数理解析研究所講究録
第 1772 巻 2011 年 91-97

On a broad class of univalent functions

Mamoru Nunokawa[†], Oh Sang Kwon[‡] and Nak Eun Cho[§]

[†] *Department of Mathematics, Gunma University*
Hoshikuki 798-8, Chuou-Ward, Chiba, 260-0808, Japan
e-mail : mamoru_nuno@doctor.nifty.jp
[‡] *Department of Mathematics, Kyungsung University*
Busan 608-736, Korea
e-mail : oskwon@ks.ac.kr
[§] *Department of Applied Mathematics, Pukyong National University*
Busan 608-737, Korea
e-mail : necho@pknu.ac.kr

Abstract

The purpose of the present paper is to give some univalence conditions for a broad class of analytic functions. Moreover, we consider some special cases as corollaries of the main results.

2000 Mathematics Subject Classification. 30C45.

Key Words and Phrases. univalent function, convex function, starlike function, close-to convex function, Bazilevič function of type β and ϕ-like function.

1. Introduction

It is well known that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $D = \{z \mid |z| < 1\}$ and we suppose that $f(z)$ satisfies one of the following conditions

$$\operatorname{Re} f'(z) > 0 \qquad \text{in } D \tag{1.1}$$

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \qquad \text{in } D, \tag{1.2}$$

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \qquad \text{in } D, \tag{1.3}$$

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0 \qquad \text{in } D, \tag{1.4}$$

$$\operatorname{Re} \frac{zf'(z)}{\phi(f(z))} > 0 \qquad \text{in } D, \tag{1.6}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is analytic and satisfies the condition

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > 0 \quad \text{in } D$$

or $g(z)$ is starlike in D, $0 < \beta$ and ϕ is analytic on $f(D)$ with $\phi(0) = 0$ and $\operatorname{Re} \phi'(0) > 0$, then $f(z)$ is univalent in D and we call $f(z)$ when $f(z)$ satisfies the condition (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6) as a Noshiro-Warschawski function, a convex function, a starlike function, a close-to convex function, a Bazilevič function of type β and ϕ-like function, respectively.

It is the prupose of the present paper to introduce a broad class of analytic functions and to investigate some sufficient conditions for univalence of the class.

2. Main Results

Theorem 1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in D and suppose that*

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} > 0 \quad \textit{in } D,$$

where $\varphi(f(z), z)$ is analytic in $(f(D), D)$ and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} > 0 \quad \textit{in } (f(D), D)$$

for $z = re^{i\theta}$, $0 < r < 1$ and $0 \le \theta < 2\pi$. Then $f(z)$ is univalent in D.

Proof. If there exists a point z_0, $|z_0| < 1$ such that

$$f(z) \text{ is univalent for } |z| < |z_0|$$

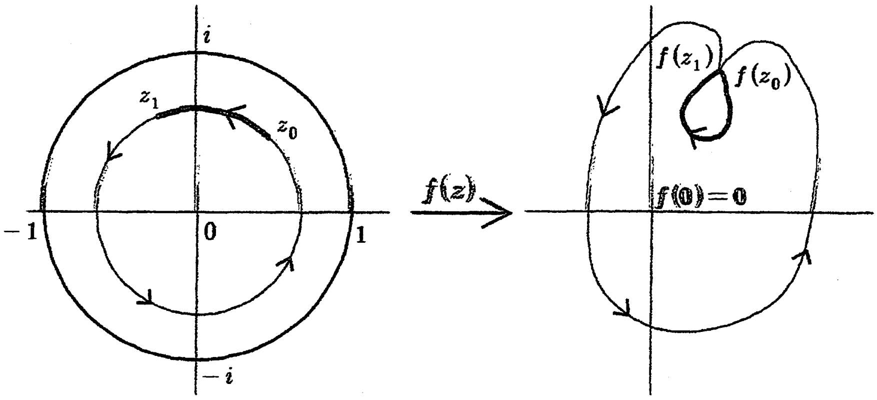
and

$$f(z) \text{ is not univalent for } |z| \le |z_0|,$$

then there exists a point z_1, $z_0 \ne z_1$, $|z_0| = |z_1|$, $z_0 = |z_0|e^{i\theta_0}$, $z_1 = |z_0|e^{i\theta_1}$ and $0 \le \theta_0 < \theta_1 < 2\pi$ for which

$$f(z_0) = f(z_1),$$

as we see in the following figures.



Let

$$C_z = \{z \mid |z| = |z_0|,\ z = |z_0|e^{i\theta} \text{ and } \theta_0 \leq \theta \leq \theta_1\}.$$

Then from the hypothesis, we have

$$\begin{aligned}
-\pi &= \int_{C_z} d \arg df(z) = \int_{C_z} d \arg \frac{df(z)}{dz} dz \\
&= \int_{C_z} d \arg \left(\frac{zf'(z)}{\varphi(f(z), z)} \right) + \int_{C_z} d \arg \left(\frac{dz}{z} \right) + \int_{C_z} d \arg \varphi(f(z), z) \\
&> -\pi + (\arg \varphi(f(z_1), z_1) - \arg \varphi(f(z_0), z_0)) \\
&= -\pi + (\arg \varphi(f(z_0), z_1) - \arg \varphi(f(z_0), z_0)) \\
&= -\pi + \int_{\theta_0}^{\theta_1} \frac{d \arg \varphi(f(z_0), |z_0|e^{i\theta})}{d\theta} d\theta \\
&> -\pi.
\end{aligned}$$

This is contradiction and so, we completes the proof. □

Corollary 1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in D and suppose that*

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} > 0 \quad \textit{in } D,$$

where

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} > 0 \quad \textit{in } (f(D), D)$$

and $\varphi(f(z), z)$ satisfies one of the following conditions

$$\varphi(f(z), z) = z = re^{i\theta} \qquad [5, 8], \tag{2.1}$$

$$\varphi(f(z), z) = zf'(z) = re^{i\theta} f'(e^{i\theta}) \qquad [7], \qquad (2.2)$$

$$\varphi(f(z), z) = f(z) = f(re^{i\theta}) \qquad [4], \qquad (2.3)$$

$$\varphi(f(z), z) = g(z) = g(re^{i\theta}) \qquad [2, 3, 6], \qquad (2.4)$$

$$\varphi(f(z), z) = \varphi(w, re^{i\theta}) = w^{1-\beta} g(z)^{\beta} = w^{1-\beta} g(re^{i\theta}) \qquad [1], \qquad (2.5)$$

where $z = re^{i\theta}$, $0 \le r < 1$, $0 \le \theta < 2\pi$, $0 < \beta$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *is analytic and starlike in* D *or*

$$\mathrm{Re} \frac{zg'(z)}{g(z)} > 0 \quad \textit{in } D.$$

Then $f(z)$ *is univalent in* D.

Proof. For the case (2.1), from hypothesis we have

$$\mathrm{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \mathrm{Re} \frac{zf'(z)}{z} = \mathrm{Re}\, f'(z) > 0 \quad \text{in } D$$

and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} = \frac{d\theta}{d\theta} = 1 > 0 \quad \text{in } D.$$

Applying Theorem 1, $f(z)$ is univalent in D.

For the case (2.2), we have

$$\mathrm{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \mathrm{Re} \frac{zf'(z)}{zf'(z)} = 1 > 0 \quad \text{in } D$$

and

$$\begin{aligned} \frac{d \arg \varphi(w, re^{i\theta})}{d\theta} = \frac{d \arg zf'(z)}{d\theta} &= \frac{d \arg(\frac{z}{dz})}{d\theta} + \frac{d \arg df(z)}{d\theta} \\ &= \frac{d \arg df(z)}{d\theta} = 1 + \mathrm{Re} \frac{zf''(z)}{f'(z)} 0 \quad \text{in } D. \end{aligned}$$

This shows that $f(z)$ is convex and univalent in D.

For the case (2.3), we have

$$\mathrm{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \mathrm{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D$$

and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} = \frac{d \arg f(z)}{d\theta} = \mathrm{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D.$$

This shows that $f(z)$ is starlike in D.

For the case (2.4), we have

$$\operatorname{Re}\frac{zf'(z)}{\varphi(f(z),z)} = \operatorname{Re}\frac{zf'(z)}{g(z)} > 0 \quad \text{in } D$$

and

$$\frac{d\arg\varphi(w,re^{i\theta})}{d\theta} = \frac{d\arg g(z)}{d\theta} = \operatorname{Re}\frac{zg'(z)}{g(z)} > 0 \quad \text{in } D.$$

This shows that $f(z)$ is univalent in D and close-to-convex in D.

For the case (2.5), we have

$$\operatorname{Re}\frac{zf'(z)}{\varphi(f(z),z)} = \operatorname{Re}\frac{zf'(z)}{f(z)^{1-\beta}g(z)^{\beta}} > 0 \quad \text{in } D$$

and

$$\frac{d\arg\varphi(w,re^{i\theta})}{d\theta} = \frac{d\arg w^{1-\beta}g(re^{i\theta})^{\beta}}{d\theta} = \beta\frac{d\arg g(re^{i\theta})}{d\theta} = \beta\operatorname{Re}\frac{zg'(z)}{g(z)} > 0 \quad \text{in } D$$

where $0 < \beta$. This shows that $f(z)$ is univalent in D and $f(z)$ is Bazilevič function of type $0 < \beta$. □

If $f(z)$ is a Bazilevič function of type β, then β must be a positive real number. But we can obtain the following theorem.

Theorem 2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in D and suppose that*

$$\left|\arg\frac{zf'(z)}{f(z)^{1-\beta}g(z)^{\beta}}\right| < \frac{\pi}{2}\alpha \quad \textit{in } D,$$

where $0 < \alpha < 1$, $\beta < 0$, $\alpha - 2\beta < 1$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is analytic and starlike in D. Then $f(z)$ is univalent in D.

Proof. If there exists a point z_0, $|z_0| < 1$ such that

$$f(z) \text{ is univalent for } |z| < |z_0|$$

and

$$f(z) \text{ is not univalent for } |z| \leq |z_0|,$$

then there exists a point z_1, $z_0 \neq z_1$, $|z_1| = |z_0|$, $z_0 = |z_0|e^{i\theta_0}$, $z_1 = |z_0|e^{i\theta_1}$ and $0 \leq \theta_0 < \theta_1 < 2\pi$ for which

$$f(z_0) = f(z_1).$$

Then the image picture under the mapping $w = f(z)$ for $|z| = |z_0|$ is the same as the picture of the proof of Theorem 1.

Let

$$C_z = \{z \mid |z| = |z_0|,\ z = |z_0|e^{i\theta} \text{ and } \theta_0 \le \theta \le \theta_1\},$$
$$C'_z = \{z \mid |z| = |z_0|\} - C_z,$$

and

$$\Gamma'_w = f(C'_z).$$

Then we have

$$\begin{aligned}
3\pi &= \int_{\Gamma'_w} d\arg dw = \int_{\Gamma'_w} d\arg df(z) \\
&= \int_{C'_z} d\arg\left(\frac{zf'(z)}{f(z)^{1-\beta}g(z)^{\beta}}\right) + \int_{C'_z} d\arg\left(\frac{dz}{z}\right) + \int_{C'_z} d\arg f(z)^{1-\beta} + \int_{C'_z} d\arg g(z)^{\beta} \\
&= \int_{C'_z} d\arg\left(\frac{zf'(z)}{f(z)^{1-\beta}g(z)^{\beta}}\right) + (1-\beta)\int_{C'_z} d\arg f(z) + \beta\int_{C'_z} d\arg g(z) \\
&< \alpha\pi + (1-\beta)2\pi = \pi(2+\alpha-2\beta) < 3\pi.
\end{aligned}$$

This is a contradiction and so, we completes the proof. □

References

[1] I. E. Bazilevič, *On a case of integrability inquardratures of the Loewner-Kutarev equation*, Mat. Sb. 37 (1955), 471–476 (Russian).

[2] A. W. Goodman, *Univalent Function II*, Mariner Publishing Com. Inc. Tampa Florida. 1983.

[3] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. 1 (1952), 169–185.

[4] R. Nevanlinna, *Über die konforme Abbildung Sterngebieten*, Oeversikt av Finska-Vetenskaps Societen Forhandlinger 63(A), No. 6 (1921), FM 48–403.

[5] Noshiro, *On the theory of schlicht functions*, J. Fac. Sci. Hokkaido Univ. (1) 2 (1943–1935), 129–155.

[6] S. Ozaki, *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku. Sect. A, 2 (1935), 167–188.

[7] E. Strudy, *Konforme Abbildung Einfachzusammenhangender Bereiche*, B. C. Tueber, Leipzig and Berlin, 1913, FM 44–755.

[8] S. Warshawski, *On the higher derivatives at the boundary in confimal mappings*, Trans. Amer. Math. Soc. 38 (1935), 310–340.