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# A note on quantum fundamental groups and quantum representation varieties for 3-manifolds

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This informal note is based on the author's talk "Quantum fundamental groups and quantum representation varieties for 3-manifolds" given in the workshop "Geometric and analytic approaches to representations of a group and representation spaces", held at RIMS during June 20 – June 24, 2011. Details of this note will appear in papers in preparation.

## 1 Cobordism categories and embedding categories

### 1.1 Cobordism categories and TQFTs

In Quantum Topology, one considers the *cobordism category* $\mathbf{Cob}_d$, whose objects are compact, oriented $(d-1)$-manifolds and whose morphisms are homeomorphism classes of $d$-dimensional cobordisms.

A $d$-dimensional Topological Quantum Field Theory is a functor

$$F\colon \mathbf{Cob}_d \to \mathbf{Vect}$$

from $\mathbf{Cob}_d$ to the category $\mathbf{Vect}$ of vector spaces.

### 1.2 Embedding category $\mathbf{Emb}_d$

In this note, we consider embedding categories, which are another type of categories closely related to manifold topology. Let $d \geq 1$ be an integer. The $d$-dimensional *embedding category* $\mathbf{Emb}_d$ is the category whose objects are compact, oriented $d$-manifolds, and whose morphisms are isotopy classes of embeddings. Composition of morphisms is induced by composition of embeddings, and the identity morphisms is represented by the identity homeomorphisms.

In what follows, we often confuse an embedding $f\colon M \to N$ and its isotopy class $[f]\colon M \to N$.

## 1.3 Relation between embedding categories and cobordism categories

The embedding category is related to the cobordism category as follows. There is a functor

$$\begin{aligned}\partial\colon\ &\mathbf{Emb}_d \to \mathbf{Cob}_d,\\ &M \mapsto \partial M\\ &[f\colon M \hookrightarrow N] \mapsto [N \setminus (\operatorname{int} f(M))].\end{aligned}$$

More precisely, the functor $\partial$ maps each $(d-1)$-manifold $M$ to its boundary $\partial M$, and each morphism $[f]\colon M \to N$ (represented by an embedding $f\colon M \to N$) to its "complement" $[N \setminus (\operatorname{int} f(M))]$, where $f$ is chosen so that $f(M)$ is contained in the interior of $M$.

## 1.4 Functors from $\mathbf{Emb}_d$

Note that a homeomorphism $f\colon M \to M'$ between two $d$-manifolds $M, M' \in \mathrm{Ob}(\mathbf{Emb}_d)$ represents an isomorphism in $\mathbf{Emb}_d$. Therefore, for each functor $F\colon \mathbf{Emb}_d \to C$ from $\mathbf{Emb}_d$ to a category $C$, the isomorphism class of $F(M) \in \mathrm{Ob}(C)$ for $M \in \mathrm{Ob}(\mathbf{Emb}_d)$ is an invariant of $M$.

### 1.4.1 The functor $U\colon \mathbf{Emb}_d \to \mathbf{Toph}$

Let

$$U\colon \mathbf{Emb}_d \to \mathbf{Toph} := \mathbf{Top}/\text{homotopy}$$

denote the functor which maps $M \in \mathrm{Ob}(\mathbf{Emb}_d)$ to the underlying topological space of $M$ and which maps each morphism $[f]\colon M \to M'$, which is an isotopy class, to the homotopy class of $f$. Composing $U$ with various functors from $\mathbf{Toph}$ defined in Algebraic Topology, one obtains many functors from $\mathbf{Emb}_d$. For example,

$$\mathbf{Emb}_d \xrightarrow{U} \mathbf{Toph} \xrightarrow{H_k(-,\mathbb{Z})} \mathbf{Ab}$$

$$\mathbf{Emb}_d^* \xrightarrow{U} \mathbf{Toph}^* \xrightarrow{\pi_1} \mathbf{Grp} \xrightarrow{\mathrm{Hom}(-,G)} \mathbf{Set}^{\mathrm{op}}$$

Here $\mathbf{Emb}_d^*$ and $\mathbf{Toph}^*$ are the basepointed versions of $\mathbf{Emb}_d$ and $\mathbf{Toph}$, respectively, and $G$ is a fixed group.

### 1.4.2 Skein modules

Another important class of functors defined on $\mathbf{Emb}_d$ is defined by skein modules. Roughly speaking, a skein module associated with a manifold $M$ is

$$A(M) = k\{\text{"links" in } M\}/(\text{ambient isotopy and local relations})$$

Here "links" are a certain kind of subcomplexes in $M$, possibly with framings, coloring, etc. It is clear that $A(M)$ is functorial in embeddings, and hence we have a functor

$$A\colon \mathbf{Emb}_d \to k\text{-}\mathbf{Mod}.$$

# 2 The category $\mathcal{E}$

In the rest of this note, we restrict to the case $d = 3$.

In this section, we define the "category of disc-based 3-manifolds and disc-based embeddings", denoted by $\mathcal{E}$, which is the main object of study in this note.

In what follows, all manifolds are oriented and all codimension 0 embeddings are orientation-preserving.

## 2.1 Disk-based 3-manifolds and disk-based embeddings

A *disk-based 3-manifold* $(M, i)$ consists of

- a connected 3-manifold $M$, and
- an embedding $i\colon D^2 \hookrightarrow \partial M$.

The embedding $i$ is called the *disc-basing.*

A *(disk-based) embedding* $f\colon (M, i) \hookrightarrow (N, j)$ is an embedding $f\colon M \hookrightarrow N$ which is compatible with the disc-basing, i.e., $j = (f|_{\partial M}) \circ i$.

## 2.2 The category $\mathcal{E}$

Define $\mathcal{E}$ to be the category as follows. The objects are disc-based 3-manifolds, the morphisms are the equivalence classes of disk-based embeddings, where two disc-based 3-manifolds are equivalent if there is an isotopy between them through disk-based embeddings. The composition in $\mathcal{E}$ is induced by composition of embeddings. The identity morphisms are defined by $1_{(M,i)} = [\mathrm{id}_M]$.

For simplicity, we often write $M$ for $(M, i)$ by dropping the disc-basing $i$, and we often confuse embeddings and their isotopy classes.

## 2.3 Based-homeomorphisms as isomorphisms in $\mathcal{E}$

Clearly, a based-*homeomorphism* is an isomorphism in $\mathcal{E}$.

Thus, given a functor $F\colon \mathcal{E} \to C$ from $\mathcal{E}$ to a category $C$, the isomorphism class of $F(M)$ in $C$ is a topological invariant of $M$.

**Proposition 1.** *A morphism $f\colon M \to M'$ in $\mathcal{E}$ is an isomorphism if and only if $f$ is represented by a disk-based homeomorphism.*

**Corollary 2.** *For $M \in \mathrm{Ob}(\mathcal{E})$, the group $\mathrm{Aut}_{\mathcal{E}}(M)$ is isomorphic to the "disc-based mapping class group of $M$", i.e., the group of the disc-based ambient isotopy classes of the disc-based self-homeomorphisms of $M$.*

### 2.4 Braided monoidal structure of $\mathcal{E}$

The category $\mathcal{E}$ has a braided monoidal category structure.

- The tensor functor
$$\otimes\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$$
is given by a kind of boundary connected sum.
- The monoidal unit given by the 3-ball $B^3$.
- The braidings
$$\psi_{M,M'}\colon M \otimes M' \xrightarrow{\cong} M' \otimes M$$
is represented by a homeomorphism which switches the $M$-part and the $M'$-part in $M \otimes M'$ and $M' \otimes M$.

## 3 The category $\mathcal{H}$ of handlebody embeddings

### 3.1 The full subcategory $\mathcal{H}$ of $\mathcal{E}$

Let $\mathcal{H}$ denote the full subcategory of $\mathcal{E}$ such that
$$\mathrm{Ob}(\mathcal{H}) = \{V_0, V_1, V_2, \ldots\},$$
where $V_g$ is a fixed genus $g$ handlebody obtained from a cylinder $D^2 \times [0,1]$ with $g$ 1-handles on the top.

We identify $\mathrm{Ob}(\mathcal{H})$ with $\{0,1,2,\ldots\}$.

In other words, $\mathcal{H}$ is the category with $\mathrm{Ob}(\mathcal{H}) = \{0,1,2,\ldots\}$ and
$$\begin{aligned}\mathcal{H}(m,n) &= \{\text{d.b. embeddings } V_m \hookrightarrow V_n\}/\text{isotopy}\\ &= \{m\text{-component } \textit{bottom tangles} \text{ in } V_n\}/\text{isotopy}.\end{aligned}$$

### 3.2 Relations of $\mathcal{H}$ and other categories

Let $\mathcal{C}$ denote the Crane–Kerler–Yetter (CKY) cobordism category [3, 5]:

- objects — surface with boundary parametrized by $S^1$,
- morphisms — homeomorphism classes of connected cobordisms.

*Remark* 3. $\mathcal{H}^{\mathrm{op}}$ is isomorphic to

- the "category of bottom tangles in handlebodies" ([4])
- the "category of special Lagrangian cobordisms" ([2])

which are (isomorphic to) a subcategory of the CKY category $\mathcal{C}$.

### 3.3 Some structures of $\mathcal{H}$

*Fact.* $\mathcal{H}$ is a braided monoidal subcategory of $\mathcal{E}$. In particular, $V_g \otimes V_{g'} \cong V_{g+g'}$ in $\mathcal{E}$.

*Fact.* In $\mathcal{H}$, there is a braided Hopf algebra structure

$$H = (V_1, \mu, \eta, \Delta, \epsilon, S).$$

(Crane–Yetter [3] and Kerler [5] had introduced the same structure in $\mathcal{C} (\cong \mathcal{C}^{\mathrm{op}})$.)

## 4 Quantum fundamental groups

### 4.1 Definition of quantum fundamental groups

The *quantum fundamental group* (QFG) of $M \in \mathrm{Ob}(\mathcal{E})$ is the functor

$$P(M) = \mathcal{E}(i(-), M)\colon\ \mathcal{H}^{\mathrm{op}} \to \mathbf{Set}.$$

Clearly, $P(M)$ is functorial in $M$. Thus, we have a functor

$$P\colon\ \mathcal{E} \to \widehat{\mathcal{H}} := \mathbf{Set}^{\mathcal{H}^{\mathrm{op}}}.$$

Note that

$$\begin{aligned} P(M)(n) = \mathcal{E}(i(n), M) = \mathcal{E}(V_n, M) &= \{[V_n \hookrightarrow M]\} \\ &= \{[n\text{-component bottom tangle in } M]\} \end{aligned}$$

maps surjectically onto the direct product $\pi_1(M)^n$. Thus, $P(M)$ is a refinement of the set $\pi_1(M)$.

### 4.2 Goal

I would like to generalize everything about $\pi_1$ into QFGs.

In the rest of this talk, I will explain attempts to generalizing

- representation spaces $\mathrm{Rep}^G(\pi_1 M) = \mathrm{Hom}_{\mathbf{Grp}}(\pi_1 M, G)$,
- van Kampen Theorem.

## 5 Kan extension

For the definitions and properties of the Kan extensions, see Mac Lane's book [7].

### 5.1 Left Kan extension along $i\colon \mathcal{H} \to \mathcal{E}$

Let $\mathcal{V}$ be a cocomplete category, such as **Set**, **Vect**, **Grp**, **Ab**, ....

If we are given a functor $Q\colon \mathcal{H} \to \mathcal{V}$, then there is the *left Kan extension* of $Q$ along $i$

$$\mathrm{Lan}_i\, Q\colon \mathcal{E} \to \mathcal{V}.$$

**Example 4.** 1. For the fundamental groups, we have

$$\mathrm{Lan}_i(\pi_1\colon \mathcal{H} \to \mathbf{Grp}) \cong (\pi_1\colon \mathcal{E} \to \mathbf{Grp}).$$

2. For the QFGs, we have

$$\mathrm{Lan}_i(Pi = Y\colon \mathcal{H} \to \widehat{\mathcal{H}}) \cong (P\colon \mathcal{E} \to \widehat{\mathcal{H}}).$$

Thus, the QFG is the left Kan extension along $i$ of the Yoneda embedding $Y\colon \mathcal{H} \to \widehat{\mathcal{H}}$.

### 5.2 Kan extension as coend

For simplicity, consider the case $\mathcal{V} = \mathbf{Vect} = \mathbf{Vect}_k$.

Let $k(-)\colon \mathbf{Set} \to \mathbf{Vect}$, $S \mapsto k \cdot S$. For $M \in \mathrm{Ob}(\mathcal{E})$, we have a functor

$$kP(M)\colon \mathcal{H}^{\mathrm{op}} \to \mathbf{Vect}.$$

If $Q\colon \mathcal{H} \to \mathbf{Vect}$ is a functor, then $(\mathrm{Lan}_i\, Q)(M)$ can be computed as the *coend*, or the "tensor product" of $kP(M)\colon \mathcal{H}^{\mathrm{op}} \to \mathbf{Vect}$ and $Q\colon \mathcal{H} \to \mathbf{Vect}$

over $\mathcal{H}$

$$\begin{aligned}(\mathrm{Lan}_i\, Q)(M) &= kP(M) \otimes_{\mathcal{H}} Q \\ &:= \int^{n\in\mathcal{H}} kP(M)(n) \otimes_k Q(n) \\ &= \left( \bigoplus_{n\in \mathsf{Ob}(\mathcal{H})} kP(M)(n) \otimes_k Q(n) \right) /\mathsf{Relations}\end{aligned}$$

where $\mathsf{Relations}$ is spanned by

$$x \otimes Q(f)(y) - kP(M)(f)(x) \otimes y$$

for $f \in \mathcal{H}(n, n')$, $y \in Q(n)$, $x \in kP(M)(n')$, and $n, n' \in \mathrm{Ob}(\mathcal{H})$.

## 5.3 Problem

$\mathrm{Lan}_i\, Q$ can be denoted

$$\mathrm{Lan}_i\, Q = kP \otimes_{\mathcal{H}} Q = \mathrm{Ind}_{\mathcal{H}}^{\mathcal{E}}\, Q \colon\ \mathcal{E} \to \mathbf{Vect}.$$

**Problem.** Construct interesting functors

$$Q\colon\ \mathcal{H} \to \mathbf{Vect}$$

which induce interesting functors on $\mathcal{E}$

$$\mathrm{Lan}_i\, Q\colon\ \mathcal{E} \to \mathbf{Vect}.$$

## 5.4 Co-ribbon Hopf algebras

The notion of co-ribbon Hopf algebra is the dual to that of ribbon Hopf algebra:

A *co-ribbon Hopf algebra* is a Hopf algebra $H = (H, \mu, \eta, \Delta, \epsilon, S)$ equipped with

- a universal $R$-form $R\colon\ H \otimes H \to H$,
- a co-ribbon element $r\colon\ H \to k$.

## 5.5 Examples of co-ribbon Hopf algebras

- The dual $H^* = \mathrm{Hom}(H, k)$ of a finite dimensional ribbon Hopf algebra $H$.
- Commutative Hopf algebras. $(R = \epsilon \otimes \epsilon,\ r = \epsilon)$
  - The algebra $\mathrm{Fun}_k(G)$ of functions on a finite group $G$.
  - The algebra $k(G)$ of regular functions on a linear algebraic group $G$.
- The quantized algebra of regular functions, $k_q(G)$, for $G = SL(N), \ldots$.

## 5.6 The category $\mathbf{Comod}_H$

Let $\mathbf{Comod}_H$ denote the category of left $H$-comodules.

*Fact.*
- If $H$ is a Hopf algebra, then $\mathbf{Comod}_H$ is a monoidal category.
- (Majid) If $H$ is co-quasitriangular, then $\mathbf{Comod}_H$ is a braided category. The object $\underline{H} := (H, \mathrm{coad}) \in \mathrm{Ob}(\mathbf{Comod}_H)$ has a braided Hopf algebra structure. Here
$$\mathrm{coad}\colon H \to H \otimes H, \quad x \mapsto \sum x_{(1)} S(x_{(3)}) \otimes x_{(2)}$$
is the left coadjoint coaction.

**Theorem 5** (Cf. [6]). *If $H$ is a co-ribbon Hopf algebra, then there is a braided monoidal functor*
$$Q^H\colon \mathcal{H} \to \mathbf{Comod}_H,$$
*which maps the braided Hopf algebra structure in $\mathcal{H}$ to that in $\mathbf{Comod}_H$.*

## 5.7 Quantum representation variety

Since $\mathbf{Comod}_H$ is cocomplete, we have the following.

**Corollary**. If $H$ is a co-ribbon Hopf algebra, then we have a functor
$$\mathsf{Rep}^H := \mathrm{Lan}_i\, Q^H\colon \mathcal{E} \to \mathbf{Comod}_H.$$

We call $\mathsf{Rep}^H(M)$ the *quantum representation variety* of $M$ associated to $H$.

## 5.8 Examples

- If $H = \mathrm{Fun}_k(G)$ with $G$ a finite group, then $\mathsf{Rep}^{\mathrm{Fun}(G)}(M) = \mathrm{Fun}(\mathrm{Hom}_{\mathbf{Grp}}(\pi_1 M, G))$.
- If $H = k(G)$ with $G$ a linear algebraic group, then $\mathsf{Rep}^{k(G)}(M) = \mathrm{Reg}(\mathsf{Rep}^G(\pi_1 M))$, the algebra of regular functions on the representation variety $\mathsf{Rep}^G(\pi_1 M)$.
- If $H = k_q(G)$, then $\mathsf{Rep}^{k_q(G)}$ is a $q$-deformation of $\mathsf{Rep}^{k(G)}(\pi_1 M)$. It is simply an $H$-comodule. It is not an algebra.

*Remark* 6. Recall that the Kauffman bracket skein module is a $q$-deformation of the $SL_2(\mathbf{C})$-character variety ([1], etc.).

$\mathsf{Rep}^{k_q(SL_2)}$ is closely related to the Kauffman skein modules.

In fact, one can define "quantum character variety" $X^H(M) \subset \mathsf{Rep}^H(M)$ for a co-ribbon Hopf algebra $H$ and $M \in \mathcal{E}$ as the $H$-invariant part of the $H$-comodule $\mathsf{Rep}^H(M)$.

When $H = k_q(SL_2)$, $X^{k_q(SL_2)}(M)$ seems almost isomorphic to the Kauffman bracket skein module of $M$.

### 5.9 Quantum van Kampen (sketch)

There is a gluing formula for the QFG, or "quantum van Kampen theorem", of the disk-based 3-manifold $M_1 \cup_\Sigma M_2 \in \mathcal{E}$ obtained from $M_1, M_2 \in \mathcal{E}$ by gluing along a connected surface $\Sigma$ on the boundaries of $M_1, M_2$.

We have

$$P(M_1 \cup_\Sigma M_2) \cong P(M_1) \otimes_{P(\Sigma)} P(M_2).$$

Here $P(\Sigma) = P(\Sigma \times [0,1])$ equipped with a monoid structure in the cocompletion $\widehat{\mathcal{H}} = \mathbf{Set}^{\mathcal{H}^{\mathrm{op}}}$, which is a monoidal category.

$\otimes_{P(\Sigma)}$ denotes "tensor product over the monoid $P(\Sigma)$", which exists since $\widehat{\mathcal{H}}$ is a cocomplete monoidal category.

### 5.10 Gluing formula for QRVs (sketch)

The Quantum van Kampen Theorem for QFGs implies a gluing formula for QRV associated to a co-ribbon Hopf algebra $H$.

For a connected surface $\Sigma$, there is an (ordinary) algebra structure for $A_\Sigma := \mathsf{Rep}^H(\Sigma \times [0,1])$.

If $M$ is a "cobordism" from $\Sigma$ to $\Sigma'$, then $\mathsf{Rep}^H(M)$ is equipped with an $(A_\Sigma, A_{\Sigma'})$-bimodule structure. Then the gluing formula for QRVs states that

$$\mathsf{Rep}^H(M_1 \cup_\Sigma M_2) \cong \mathsf{Rep}^H(M_1) \otimes_{A_\Sigma} \mathsf{Rep}^H(M_2).$$

These constructions give a 2-functor

$$\mathsf{Rep}^H : \left\{ \begin{array}{c} \text{surfaces} \\ \text{cobordisms} \\ \text{embeddings} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{algebras} \\ \text{bimodules} \\ \text{homomorphisms} \end{array} \right\}$$

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