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無限個の錐制約付き半無限計画問題における KKT条件の導出

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Abstract

The semi-infinite program (SIP) is normally represented with infinitely many inequality constraints, and has been studied extensively so far. However, there have been very few studies on the SIP involving conic constraints, even though it has important applications such as Chebyshev-like approximation, filter design, and so on.

In this paper, we focus on the SIP with a convex objective function and infinitely many conic constraints, called an SICP for short. We show that, under Slater's constraint qualification, an optimum of the SICP satisfies the KKT conditions that can be represented only with a *finite* subset of the conic constraints.

1 Introduction

In this paper, we focus on the following optimization problem with an infinite number of *conic* constraints:

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & A(t)^\top x - b(t) \in C \quad \text{for all } t \in T, \end{array} \tag{1.1}$$

where $f:\mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable convex function, $A: T \to \mathbb{R}^{n\times m}$ and $b: T \to \mathbb{R}^m$ are continuous functions, $T \subset \mathbb{R}^\ell$ is a given compact set, and $C \subset \mathbb{R}^m$ is a closed convex cone with nonempty interior. We call this problem the semi-infinite conic program, SICP for short. Throughout this paper, we assume that SICP (1.1) has a nonempty solution set.

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When $m = 1$ and $C = \mathbb{R}_+ := \{z \in \mathbb{R} \mid z \geq 0\}$, SICP (1.1) reduces to the classical semi-infinite program (SIP) [3, 5, 7, 9, 10, 13], which has wide applications in engineering, e.g., the air pollution control, the robot trajectory planning, the stress of materials, etc.[7, 9]. So far, many algorithms have been proposed for solving SIPs, such as the discretization method [3], the local reduction based method [4, 8, 14] and the exchange method [5, 6, 13]. A more general choice for C is the symmetric cone such as the second-order cone (SOC) $\mathcal{K}^m := \left\{(z_1, z_2, \ldots, z_m)^\top \in \mathbb{R}^m \mid z_1 \geq \|(z_2, z_3, \ldots, z_m)^\top\|_2\right\}$ and the semi-definite cone $\mathcal{S}^m_+ := \{Z \in \mathbb{R}^{m\times m} \mid Z = Z^\top,\ Z \succeq 0\}$.

There are some important applications of SICP (1.1). For example, when C is an SOC, SICP (1.1) can be used to formulate a Chebyshev-like approximation problem involving vector-valued functions. Specifically, let $Y \subseteq \mathbb{R}^n$ be a given compact set, and $\Phi : Y \to \mathbb{R}^m$ and $F : \mathbb{R}^\ell \times Y \to \mathbb{R}^m$ be given functions. Then, we want to determine a parameter $u \in \mathbb{R}^\ell$ such that $\Phi(y) \approx F(u, y)$ for all $y \in Y$. One relevant approach is to solve the following problem:

$$\underset{u}{\text{Minimize}} \quad \max_{y\in Y} \|\Phi(y) - F(u, y)\|_2.$$

By introducing the auxiliary variable $r \in \mathbb{R}$, we can transform the above problem to

$$\begin{array}{ll} \underset{u,r}{\text{Minimize}} & r \\ \text{subject to} & \begin{pmatrix} r \\ \Phi(y) - F(u, y) \end{pmatrix} \in \mathcal{K}^{m+1} \ \text{ for all } y \in Y, \end{array}$$

which is of the form (1.1) when F is affine with respect to u.

The main purpose of the paper is to study the Karush-Kuhn-Tucker (KKT) conditions for SICP (1.1). Although the original KKT conditions for SICP (1.1) could be described by means of integration and Borel measure, we show that they can be represented by a *finite* number of elements in T under Slater's constraint qualification.

Throughout the paper, we use the following notations. $\|\cdot\|$ denotes the Euclidean norm defined by $\|z\| := \sqrt{z^\top z}$ for $z \in \mathbb{R}^m$. For a given cone $C \subseteq \mathbb{R}^m$, C^d denotes the dual cone defined by $C^d := \{z \in \mathbb{R}^m \mid z^\top w \geq 0,\ \forall w \in C\}$. For vectors $z \in \mathbb{R}^m$ and $w \in \mathbb{R}^m$, the conic complementarity condition, $z^\top w = 0,\ z \in C$ and $w \in C^d$, is also written as $C \ni z \perp w \in C^d$. For a nonempty set $D \subseteq \mathbb{R}^m$ and a function $h : \mathbb{R}^m \to \mathbb{R}$, $\text{argmin}_{z\in D} h(z)$ denotes the set of minimizers of h over D. In addition, for $z \in \mathbb{R}^m$ and $\delta > 0$, $B(z, \delta) \subseteq \mathbb{R}^m$ denotes the closed ball with center z and radius δ, i.e., $B(z, \delta) := \{w \in \mathbb{R}^m \mid \|w - z\| \leq \delta\}$.

2 Karush-Kuhn-Tucker Conditions

In this section, we provide the optimality conditions for SICP (1.1). When $m = 1$ and $C = \mathbb{R}_+$, SICP (1.1) reduces to the classical semi-infinite program and the optimality conditions are given as follows [9, Theorem 2].

Let $\bar{x}$ be an optimum of SICP (1.1) with $m = 1$ and $C = \mathbb{R}_+$. Suppose that the Slater constraint qualification holds for SICP (1.1) with $C = \mathbb{R}_+$, i.e., there exists an $x_0 \in \mathbb{R}^n$ such that $A(t)^\top x_0 - b(t) > 0$ $(\forall t \in T)$. Then, there exist p elements $t_1, t_2, \ldots, t_p \in T$ such that $p \leq n$ and

$$
\begin{aligned}
&\nabla f(\bar{x}) - \sum_{i=1}^{p} \eta_i A(t_i) = 0, \\
&\mathbb{R}_+ \ni \eta_i \perp A(t_i)^\top \bar{x} - b(t_i) \in \mathbb{R}_+ \quad (i = 1, 2, \ldots, p).
\end{aligned} \tag{2.1}
$$

In this section, we define the generalized Slater constraint qualification (GSCQ), and show that the optimality conditions can be represented with finitely many conic constraints under the GSCQ.

This section consists of two subsections. In Subsection 2.1, we define the GSCQ and the generalized Abadie constraint qualifications (GACQ) and show that the GACQ holds under the GSCQ. In Subsection 2.2, we derive the optimality conditions for SICP (1.1) by using the results of Subsection 2.1 and Carathéodory's Theorem.

Before going to the subsections, we provide some propositions, which play important roles in proving the propositions and theorems.

Proposition 2.1. [11] *Let $C \subseteq \mathbb{R}^n$ be an arbitrary nonempty cone. Then, we have*

$$C^{dd} = \operatorname{cl} \operatorname{co} C.$$

Particularly, when C is a closed convex cone, we have $C = C^{dd}$.

Proposition 2.2. *Let $D \subseteq \mathbb{R}^n$ be an arbitrary convex set with nonempty interior. Then, we have*

$$x \in \operatorname{int} D,\ y \in \operatorname{cl} D,\ \lambda \in [0, 1) \implies (1 - \lambda)x + \lambda y \in \operatorname{int} D. \tag{2.2}$$

Proof. Choose $x \in \operatorname{int} D$, $y \in \operatorname{cl} D$ and $\lambda \in [0, 1)$ arbitrarily. We will show that there exists an $\varepsilon > 0$ such that $(1 - \lambda)x + \lambda y + B(0, \varepsilon) \subseteq D$, where $B(0, \varepsilon) := \{x \in \mathbb{R}^n \mid \|x\| \leq \varepsilon\}$. From $y \in \operatorname{cl} D$, we have $y \in D + B(0, \varepsilon)$ for any $\varepsilon > 0$. Therefore, by choosing a sufficiently small $\varepsilon > 0$, we have

$$
\begin{aligned}
(1 - \lambda)x + \lambda y + B(0, \varepsilon) &\subseteq (1 - \lambda)x + \lambda(D + B(0, \varepsilon)) + B(0, \varepsilon) \\
&= (1 - \lambda)(x + (1 - \lambda)^{-1}(1 + \lambda)B(0, \varepsilon)) + \lambda D \\
&\subseteq (1 - \lambda)D + \lambda D = D,
\end{aligned}
$$

where the equalities hold since $\alpha X + \beta X = (\alpha + \beta)X$ for any $\alpha, \beta \geq 0$ and any convex set X, and the last inclusion is due to $x \in \operatorname{int} D$. $\square$

2.1 Generalized Slater and Abadie constraint qualifications

In the case of the convex optimization problem with finitely many inequality constraints, it is known that Abadie's constraint qualification holds under Slater's constraint qualification, and then the KKT conditions serve as a necessary and sufficient condition for the global optimality [1]. In this subsection, we define the generalized Slater and Abadie constraint qualifications (GSCQ and GACQ) for SICP (1.1), and show that the GACQ always holds under the GSCQ. Let $\bar{x}$ be an arbitrary feasible solution of SICP (1.1), and S be the feasible solution set of SICP (1.1), that is,

$$S := \{x \in \mathbb{R}^n \mid A(t)^\top x - b(t) \in C \ (\forall t \in T)\}.$$

We define the following cones:

$$\begin{aligned} G_t(\bar{x}) &:= \{\alpha(A(t)^\top \bar{x} - b(t)) \mid \alpha \leq 0\}, \\ \Lambda_t(\bar{x}) &:= C + G_t(\bar{x}), && (2.3) \\ C_t(\bar{x}) &:= \left\{y \in R^n \mid A(t)^\top y \in \operatorname{cl} \Lambda_t(\bar{x})\right\}, && (2.4) \\ C_S(\bar{x}) &:= \bigcap_{t \in T} C_t(\bar{x}). && (2.5) \end{aligned}$$

We note that the closure of $\Lambda_t(\bar{x})$ is the tangent cone of C at $A(t)^\top \bar{x} - b(t)$, and the dual cone of $\Lambda_t(\bar{x})$ characterizes the directions satisfying the conic complementarity conditions for $A(t)^\top \bar{x} - b(t)$, i.e., $\Lambda_t(\bar{x})^d = \{y \in \mathbb{R}^m \mid C \ni y \perp A(t)^\top \bar{x} - b(t) \in C\}$. (See Proposition 2.9 below.) Also, $C_S(\bar{x})$ is a generalization of the linearized cone as defined in [2], for the case where $|T| < \infty$ and $C = \mathbb{R}_+$.

Now, we define GSCQ and GACQ by using the above cones.

Definition 2.3 (GSCQ). *We say that the generalized Slater constraint qualification (GSCQ) holds for SICP* (1.1) *if there exists some $x_0 \in \mathbb{R}^n$ such that*

$$A(t)^\top x_0 - b(t) \in \operatorname{int} C \quad (\forall t \in T). \tag{2.6}$$

Definition 2.4 (GACQ). *Let S and $\bar{x} \in S$ be the feasible set and a feasible solution of SICP* (1.1)*, respectively. Then, we say that the generalized Abadie constraint qualification GACQ holds at $\bar{x} \in S$ if*

$$C_S(\bar{x}) \subseteq T_S(\bar{x}), \tag{2.7}$$

where $C_S(\bar{x})$ is defined by (2.5) *and $T_S(\bar{x})$ is the tangent cone to S at $\bar{x}$.*

Next, we show that the GACQ holds under the GSCQ. To this end, we show the following two lemmas by using the following set:

$$C_S^\circ(\bar{x}) := \bigcap_{t \in T} \left\{y \in \mathbb{R}^n \mid A(t)^\top y \in \operatorname{int} C + G_t(\bar{x})\right\}. \tag{2.8}$$

Notice that $C_S^\circ(\bar{x})$ is not empty for the GSCQ.

Lemma 2.5. *Assume that the GSCQ holds for SICP* (1.1)*. Let $\bar{x}$ be an arbitrary feasible solution of SICP* (1.1)*. Let $C_S(\bar{x})$ and $C_S^\circ(\bar{x})$ be defined by* (2.5) *and* (2.8)*, respectively. Then, $C_S^\circ(\bar{x})$ is nonempty and $C_S(\bar{x}) = \operatorname{cl} C_S^\circ(\bar{x})$.*

Proof. If we have $C_S(\bar{x}) = \mathrm{cl}\, C_S^\circ(\bar{x})$, then $C_S^\circ(\bar{x})$ must be nonempty since $0 \in C_S(\bar{x})$. So, we only show $C_S(\bar{x}) = \mathrm{cl}\, C_S^\circ(\bar{x})$. Notice that $C_S(\bar{x}) \supseteq C_S^\circ(\bar{x})$. Then, we have $C_S(\bar{x}) \supseteq \mathrm{cl}\, C_S^\circ(\bar{x})$, since $C_S(\bar{x})$ is closed. Thus, it suffices to show $C_S(\bar{x}) \subseteq \mathrm{cl}\, C_S^\circ(\bar{x})$. Let $y \in C_S(\bar{x})$ be chosen arbitrarily. Then, we have to show that there exists some $\{y^k\} \subseteq C_S^\circ(\bar{x})$ such that $y^k \to y$ as $k \to \infty$. By the GSCQ, there is an $x_0 \in \mathbb{R}^n$ such that $A(t)^\top x_0 - b(t) \in \mathrm{int}\, C$ for any $t \in T$. Let $y_0 := x_0 - \bar{x}$. Then, we have $A(t)^\top y_0 = (A(t)^\top x_0 - b(t)) - (A(t)^\top \bar{x} - b(t)) \in \mathrm{int}\, C + G_t(\bar{x})$. Since $\mathrm{int}\, C + G_t(\bar{x})$ is an open convex set, we have

$$A(t)^\top y_0 \in \mathrm{int}\, C + G_t(\bar{x}) = \mathrm{int}\,(\mathrm{int}\, C + G_t(\bar{x})).$$

for any $t \in T$. Since $y \in C_S(\bar{x})$ and $\mathrm{cl}\, \Lambda_t(\bar{x}) = \mathrm{cl}\,(C + G_t(\bar{x})) = \mathrm{cl}\,(\mathrm{int}\, C + G_t(\bar{x}))$[1], we have

$$A(t)^\top y \in \mathrm{cl}\,(\mathrm{int}\, C + G_t(\bar{x})).$$

Applying Proposition 2.2 with $D := \mathrm{int}\, C + G_t(\bar{x})$, $x := A(t)^\top y_0$, $\lambda := 1 - \eta$ and $y := A(t)^\top y$, we have

$$A(t)^\top ((1-\eta) y + \eta y_0) \in \mathrm{int}\, C + G_t(\bar{x}) \tag{2.9}$$

for any $t \in T$ and $\eta \in (0,1]$. Let $\{\eta_k\} \subseteq (0,1]$ be a sequence such that $\lim_{k\to\infty} \eta_k = 0$ and $\{y^k\}$ be defined by $y^k := (1-\eta_k) y + \eta_k y_0$. Then, (2.9) implies that $A(t)^\top y^k \in \mathrm{int}\, C + G_t(\bar{x})$ for any k and $t \in T$. Therefore, $\{y^k\} \subseteq C_S^\circ(\bar{x})$ and $\lim_{k\to\infty} y^k = y$. This completes the proof. □

Lemma 2.6. *Assume that the GSCQ holds for SICP* (1.1). *Let $\bar{x}$ be an arbitrary feasible solution of SICP*(1.1). *For $y \in \mathbb{R}^n$ and $t \in T$, let $\alpha_y(t) \in \mathbb{R}$ be defined by*

$$\alpha_y(t) := \max_{\alpha \in [0,1]} \{\alpha \mid A(t)^\top (\bar{x} + \alpha y) - b(t) \in C\}. \tag{2.10}$$

Then, for any $y \in C_S^\circ(\bar{x})$, we have

$$\inf_{t \in T} \alpha_y(t) > 0.$$

Proof. Let $y \in C_S^\circ(\bar{x})$ and $t \in T$ be chosen arbitrarily. First note that $\alpha_y(t) \geq 0$, since $\bar{x}$ is feasible to SICP (1.1). Then, we first prove $\alpha_y(t) > 0$. To this end, it suffices to show the existence of $\alpha \in (0,1]$ such that

$$A(t)^\top (\bar{x} + \alpha y) - b(t) \in \mathrm{int}\, C. \tag{2.11}$$

Since $y \in C_S^\circ(\bar{x})$, we have $A(t)^\top y \in \mathrm{int}\, C + G_t(\bar{x})$, which together with the definition of $G_t(\bar{x})$ implies the existence of some $\beta \geq 0$ such that

$$\beta \left(A(t)^\top \bar{x} - b(t)\right) + A(t)^\top y \in \mathrm{int}\, C. \tag{2.12}$$

When $\beta = 0$, (2.12) reduces to $A(t)^\top y \in \mathrm{int}\, C$, which together with $A(t)^\top \bar{x} - b(t) \in C$ and Proposition 2.2 implies $\frac{1}{2} A(t)^\top y + \frac{1}{2}\left(A(t)^\top \bar{x} - b(t)\right) = \frac{1}{2}\left(A(t)^\top (\bar{x} + y) - b(t)\right) \in$

[1]This equality can be obtained easily from the fact that $\mathrm{cl}\,(\mathrm{int}\, C) = C$

$\operatorname{int} C$, and hence, $A(t)^\top(\bar{x}+y)-b(t) \in \operatorname{int} C$. We thus have (2.11) with $\alpha = 1$. When $\beta > 0$, by multiplying (2.12) by β^{-1}, we have $A(t)^\top(\bar{x} + \beta^{-1}y) - b(t) \in \operatorname{int} C$. Due to Proposition 2.2, we have $A(t)^\top(\bar{x} + sy) - b(t) \in \operatorname{int} C$ for any $s \in (0, \beta^{-1}]$, which implies $A(t)^\top(\bar{x} + \min(\beta^{-1}, 1)y) - b(t) \in \operatorname{int} C$. Hence, we also have (2.11).

In what follows, we show $\inf_{t\in T} \alpha_y(t) > 0$. Suppose to the contrary that there exists a sequence $\{t^k\} \subseteq T$ such that $\alpha_y(t^k) \to 0$ as $k \to \infty$. Let t^* be an arbitrary accumulation point of $\{t^k\}$. Then, by taking an appropriate subsequence, we have

$$\lim_{k\to\infty} t^k = t^*, \qquad \lim_{k\to\infty} \alpha_y(t^k) = 0. \tag{2.13}$$

From (2.11), there exists an $\bar{\alpha} > 0$ such that

$$A(t^*)^\top(\bar{x} + \bar{\alpha}y) - b(t^*) \in \operatorname{int} C. \tag{2.14}$$

Hence, by the continuity of functions A and b, we have

$$A(t^k)^\top(\bar{x} + \bar{\alpha}y) - b(t^k) \in \operatorname{int} C \tag{2.15}$$

for all k sufficiently large. From (2.15) and (2.10), we have $0 < \bar{\alpha} \leq \alpha_y(t^k)$, which together with (2.13) implies $\bar{\alpha} = 0$. However, this contradicts $\bar{\alpha} > 0$. Hence, we have $\inf_{t\in T} \alpha_y(t) > 0$.

□

Now, we show the main theorem of this section, which claims that the GSCQ implies the GACQ for SICP (1.1).

Theorem 2.7. *Let $\bar{x}$ be an arbitrary feasible solution of SICP* (1.1). *Assume that the GSCQ holds. Then, the GACQ holds at $\bar{x}$.*

Proof. Let $C_S^\circ(\bar{x})$ be defined by (2.8). Then we have $\operatorname{cl} C_S^\circ(\bar{x}) = C_S(\bar{x})$ from Lemma 2.5. Therefore, due to the closedness of $T_S(\bar{x})$, we only have to show

$$C_S^\circ(\bar{x}) \subseteq T_S(\bar{x}).$$

Let $y \in C_S^\circ(\bar{x})$ be chosen arbitrarily and $\alpha_y := \inf_{t\in T} \alpha_y(t)$, where $\alpha_y(t)$ is given by (2.10). Then, we have

$$A(t)^\top(\bar{x} + \beta y) - b(t) \in C \tag{2.16}$$

for any $\beta \in [0, \alpha_y]$ and $t \in T$, since $A(t)^\top\bar{x} - b(t) \in C$ and C is convex.

By Lemma 2.6, we have $\alpha_y > 0$. Hence, we can choose $\{b_k\} \subseteq (0, \alpha_y]$ such that $\lim_{k\to\infty} b_k = 0$. By (2.16), we have

$$A(t)^\top(\bar{x} + b_k y) - b(t) \in C \qquad (\forall t \in T),$$

which implies $\bar{x} + b_k y \in S$ for all k. Now, recall that the definition of $T_S(\bar{x})$ is given by

$$T_S(\bar{x}) := \left\{y \in \mathbb{R}^n \mid \lim_{k\to\infty} a_k(x_k - \bar{x}) = y,\ \lim_{k\to\infty} x_k = \bar{x},\ x_k \in S,\ a_k \geq 0\ (k = 1, 2, \ldots)\right\}. \tag{2.17}$$

Thus, by setting $x_k := \bar{x} + b_k y$ and $a_k := 1/b_k$, we have $y \in T_S(\bar{x})$. The proof is completed. □

2.2 The KKT conditions for SICP

As we have shown in the previous subsection, the GACQ holds under the GSCQ. In this subsection, by using this result, we show that the optimality condition for SICP (1.1) can be represented as the KKT conditions with finitely many conic constraints. It is well known that the following Carathéodory's Theorem plays a significant role in deriving the optimality condition for the ordinary SIP with inequality constraints. The theorem is also important in deriving the optimality conditions for SICP (1.1).

Lemma 2.8. (Carathéodory's Theorem [11, Theorem 17.1]) *Let $D \subseteq \mathbb{R}^n$ be an arbitrary nonempty set, and $\mathrm{co}\, D$ be the convex hull of D. Then, for any $x \in \mathrm{co}\, D$, there exist p elements $s_1, s_2, \dots, s_p \in D$ and p positive numbers $\lambda_1, \lambda_2, \dots, \lambda_p > 0$ such that $p \leq n+1$, $\sum_{i=1}^{p} \lambda_i = 1$, and $x = \lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_p s_p$.*

The conic complementarity condition that appears in the KKT conditions is written as $C \ni y(t) \perp A(t)^\top x - b(t) \in C$ with a Lagrange multiplier vector $y(t)$. The next proposition claims that the dual cone of $\Lambda_t(\bar{x})$ defined by (2.3) characterizes the Lagrange multiplier $y(t)$.

Proposition 2.9. *Let $t \in T$ be chosen arbitrarily, and $\bar{x}$ be an arbitrary feasible solution of SICP (1.1). Let $\Lambda_t(\bar{x})$ be defined by (2.3). Then, we have*

$$\begin{aligned}
\Lambda_t(\bar{x}) &= \mathrm{co}\,(C \cup G_t(\bar{x})), \\
\Lambda_t(\bar{x})^d &= C \cap G_t(\bar{x})^d \\
&= \{y \in \mathbb{R}^m \mid C \ni y \perp A(t)^\top \bar{x} - b(t) \in C\}.
\end{aligned}$$

Proof. First, we show $\Lambda_t(\bar{x}) = C + G_t(\bar{x}) = \mathrm{co}\,(C \cup G_t(\bar{x}))$. Since $0 \in G_t(\bar{x})$ and $0 \in C$, we have $C + G_t(\bar{x}) \supseteq C$ and $C + G_t(\bar{x}) \supseteq G_t(\bar{x})$, that is, $C + G_t(\bar{x}) \supseteq C \cup G_t(\bar{x})$. Noticing the convexity of $C + G_t(\bar{x})$, we have $C + G_t(\bar{x}) \supseteq \mathrm{co}\,(C \cup G_t(\bar{x}))$. Conversely, we show $C + G_t(\bar{x}) \subseteq \mathrm{co}\,(C \cup G_t(\bar{x}))$. Choose $y \in C + G_t(\bar{x})$ arbitrarily. Then, there exist some $k \in C$ and $g \in G_t(\bar{x})$ such that $y = k + g$. Since C and $G_t(\bar{x})$ are cones, we have $2k \in C$ and $2g \in G_t(\bar{x})$, and hence $y = (2k + 2g)/2 \in \mathrm{co}\,(C \cup G_t(\bar{x}))$. This shows $C + G_t(\bar{x}) \subseteq \mathrm{co}\,(C \cup G_t(\bar{x}))$.

We can readily show $\Lambda_t(\bar{x})^d = C \cap G_t(\bar{x})^d$ since $\Lambda_t(\bar{x})^d = (\mathrm{co}\,(C \cup G_t(\bar{x})))^d = (C \cup G_t(\bar{x}))^d = C^d \cap G_t(\bar{x})^d = C \cap G_t(\bar{x})^d$, where the second equality follows since $(\mathrm{co}\, C)^d = C^d$ for any cone C, the third equality holds since $(C_1 \cup C_2)^d = C_1^d \cap C_2^d$ for any cones C_1 and C_2, and the last equality holds since C is self-dual.

Finally we show $C \cap G_t(\bar{x})^d = \{y \in \mathbb{R}^m \mid C \ni y \perp A(t)^\top \bar{x} - b(t) \in C\}$. Note that $\{y \in \mathbb{R}^m \mid C \ni y \perp A(t)^\top \bar{x} - b(t) \in C\} = C \cap (A(t)^\top \bar{x} - b(t))^\perp$, where $v^\perp$ denotes the hyper plane orthogonal to vector v. Since it is not difficult to see $G_t(\bar{x})^d \supseteq (A(t)^\top \bar{x} - b(t))^\perp$, we have $C \cap G_t(\bar{x})^d \supseteq C \cap (A(t)^\top \bar{x} - b(t))^\perp$. Therefore, the proof will be complete if we show the converse inclusion. Choose $z \in C \cap G_t(\bar{x})^d$ arbitrarily. Since $z \in G_t(\bar{x})^d$, we have $z^\top (A(t)^\top \bar{x} - b(t)) \leq 0$. On the other hand, $z \in C$ and $A(t)^\top \bar{x} - b(t) \in C$ imply $z^\top (A(t)^\top \bar{x} - b(t)) \geq 0$. Hence, $z^\top (A(t)^\top \bar{x} - b(t)) = 0$, i.e., $z \in C \cap (A(t)^\top \bar{x} - b(t))^\perp$. This completes the proof. $\square$

Now, in order to obtain the optimality condition for SICP (1.1), we introduce the following cones:

$$H_t(\bar{x}) := \{z \in \mathbb{R}^n \mid z = A(t)\lambda,\ \lambda \in \Lambda_t(\bar{x})^d\}, \tag{2.18}$$

$$H(\bar{x}) := \bigcup_{t \in T} H_t(\bar{x}), \tag{2.19}$$

where $t \in T$ and $\bar{x}$ is a feasible solution. Note that $H_t(\bar{x})$ is a convex cone but may not be closed, and $H_t(\bar{x})$ is a cone but may not be closed or convex.

The next proposition shows the relation between $H(\bar{x})$ and $C_S(\bar{x})$.

Proposition 2.10. *Let $\bar{x} \in S$ be an arbitrary feasible solution of SICP* (1.1)*. Let $C_S(\bar{x})$ and $H(\bar{x})$ be defined by* (2.5) *and* (2.19)*, respectively. Then, we have*

$$C_S(\bar{x})^d \subseteq \operatorname{cl}\operatorname{co} H(\bar{x}).$$

Proof. It suffices to prove $C_S(\bar{x}) \supseteq H(\bar{x})^d$. Choose $y \in H(\bar{x})^d$, $t \in T$ and $\lambda \in \Lambda_t(\bar{x})^d$ arbitrarily. Since $y \in H(\bar{x})^d$ and $A(t)\lambda \in H_t(\bar{x}) \subseteq H(\bar{x})$, we have $\langle A(t)^\top y, \lambda\rangle = \langle y, A(t)\lambda\rangle \geq 0$. Note that $t \in T$ and $\lambda \in \Lambda_t(\bar{x})^d$ were chosen arbitrarily. Therefore, we have $A(t)^\top y \in \Lambda_t(\bar{x})^{dd} = \operatorname{cl}\operatorname{co}\Lambda_t(\bar{x}) = \operatorname{cl}\Lambda_t(\bar{x})$ for any $t \in T$, which implies $y \in C_S(\bar{x})$. □

The following lemma is also important for the proof of the subsequent theorem.

Lemma 2.11. *Assume that the GSCQ holds for SICP* (1.1)*. Let x_0 be an arbitrary point satisfying* (2.6) *and $z \in C$ be an arbitrary vector. Then, there exists some $\varepsilon > 0$ such that*

$$(A(t)^\top x_0 - b(t))^\top z \geq \varepsilon\|z\| \tag{2.20}$$

for any $t \in T$.

Proof. For simplicity, let $y(t) := A(t)^\top x_0 - b(t)$. When $z = 0$, inequality (2.20) holds obviously for any $t \in T$. So we only consider the case where $z \neq 0$. Let

$$\delta(t) := \frac{y(t)^\top z}{\|z\|}. \tag{2.21}$$

To show (2.20), it suffices to prove $\inf_{t\in T}\delta(t) > 0$. Suppose that $\inf_{t\in T}\delta(t) \leq 0$ for contradiction. Then, we must have $\inf_{t\in T}\delta(t) = 0$ since $y(t) \in \operatorname{int} C$ and $z \in C$ implies $\delta(t) \geq 0$. Due to the compactness of T, there exist some subsequence $\{t^k\} \subseteq T$ and $t^* \in T$ such that $\lim_{k\to\infty}\delta(t^k) = 0$ and $\lim_{k\to\infty} t^k = t^*$. Moreover, the continuity of $y(t)$ yields $\lim_{k\to\infty} y(t^k) = y(t^*)$. Then, by (2.21), we obtain $y(t^*)^\top z = 0$. However, this contradicts $0 \neq z \in C$ and $y(t^*) \in \operatorname{int} C$. Therefore, we have $\inf_{t\in T}\delta(t) > 0$. □

Now, we are in the position to show the theorem on the optimality condition for SICP (1.1).

Theorem 2.12 (Optimality condition). *Assume that the GSCQ holds for SICP (1.1). Let x^* be an arbitrary optimizer of SICP (1.1). Then, there exist $t_1, t_2, \ldots, t_p \in T$ and $y_1, y_2, \ldots, y_p \in \mathbb{R}^m$ such that $p \le n+1$ and*

$$\nabla f(x^*) - \sum_{i=1}^{p} A(t_i) y_i = 0, \tag{2.22}$$

$$C \ni y_i \perp A(t_i)^\top x^* - b(t_i) \in C \quad (i = 1, 2, \ldots, p). \tag{2.23}$$

Proof. From $x^* \in \operatorname{argmin}_{x \in S} f(x)$ and [12, Theorem 3.6], we have $\nabla f(x^*) \in T_S(x^*)^d$. Also we have $T_S(x^*)^d \subseteq C_S(x^*)^d \subseteq \operatorname{cl} \operatorname{co} H(x^*)$, where the first inclusion holds since $C_S(x^*) \subseteq T_S(x^*)$ from Theorem 2.7, and the second inclusion follows from Proposition 2.10. Therefore, we have

$$\nabla f(x^*) \in \operatorname{cl} \operatorname{co} H(x^*),$$

which indicates the existence of a sequence $\{z^k\} \subseteq \operatorname{co} H(x^*)$ such that

$$\lim_{k \to \infty} z^k = \nabla f(x^*).$$

By Lemma 2.8, (2.18) and (2.19), there exist $n+1$ nonnegative scalars[2] $\alpha_1^k, \alpha_2^k, \ldots, \alpha_{n+1}^k \ge 0$ such that $\sum_{i=1}^{n+1} \alpha_i^k = 1$ and

$$z^k = \sum_{i=1}^{n+1} A(t_i^k) \alpha_i^k \lambda_i^k, \quad \lambda_i^k \in \Lambda_{t_i^k}(x^*)^d. \tag{2.24}$$

Denote $y_i^k := \alpha_i^k \lambda_i^k \in \Lambda_{t_i^k}(x^*)^d$ for each i in (2.24).

In what follows, we show that the sequence $\{y_i^k\}$ is bounded and any accumulation point satisfies (2.22) and (2.23). From the GSCQ, there exists an $x_0 \in \mathbb{R}^n$ such that $A(t_i^k)^\top x_0 - b(t_i^k) \in \operatorname{int} C$ for each i. By $y_i^k \in \Lambda_{t_i^k}(x^*)^d \subseteq C$ and Lemma 2.11, there exists $\varepsilon > 0$ such that

$$\langle y_i^k, \ A(t_i^k)^\top x_0 - b(t_i^k) \rangle \ge \varepsilon \|y_i^k\| \tag{2.25}$$

for each i. Since $y_i^k \in \Lambda_{t_i^k}(x^*)^d \subseteq G_{t_i^k}(x^*)^d$ from Proposition 2.9, we have

$$\langle y_i^k, A(t_i^k)^\top x^* - b(t_i^k) \rangle \le 0. \tag{2.26}$$

It, then, follows from (2.26) and (2.25) that

$$\langle y_i^k, A(t_i^k)^\top (x_0 - x^*) \rangle \ge \varepsilon \|y_i^k\|. \tag{2.27}$$

From (2.24), (2.27) and $y_i^k = \alpha_i^k \lambda_i^k$, we have $(z^k)^\top (x_0 - x^*) = \sum_{i=1}^{n+1} \langle y_i^k, A(t_i^k)^\top (x_0 - x^*) \rangle \ge \sum_{i=1}^{n+1} \varepsilon \|y_i^k\|$. Moreover, since $\{z^k\}$ is convergent, there exists $M > 0$ such that $(z^k)^\top (x_0 - x^*) \le M$ for all k. Therefore, we have

$$M \ge \varepsilon \sum_{i=1}^{n+1} \|y_i^k\|.$$

[2]If we have $p < n+1$ scalars, then we can set $\alpha_{p+1}^k = \alpha_{p+2}^k = \cdots = \alpha_{n+1}^k = 0$ without loss of generality.

which implies the boundedness of $\{y_i^k\}$. Now, let y_i and t_i be arbitrary accumulation points of $\{y_i^k\}$ and $\{t_i^k\}$, respectively. Then there exist subsequences such that $z^k \to \nabla f(x^*)$, $t_i^k \to t_i$ and $y_i^k \to y_i$ for each $i = 1, 2, \ldots, n+1$. From (2.24) with $y_i^k = \alpha_i^k \lambda_i^k$ and the continuity of function A, we obtain $\nabla f(x^*) = \sum_{i=1}^{n+1} A(t_i) y_i$. Hence, we have (2.22). From $y_i^k \in \Lambda_{t_i^k}(x^*)^d$ and Proposition 2.9, it follows that $C \ni y_i^k \perp A(t_i^k)^\top x^* - b(t_i^k) \in C$ for each k. Since C is closed, we have $y_i \in C$ and $A(t_i)^\top x^* - b(t_i) \in C$. Moreover, we have $\langle y_i, A(t_i)^\top x^* - b(t_i) \rangle = 0$, since the function defined by $\theta(y, t) := \langle y, A(t)^\top x^* - b(t) \rangle$ is continuous at any $y \in \mathbb{R}^m$ and $t \in T$. Therefore, (2.23) is obtained. □

3 Concluding remarks

For the semi-infinite program with an infinite number of conic constraints (SICP), we have shown that the KKT conditions can be represented with finitely many conic constraints, as long as the Slater constraint qualification holds. It is an interesting subject of future research to extend the result to the more general SICP without the convexity assumption on the objective and constraint functions.

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