)}}{1 + z \frac{h'(z)}{h(z)}}.
\]

Therefore we have \( w(0) = 0 \). This shows that the second dilatation function satisfies the conditions of Schwarz Lemma and

\[
1 - w(z) = \frac{\varphi'(z)/\varphi(z)}{f_z/f}, \quad \frac{w(z)}{1 - w(z)} = \frac{\overline{f_z}/\overline{f}}{\varphi'(z)/\varphi(z)}.
\]

(2.2)

Using the subordination principle, the equalities (2.2) can be written in the following form

\[
\frac{\varphi'(z)/\varphi(z)}{f_z/f} \prec 1 - z \quad \text{and} \quad \frac{\overline{f_z}/\overline{f}}{\varphi'(z)/\varphi(z)} \prec \frac{z}{1 - z}.
\]
Theorem 2.2. Let $f(z) = zh(z)\overline{g(z)} \in S^*_{LH}$, then

$$e^{-\frac{4r}{1+r}} \frac{1-r}{(1+r)^2} \leq |f_z| \leq e^{\frac{4r}{1-r}} \frac{1+r}{(1-r)^2},$$  \hspace{1cm} (2.3)

$$0 \leq |f_z| \leq e^{\frac{4r}{1-r}} \frac{r(1+r)}{(1-r)^2},$$  \hspace{1cm} (2.4)

for all $|z| = r < 1$.

Proof. Since the transformations $w_1(z) = 1 - z$ and $w_2(z) = \frac{z}{1-z}$ map $|z| = r$ onto the discs with the centers $C_1(r) = (1,0)$, $C_2(r) = \left(\frac{r^2}{1-r^2},0\right)$ and radius $\rho_1(r) = r$, $\rho_2(r) = \frac{r}{1-r^2}$ respectively. Using Lemma 2.1 and subordination principle then we can write

$$\left| \frac{\varphi'(z)/\varphi(z)}{f_z/f} - 1 \right| \leq r$$

and

$$\left| \frac{\overline{f_z}/\overline{f}}{\varphi(z)/\varphi(z)} - \frac{r^2}{1-r^2} \right| \leq \frac{r}{1-r^2}. \hspace{1cm} (2.5)$$

Using Theorem 1.1, Theorem 1.2, Theorem 1.3 and inequalities (2.5) and after the straightforward calculations we obtain (2.3) and (2.4). \hfill \Box

As a consequence of Theorem 2.2 we have the following corollary:

Corollary 2.3. Let $f(z) = zh(z)\overline{g(z)}$ be element of $S^*_{LH}$, then

$$e^{-\frac{8r}{1+r}} \frac{(1-r)^2}{(1+r)^4} - e^{\frac{8r^2}{1-r^2}} \frac{r}{(1-r^2)^2} \leq J_f(z) \leq e^{\frac{8r}{1-r}} \frac{(1+r)^3}{(1-r)^4},$$

for all $|z| = r < 1$.

Theorem 2.4. Let $f(z) = zh(z)\overline{g(z)}$ be an element of $S^*_{LH}$, then

$$|h(z)| \leq e^{\frac{2}{1-r}} \frac{1}{1-r}, \hspace{1cm} (2.6)$$

$$|g(z)| \leq (1-r)e^{\frac{2}{1-r}}, \hspace{1cm} (2.7)$$

for all $|z| = r < 1$. 

Proof. Using standart inequalities for complex numbers, we can write

\[
\Re \left( \frac{zf_z}{f} \right) \leq \left| \frac{zf_z}{f} \right| \tag{2.8}
\]
and

\[
\Re \left( \frac{\overline{z}f_{\overline{z}}}{f} \right) \leq \left| \frac{\overline{z}f_{\overline{z}}}{f} \right| \tag{2.9}
\]
for all \( z \in \mathbb{D} \). On the other hand,

\[
\Re \left( \frac{zf_z}{f} \right) = \Re \left( 1 + z \frac{h'(z)}{h(z)} \right) = 1 + \Re \left( z \frac{h'(z)}{h(z)} \right) = 1 + r \frac{\partial}{\partial r} \log |h(z)| \tag{2.10}
\]
and

\[
\Re \left( \frac{\overline{z}f_{\overline{z}}}{f} \right) = \Re \left( \overline{z} \frac{g'(z)}{g(z)} \right) = \Re \left( z \frac{g'(z)}{g(z)} \right) = r \frac{\partial}{\partial r} \log |g(z)| \tag{2.11}
\]
for all \( z \in \mathbb{D} \).

Using Theorem 2.2 and the inequalities (2.8), (2.9), (2.10) and (2.11), we find

\[
\frac{\partial}{\partial r} \log |h(z)| \leq \frac{1 + r}{r(1 - r)^2} - \frac{1}{r} \tag{2.12}
\]
and

\[
\frac{\partial}{\partial r} \log |g(z)| \leq \frac{1 + r}{r(1 - r)^2} \tag{2.13}
\]
Integrating from zero to \( r \) we obtain (2.6) and (2.7). \( \square \)

**Theorem 2.5.** If \( f(z) = zh(z)\overline{g(z)} \) is in \( S_{LH}^* \) and \( a \) is in \( \mathbb{D} \), then

\[
\varphi_*(z) = \frac{zg(a)h(z)}{h(a)(1 + \overline{a}z)^2g(z)} \tag{2.14}
\]
is likewise in \( S^* \).

**Proof.** For \( \rho \) real, \( 0 < \rho < 1 \), let

\[
\varphi_\rho(z) = \frac{zg(\rho a)h(\rho \frac{z+a}{1+\overline{a}z})}{h(\rho a)(1 + \overline{a}z)^2g(\rho \frac{z+a}{1+\overline{a}z})} \tag{2.15}
\]
is likewise in \( S^* \).
then
\[
\frac{\varphi'_p(z)}{\varphi_p(z)} = \frac{1 - |a|^2}{1 + |a|^2} \frac{z}{(1 + a\overline{a})(z + a)} \cdot \left[ \rho \left( z + a \frac{z + a}{1 + a\overline{a}} \right) \frac{h'(\rho(z))}{h(\rho(z))} - \rho \left( z + a \frac{z + a}{1 + a\overline{a}} \right) \frac{g'(\rho(z))}{g(\rho(z))} \right] .
\]

(2.14)

Letting \( z = e^{i\theta}, a = |a|e^{i\phi} \) and \( \nu = \rho \left( \frac{e^{i\theta} + a}{1 + \overline{a}e^{i\theta}} \right) \) and after the simple calculations we get
\[
\frac{\varphi'_p(z)}{\varphi_p(z)} = \frac{1 - |a|^2}{|1 + ae^{-i\theta}|^2} \left( 1 + \nu \frac{h'(\nu)}{h(\nu)} - \nu \frac{g'(\nu)}{g(\nu)} \right) + i\frac{2|a|\sin(\phi - \theta)}{|1 + ae^{-i\theta}|^2} .
\]

Therefore for \(|z| = 1\), we have
\[
\text{Re} \left( \frac{\varphi'_p(z)}{\varphi_p(z)} \right) = \frac{1 - |a|^2}{|1 + ae^{-i\theta}|^2} \text{Re} \left( 1 + \nu \frac{h'(\nu)}{h(\nu)} - \nu \frac{g'(\nu)}{g(\nu)} \right) > 0
\]

(2.15)

and we conclude that \( \varphi_p(z) \) is in \( S^* \) for admissible \( \rho \). From the compactness of \( S^* \) and (2.15) we infer that \( \varphi_p(z) = \lim_{\rho \to 1} \varphi_p(z) \) is in \( S^* \).  

We also note that if we take \( a = v, u = \frac{z + a}{1 + \overline{a}z} = \frac{z + v}{1 + \overline{v}z} \Leftrightarrow z = \frac{u - v}{1 - \overline{v}u} \) and using Theorem 2.5 and after simple calculations we obtain the following two point distortion inequalities.

**Corollary 2.6.** Let \( f(z) = zh(z)g(z) \) be an element of \( S_{LH}^* \), then
\[
e^{\frac{-4|u-v|}{|1-\overline{v}u|+|u-v|}} \frac{1 - \overline{v}u(|1 - \overline{v}u| - |u - v|)}{|1 - \overline{v}u|(|1 - \overline{v}u| + |u - v|)} \leq |f_z| \leq e^{\frac{4|u-v|}{|1-\overline{v}u|-|u-v|}} \frac{1 - \overline{v}u(|1 - \overline{v}u| + |u - v|)}{|1 - \overline{v}u|(|1 - \overline{v}u| - |u - v|)} ,
\]

and
\[
|f_z| \leq e^{\frac{4|u-v|}{|1-\overline{v}u|-|u-v|}} \frac{|u - v|(|1 - \overline{v}u| + |u - v|)}{|1 - \overline{v}u|(|1 - \overline{v}u| - |u - v|)} ,
\]
and

\[ e^{-\frac{8|u-v|}{|1-\overline{u}u|+|u-v|}} \frac{|1-\overline{v}u|^2}{(|1-\overline{v}u| + |u-v|)^4} \left( \frac{1-\overline{v}u - |u-v|}{1-\overline{v}u + |u-v|} \right)^2 \]

\[ - e^{\frac{8|u-v|^2}{|1-\overline{u}u|^d - |u-v|^2}} \frac{|1-\overline{v}u||u-v|}{|1-\overline{v}u|^2 - |u-v|^2} \leq J_f(z) \]

\[ \leq e^{\frac{8|u-v|}{|1-\Phi u|-|u-v|}} \frac{|1-\overline{v}u||1-\overline{v}u|+|u-v|}{(|1-\overline{v}u|-|u-v|)^4} \cdot \]

References


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