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The numerical radius of a weighted shift operator

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Abstract

This article briefly resumes previous works of the author, joint with Professor Hiroshi Nakazato, on the q-numerical radius of a weighted shift operator with geometric weights and periodic weights.

1. Introduction

Let T be a bounded linear operator on a complex Hilbert space H. For $0 \leq q \leq 1$, the q-numerical range $W_q(T)$ of T

$$W_q(T) = \{\langle T\xi, \eta\rangle : ||\xi|| = ||\eta|| = 1, \langle \xi, \eta\rangle = q\}.$$

$W_q(T)$ is a bounded convex subset of $\mathbf{C}$(cf. [12]). Its q-numerical radius

$$w_q(T) = \sup\{|z| : z \in W_q(T)\}.$$

When $q = 1$, $W_q(T)$ reduces to the classical numerical range of T which is defined by

$$W(T) = W_1(T) = \{\langle T\xi, \xi\rangle : ||\xi|| = 1\}.$$

Consider a weighted shift operator in infinite matrix form

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ s_1 & 0 & 0 & 0 & \ldots \\ 0 & s_2 & 0 & 0 & \ldots \\ 0 & 0 & s_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the weights $\{s_n : n = 1, 2, 3, \ldots\}$ is a bounded sequence. Define a unitary operator

$$U = \mathrm{diag}(c_1,\ c_1c_2,\ c_1c_2c_3,\ \ldots),$$

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$c_1 = 1$, $c_{n+1} = \overline{s_n}/|s_n|$ if $s_n \neq 0$, and $c_{n+1} = 1$ if $s_n = 0$. Then

$$UTU^* = |T|.$$

Hence, we may assume the weights of a weighted shift operator are nonnegative when the q-numerical range is involved.

Let T be a weighted shift operator with weights $\{s_n\}$. Shields [7] showed that $W(T)$ is a circular disk about the origin. Further, if the weights are periodic, Ridge [6] proved $W(T)$ is closed if any of weights is zero, and Stout [9] showed $W(T)$ is an open disk if all weights are nonzero. In particular, if $s_n = 1$, for all n, it is well known that $W(T)$ is the open unit disk and $w(T) = 1$. Tam [10] proved $W_q(T)$ is the closed unit disk for all $0 \leq q < 1$. It is interesting to ask what is the radius of the circular disk of $W_q(T)$? Berger-Stampfli [1] gave a partial answer showing that for weighted shift operator with weights $\{1+h, 1, 1, \ldots\}$, $1 + h > \sqrt{2}$,

$$w(T) = \frac{1}{2}\left(((1+h)^2 - 1)^{\frac{1}{2}} + ((1+h)^2 - 1)^{-\frac{1}{2}}\right).$$

In this paper, we examine the q-numerical radius of a weighted shift operator when its weights are in geometric sequence and periodic sequence.

2. Geometric weights

Let T be a linear operator, and $T = UP$ be its the polar decomposition. The Aluthge transformation of T is defined by

$$\Delta(T) = P^{\frac{1}{2}} U P^{\frac{1}{2}}.$$

Suppose T is a weighted shift operator with geometric weights $s_n = r^{n-1}$, $0 < r < 1$. Then $P = \text{diag}(1, r, r^2, r^3, \ldots, r^{n-1}, \ldots)$. and

$$\Delta(T) = \sqrt{r}\, T.$$

Applying Yamazaki inequality [13],

$$w(T) \leq ||T||/2 + w(\Delta(T))/2,$$

we obtain a bound for the numerical radius.

Theorem 2.1 (cf.[2]) Let T be a weighted shift operator with geometric weights $\{r^{n-1}, n \in \mathbf{N}\}$, $0 < r < 1$. Then $W(T)$ is a closed disk about the origin, and $w(T) \leq 1/(2 - \sqrt{r})$

Let T be a weighted shift operator with finite square sum. Denote $F_T(z)$ the determinant of $I - z(T + T^*)$ given by

$$F_T(z) = 1 + \sum_{n=1}^{\infty} (-1)^n c_n z^{2n},$$

where

$$c_n = \sum s_{i_1}^2 s_{i_2}^2 \cdots s_{i_n}^2,$$

the sum is taken over

$$i_2 - i_1 \geq 2, i_3 - i_2 \geq 2, \ldots, i_n - i_{n-1} \geq 2.$$

Stout [9] proved that $w(T) = 1/(2\lambda)$, where λ is the minimum positive root of $F_T(z)$. We present explicitly the series $F_T(z)$ if T is a weighted shift operator with geometric weights.

Theorem 2.2 (cf.[2]) Let T be a weighted shift operator with geometric weights $\{r^{n-1}, n \in \mathbf{N}\}$, $0 < r < 1$. Then

$$F_T(z) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n r^{2n(n-1)}}{(1-r^2)(1-r^4)(1-r^6)\cdots(1-r^{2n})} z^{2n}.$$

For instance, if $r = 0.2$, $s_n = (0.2)^{n-1}$, then by Theorem 2.1, $w(T) \leq 1/(2-\sqrt{r}) \approx 0.644$. While from Theorem 2.2, the minimum positive root of $F_T(z)$ is estimated by 0.980552, and thus $w(T) \approx 1/(2 \times 0.980552) = 0.50991$.

Substituting $z = ir$ into $F_T(z)$ in Theorem 2.2,

$$F_T(z) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n r^{2n(n-1)}}{(1-r^2)(1-r^4)(1-r^6)\cdots(1-r^{2n})} z^{2n},$$

$$F_T(ir) = 1 + \sum_{n=1}^{\infty} \frac{r^{2n^2}}{(1-r^2)(1-r^4)\cdots(1-r^{2n})}.$$

$$1 + \sum_{n=1}^{\infty} \frac{r^{n^2}}{(1-r)(1-r^2)\cdots(1-r^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-r^{5n+1})(1-r^{5n+4})} \qquad (1)$$

Sloane-Robinsonv [8] mentioned that the coefficients of the power series in the right-hand side of (1) are in expansion of permanent of the infinite tridiagonal matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots \\ r & 1 & 1 & 0 & \cdots \\ 0 & r^2 & 1 & 1 & \cdots \\ 0 & 0 & r^3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We consider a finite matrix of size n,

$$A(n,r) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ r & 0 & 1 & 0 & \cdots & 0 \\ 0 & r^2 & 0 & 1 & \cdots & 0 \\ 0 & 0 & r^3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & r^{n-1} & 0 \end{bmatrix}.$$

We are able to describe the numerical ranges of these tridiagonal matrices.

Theorem 2.3 (cf.[3]) For $n \geq m \geq 3$ and any real number r, $W(A(n,r)) \supset W(A(m,r))$.

Theorem 2.4 (cf.[3]) Let $n = 2\ell - 1 \geq 5$. Then $W(A(n,-1))$ is the convex hull of the two ellipses

$$\{(x,y) \in \mathbf{R}^2, x^2 \pm 2\cos(2\pi/(n+1))xy + y^2 = \sin^2(2\pi/(n+1))\}.$$

When $n = \infty$. We define the operator

$$A(\infty,-1) = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ (-1) & 0 & 1 & 0 & \dots \\ 0 & (-1)^2 & 0 & 1 & \dots \\ 0 & 0 & (-1)^3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The numerical range of this operator has a special type of shape.

Theorem 2.5 (cf.[3]) For

$W(A(\infty,-1)) = \{z \in \mathbf{C} : -1 \leq \Re(z) \leq 1, -1 \leq \Im(z) \leq 1\} \backslash \{1+i, 1-i, -1+i, -1-i\}$.

3. Periodic weights

Let T be a weighted shift operator with periodic weights $\{s_1, s_2, \dots, s_m, s_1, s_2, \dots, s_m, \dots\}$. Consider the $m \times m$ weighted cyclic matrix S with weights $\{s_1, s_2, \dots, s_m\}$

$$S = S(s_1, s_2, \dots, s_m) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & s_m \\ s_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & s_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & s_{m-1} & 0 \end{bmatrix}. \tag{2}$$

Numerical ranges of weighted cyclic matrices (2) have been developed by several authors, for examples, [4, 11].

Theorem 3.1 (cf.[11]) Let $S(s_1, s_2, \dots, s_m)$ be a weighted cyclic matrix defined in (2). Then

(*i*) $S(s_1, s_2, \dots, s_m)$ is normal if and only if $|s_1| = |s_2| = \cdots = |s_m|$, which is also equivalent to $W(S(s_1, s_2, \dots, s_m))$ is a regular m-polygonal region centered at the origin and the distance from the center to its vertices equal to $|s_1 \cdots s_m|^{1/m}$.

(*ii*) $\partial W(S(s_1, s_2, \dots, s_m))$ contains a line segment if and only if the s_j are nonzero and the numerical ranges of the $(m-1)$-by-$(m-1)$ submatrices of S are all equal.

The q-numerical radius of a weighted shift operator with periodic weights is exactly the q-numerical radius of the corresponding weighted cyclic matrix.

Theorem 3.2 (cf.[4]) Let T be a weighted shift operator with periodic weights $\{s_1, s_2, \ldots, s_m\}$ and S be the $m \times m$ weighted cyclic matrix with weights $\{s_1, s_2, \ldots, s_m\}$. Then $w_q(T) = w_q(S)$ for every $0 \leq q \leq 1$.

Notice that the case $q = 1$ of Theorem 3.2 is proved by Ridge [6]. We are capable of presenting the closed form of the q-numerical radius of a weighted shift operator with 2-periodic weights.

Theorem 3.3 (cf.[4]) Let T be a weighted shift operator with periodic weights $\{s_1, s_2\}$. Then

$$w_q(T) = \frac{s_1 + s_2}{2} + \sqrt{1 - q^2}\frac{|s_1 - s_2|}{2}.$$

Let T be a weighted shift operator with periodic weights $\{s_1, s_2, \ldots, s_m\}$. Denote $w_q(T) = w_q([s_1, s_2, \ldots, s_m])$. We have the following fundamental results of q-numerical radii.

Theorem 3.4 (cf.[4])

(a) $w_q([s_1, s_2, \ldots, s_m]) = w_q([|s_1|, |s_2|, \ldots, |s_m|])$.

(b) $w_q([cs_1, cs_2, \ldots, cs_m]) = |c| w_q([s_1, s_2, \ldots, s_m])$.

(c) If $0 \leq s_j \leq s_j', j = 1, 2, \ldots, m$, then $w_q([s_1, s_2, \ldots, s_m]) \leq w_q([s_1', s_2', \ldots, s_m'])$.

(d) $w_q([1, 1, \ldots, 1]) = w_q([1]) = 1$.

(e) $\min\{|s_1|, \ldots, |s_m|\} \leq w_q([s_1, \ldots, s_m]) \leq \max\{|s_1|, \ldots, |s_m|\}$.

(f) $w_q([s_m, s_{m-1}, \ldots, s_2, s_1]) = w_q([s_1, s_2, \ldots, s_{m-1}, s_m])$.

(g) $w_q([s_2, \ldots, s_m, s_1]) = w_q([s_1, s_2, \ldots, s_m])$.

The q-numerical radii may change while the order of the weights are changed.

Theorem 3.5(cf.[4]) Let T be a weighted shift operators with 4-periodic. Suppose that $s_4 \geq s_3 \geq s_2 \geq s_1 \geq 0$. Then

$$w_q([s_2, s_4, s_3, s_1]) \geq w_q([s_1, s_4, s_3, s_2]) \geq w_q([s_1, s_4, s_2, s_3])$$

for $0 \leq q \leq 1$.

4. Perturbations

In this section, we perturb the q-numerical radius of a weighted shift operator with periodic weights.

Theorem 4.1 (cf.[5]) Let T be a weighted shift operator with periodic nonnegative weights $\{s_1, s_2, s_3, s_4, \ldots, s_m\}$, $m \geq 5$, such that $s_3 > \max\{s_1, s_2, s_4, \ldots, s_m\}$. Then the perturbation of the q-numerical radius is

$$w_q(T) = s_3 - \frac{(s_3^2 - s_2^2)(s_3^2 - s_4^2)}{2s_3(2s_3^2 - s_2^2 - s_4^2)}q^2 + c_3^{(4)}q^4 + O(q^5),$$

where $c_3^{(4)} = c_3^{(4)}(s_1, s_2, s_3, s_4, s_5)$.

Let T be a weighted shift operator with 4-periodic. we are able to find the perturbed coefficients up to the 4th degree.

Theorem 4.2 (cf.[5]) Let T be a weighted shift operator with periodic nonnegative weights $\{s_1, s_2, s_3, s_4\}$ such that

$$s_3 > \max\{s_1, s_2, s_4\}.$$

Then the perturbation of the q-numerical radius is

$$w_q(T) = s_3 - \frac{(s_3^2 - s_2^2)(s_3^2 - s_4^2)}{2s_3(2s_3^2 - s_2^2 - s_4^2)}q^2 + c_3^{(4)}q^4 + O(q^5),$$

where

$$\begin{aligned}
c_3^{(4)} &= -\frac{1}{8\tilde{\alpha}} + \frac{\tilde{\beta}}{16\tilde{\alpha}^4} - \frac{s_3}{8}, \\
\tilde{\alpha} &= -\frac{s_3(2s_3^2 - s_2^2 - s_4^2)}{2(s_3^4 - s_2^2 s_4^2)}, \quad \tilde{\beta} = \frac{\beta_2}{\beta_1}, \\
\beta_2 &= 8s_3^3\Big(\frac{s_2^2}{(s_3^2 - s_2^2)} + \frac{s_3^2}{(s_3^2 - s_4^2)}\Big)^4, \\
\beta_1 &= -\Big(\frac{s_2^2}{s_3^2 - s_2^2} + \frac{s_3^2}{s_3^2 - s_4^2}\Big)^2 - 2\Big(\frac{s_2^2}{s_3^2 - s_2^2} + \frac{s_3^2}{s_3^2 - s_4^2}\Big)^3 \\
&\quad -\Big(\frac{s_2^2}{s_3^2 - s_2^2} + \frac{s_3^2}{s_3^2 - s_4^2}\Big)^4 + 4s_3^2\Big(- \frac{s_2^4}{(s_3^2 - s_2^2)^3} \\
&\quad + \frac{2s_1 s_2 s_3 s_4}{(s_3^2 - s_1^2)(s_3^2 - s_2^2)(s_3^2 - s_4^2)} + \frac{s_2^2}{(s_3^2 - s_2^2)^2}\Big(\frac{s_1^2}{s_3^2 - s_1^2} \\
&\quad - \frac{s_3^2(2s_3^2 - s_2^2 - s_4^2)}{(s_3^2 - s_4^2)^2}\Big) + \frac{s_3^2}{(s_3^2 - s_4^2)^3}\Big(-s_3^2 + \frac{s_4^2(s_3^2 - s_4^2)}{s_3^2 - s_1^2}\Big)\Big).
\end{aligned}$$

For 3-periodic weighted shift operator, we obtain the following perturbation.

Theorem 4.3 (cf.[5]) Let T be a weighted shift operator with periodic nonnegative weights $\{s_1, s_2, s_3\}$ such that $s_3 > \max\{s_1, s_2\}$. Then, for sufficiently small q, the perturbation of the q-numerical radius is

$$\begin{aligned}
w_q(T) &= s_3 - \frac{(s_3^2 - s_1^2)(s_3^2 - s_2^2)}{2s_3(2s_3^2 - s_1^2 - s_2^2)}q^2 + \frac{s_1 s_2 (s_3^2 - s_1^2)^2 (s_3^2 - s_2^2)^2}{s_3^3(2s_3^2 - s_1^2 - s_2^2)^3}q^3 \\
&\quad - \frac{\gamma(s_3^2 - s_1^2)^2(s_3^2 - s_2^2)^2}{8s_3^5(2s_3^2 - s_1^2 - s_2^2)^5}q^4 + O(q^5),
\end{aligned}$$

where

$$\begin{aligned}\gamma &= 16s_3^{12} - 32(s_1^2+s_2^2)s_3^{10} + (30s_1^4 + 72s_1^2s_2^2 + 30s_2^2)s_3^8 \\ &\quad -(11s_1^6 + 93s_1^4s_2^2 + 93s_1^2s_2^4 + 11s_2^6)s_3^6 + (s_1^8 + 34s_1^6s_2^2 + 162s_1^4s_2^4 \\ &\quad +34s_1^2s_2^6 + s_2^8)s_3^4 + (3s_1^8s_2^2 - 75s_1^6s_2^4 - 75s_1^4s_2^6 + 3s_1^2s_2^8)s_3^2 + 36s_1^6s_2^6.\end{aligned}$$

References

[1] C. A. Berger and J. G. Stampfli, Mapping theorems for the numerical range, American Journal of Mathematics, 89(1967), 1047-1055.

[2] Mao-Ting Chien and Hiroshi Nakazato, The numerical radius of a weighted shift operator with geometric weights, Electronic Journal of Linear Algebra, 18(Jan 2009), 58-63.

[3] Mao-Ting Chien and Hiroshi Nakazato, The numerical range of a tridiagonal operator, Journal of Mathematical Analysis and Applications, 373(2011), 297-304.

[4] Mao-Ting Chien and Hiroshi Nakazato, The q-numerical radius of weighted shift operators with periodic weights, Linear Algebra and Its Applications, 422(April 2007), 198-218.

[5] Mao-Ting Chien and Hiroshi Nakazato, Perturbation of the q-numerical radius of a weighted shift operator, Journal of Mathematical Analysis and Applications, 345(Sept 2008), 954-963.

[6] W. Ridge, Numerical range of a weighted shift with periodic weights, Proc. Amer. Math. Soc. 55 (1976) 107-110.

[7] A. L. Shields, *Weighted shift operators and analytic function theory*, Math. Surveys. Vol 13, Amer. Math. Soc., Providence, R. I., 1974.

[8] N. J. A. Sloane and H. P. Robinson, Number of partitions of n into parts $5k+1$ or $5k+4$, 2004. Available at http://www.research.att.com/~njas/sequences/A003114

[9] Q. F. Stout, The numerical range of a weighted shift. Proceedings American Mathematical Society, 88(1983), 495-502.

[10] T. Y. Tam, The q-numerical range and the real q-numerical range of the shift, preprint, Available from: http://www.auburn.edu/~tamtiny/pub.html.

[11] M. C Tsai and P Y. Wu, Numerical ranges of weighted shift matrices, Linear Algebra and Its Applications, 435(2011), 243-254.

[12] N. K. Tsing, The constrained bilinear form and the C-numerical range, Linear Algebra Appl., **56**(1984), 195-206.

[13] T. Yamazaki, On numerical range of the Aluthge transformation, Linear Algebra Appl., 341(2002), 111-117.