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OPERATOR FUNCTIONS ON CHAOTIC ORDER INVOLVING ORDER PRESERVING OPERATOR INEQUALITIES

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The fourth anniversary of Professor Masahiro Nakamura's passing

An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all vectors x in a Hilbert space, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. *Let* $\log A \geq \log B$ *and* $r_1, r_2, \ldots, r_n \geq 0$ *and any fixed* $\delta \geq 0$, *and*

$$p_1 \geq \delta,\ p_2 \geq \frac{\delta+r_1}{p_1+r_1}, \ldots, p_k \geq \frac{\delta+r_1+r_2+\ldots+r_{k-1}}{\mathfrak{q}[k-1]}, \ldots, p_n \geq \frac{\delta+r_1+r_2+\ldots+r_{n-1}}{\mathfrak{q}[n-1]}.$$

Let $\mathfrak{F}_n(p_n, r_n)$ *be defined by*

$$\mathfrak{F}_n(p_n, r_n) = A^{\frac{-r_n}{2}} \mathbb{C}_{A,B}[n]^{\frac{\delta+r_1+r_2+\ldots+r_n}{\mathfrak{q}[n]}} A^{\frac{-r_n}{2}}.$$

Then the following inequalities (i), (ii) *and* (iii) *hold:*

(i) $A^{\frac{p_{k-1}}{2}} \mathfrak{F}_{k-1}(p_{k-1}, r_{k-1}) A^{\frac{p_{k-1}}{2}} \geq \mathfrak{F}_k(p_k, r_k)$ *for* k *such that* $1 \leq k \leq n$,

(ii)
$$\begin{aligned} B^{\delta} &\geq A^{\frac{-r_1}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{\frac{\delta+r_1}{p_1+r_1}} A^{\frac{-r_1}{2}} \\ &\geq A^{\frac{-(r_1+r_2)}{2}} \{A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}\}^{\frac{\delta+r_1+r_2}{(p_1+r_1)p_2+r_2}} A^{\frac{-(r_1+r_2)}{2}} \\ &\geq A^{\frac{-(r_1+r_2+r_3)}{2}} \{A^{\frac{r_3}{2}} [A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}]^{p_3} A^{\frac{r_3}{2}}\}^{\frac{\delta+r_1+r_2+r_3}{\{(p_1+r_1)p_2+r_2\}p_3+r_3}} A^{\frac{-(r_1+r_2+r_3)}{2}} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\geq A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}} \mathbb{C}_{A,B}[n]^{\frac{\delta+r_1+r_2+\ldots+r_n}{\mathfrak{q}[n]}} A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}}, \end{aligned}$$

(iii) $\mathfrak{F}_n(p_n, r_n)$ *is a decreasing function of both* $r_n \geq 0$ *and* $p_n \geq \frac{\delta+r_1+r_2+\ldots+r_{n-1}}{\mathfrak{q}[n-1]}$,
where $\mathbb{C}_{A,B}[n]$ *and* $\mathfrak{q}[n]$ *are defined as follows:*

$$\mathbb{C}_{A,B}[n] = A^{\frac{r_n}{2}} \left\{ A^{\frac{r_{n-1}}{2}} [\ldots A^{\frac{r_3}{2}} \{A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}\}^{p_3} A^{\frac{r_3}{2}} \ldots]^{p_{n-1}} A^{\frac{r_{n-1}}{2}} \right\}^{p_n} A^{\frac{r_n}{2}}$$

and

$$\mathfrak{q}[n] = [\ldots\ldots\{(p_1+r_1)p_2+r_2\}p_3+\ldots\ldots r_{n-1}]p_n+r_n.$$

We remark that (ii) can be considered as "*a satellite inequality to chaotic order*".

This paper is early anouncement of the results in [19].

§1 Introduction

An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all vectors x in a Hilbert space, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

Theorem LH (Löwner-Heinz inequality, denoted by **LH** briefly).

If $A \geq B \geq 0$ holds, then $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0, 1]$. (LH)

This inequality LH was originally proved in [28] and then in [22]. Many nice proofs of LH are known. We mention [29] and [3]. Although LH asserts that $A \geq B \geq 0$ *ensures* $A^{\alpha} \geq B^{\alpha}$ *for any* $\alpha \in [0, 1]$, unfortunately $A^{\alpha} \geq B^{\alpha}$ *does not always hold for* $\alpha > 1$. The following result has been obtained from this point of view.

Theorem A.

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

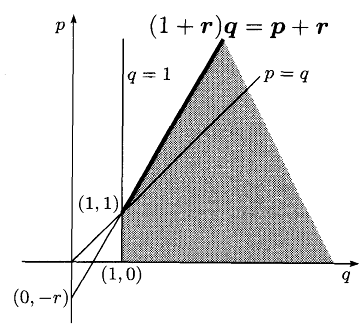

Fig. 1. Domain on p, q, r for Th. A

The original proof of Theorem A is shown in [10], an elementary one-page proof is in [11] and alternative ones are in [4], [25]. It is shown in [30] that the conditions p, q and r in FIGURE 1 are best possible.

Theorem B (e.g.,[12][6][25][26][20]). *Let $A \geq B \geq 0$ with $A > 0$, $p \geq 1$ and $r \geq 0$.*

$$G_{A,B}(p, r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1+r}{p+r}}A^{\frac{-r}{2}}$$

is a decreasing function of p and r, and $G_{A,A}[p, r] \geq G_{A,B}[p, r]$ holds, that is,

$$A^{1+r} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1+r}{p+r}} \text{ holds for } p \geq 1 \text{ and } r \geq 0. \qquad (1.1)$$

We write $A \gg B$ if $\log A \geq \log B$ for $A, B > 0$, which is called the chaotic order.

Theorem C. *For $A, B > 0$, the following* (i) *and* (ii) *hold:*

(i) $A \gg B$ *holds if and only if* $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for* $p, r \geq 0$.

(ii) $A \gg B$ *holds if and only if for any fixed* $\delta \geq 0$,

$$F_{A,B}(p,r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{\delta+r}{p+r}} A^{\frac{-r}{2}}$$

is a decreasing function of $p \geq \delta$ *and* $r \geq 0$.

(i) in Theorem C is shown in [12][6] and an excellent proof in [32] and a proof in the case $p = r$ in [1], and (ii) in [12][6] and etc.

Lemma D [13]. *Let X be a positive invertible operator and Y be an invertible operator. For any real number λ,*

$$(YXY^*)^{\lambda} = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

We state the following result on the chaotic order which inspired us.

Theorem FKN-2 [9]. *If $A \gg B$ for $A, B > 0$, then*

$$A^{t-r} \natural_{\frac{1+r-t}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^t \natural_{\frac{1-t}{p-t}} B^p \leq B$$

holds for $p \geq 1$, $s \geq 1$, $r \geq 0$ *and* $t \leq 0$.

We shall discuss further extensions of Theorem B, Theorem C and Theorem FKN-2.

The purpose of this paper is to emphasize that the chaotic order $A \gg B$ is sometimes more convenient and more useful than the usual order $A \geq B \geq 0$ for discussing some order preserving operator inequalities.

Related results in this paper are discussed in [5],[7],[8],[14],[15],[16],[21],[31],[33] and etc.

§2 Definitions of $\mathbb{C}_{A,B}[n; p_1, p_2, ..., p_{n-1}, p_n | r_1, r_2, .., r_{n-1}, r_n]$. (*denoted by* $\mathbb{C}_{A,B}[n]$ *or* $\mathbb{C}_{[n]}$ *briefly sometime*) and $\mathfrak{q}[n; p_1, p_2, ..., p_{n-1}, p_n | r_1, r_2, .., r_{n-1}, r_n]$ (*denoted by* $\mathfrak{q}[n]$ *briefly.*)

Let $A, B \geq 0$, $p_1, p_2, ..., p_n \geq 0$ *and* $r_1, r_2, ..., r_n \geq 0$ *for a natural number n.*
Let $\mathbb{C}_{A,B}[n; p_1, p_2, .., p_{n-1}, p_n | r_1, r_2, .., r_{n-1}, r_n]$ *be defined by*

$$\mathbb{C}_{A,B}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$$
$$= A^{\frac{r_n}{2}}\left\{A^{\frac{r_{n-1}}{2}}[....A^{\frac{r_3}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{p_3}A^{\frac{r_3}{2}}...]^{p_{n-1}}A^{\frac{r_{n-1}}{2}}\right\}^{p_n}A^{\frac{r_n}{2}}. \quad (2.1)$$

Denote $\mathbb{C}_{A,B}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$ by $\mathbb{C}_{A,B}[n]$ briefly.
For examples,

$$\mathbb{C}_{A,B}[1] = A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}} \text{ and } \mathbb{C}_{A,B}[2] = A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}$$

and

$$\mathbb{C}_{A,B}[4] = A^{\frac{r_4}{2}}\left[A^{\frac{r_3}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{p_3}A^{\frac{r_3}{2}}\right]^{p_4}A^{\frac{r_4}{2}}.$$

Particularly put $A = B$ in $\mathbb{C}_{A,B}[n]$ in (2.1). Then

$$\mathbb{C}_{A,A}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$$
$$= A^{\frac{r_n}{2}}\left\{A^{\frac{r_{n-1}}{2}}[....A^{\frac{r_3}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}A^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{p_3}A^{\frac{r_3}{2}}...]^{p_{n-1}}A^{\frac{r_{n-1}}{2}}\right\}^{p_n}A^{\frac{r_n}{2}} \quad (2.2)$$
$$= A^{[.....\{(p_1+r_1)p_2+r_2\}p_3+.....r_{n-1}]p_n+r_n}. \quad (2.3)$$

Next let $\mathfrak{q}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$ be defined by

$$\mathfrak{q}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$$
$$= \text{the exponential power of } A \text{ in (2.3)}$$
$$= [.....\{(p_1+r_1)p_2+r_2\}p_3+.....r_{n-1}]p_n+r_n. \quad (2.4)$$

$\mathfrak{q}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$ denoted by $\mathfrak{q}[p_1,p_2,..,p_{n-1},p_n]$ or denoted by $\mathfrak{q}[r_1,r_2,..,r_{n-1},r_n]$ for simplicity or sometimes denoted by $\mathfrak{q}[n]$ briefly.
For examples, $\mathfrak{q}[1] = p_1 + r_1$ and $\mathfrak{q}[2] = (p_1+r_1)p_2 + r_2$
and

$$\mathfrak{q}[4] = [\{(p_1+r_1)p_2+r_2\}p_3+r_3]p_4+r_4.$$

For the sake of convenience, we define

$$\mathbb{C}_{A,B}[0] = B \text{ and } \mathfrak{q}[0] = 1 \quad (2.5)$$

and these definitions in (2.5) may be reasonable by (2.1) and (2.4).

Lemma 2.1. *For $A, B \geq 0$ and any natural number n, the following* (i) *and* (ii) *hold.*

(i) $$\mathbb{C}_{A,B}[n] = A^{\frac{r_n}{2}}\mathbb{C}_{A,B}[n-1]^{p_n}A^{\frac{r_n}{2}}.$$

(ii) $$\mathfrak{q}[n] = \mathfrak{q}[n-1]p_n + r_n.$$

We state two examples using these notations of $\mathbb{C}_{A,B}[n]$ and $\mathfrak{q}[n]$ for reader's convenience.

$$A^r \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}} \iff A^r \geq \mathbb{C}_{A,B}[1]^{\frac{r}{\mathfrak{q}[1]}}.$$

$$A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \iff A^{1+r} \geq \mathbb{C}_{A,B}[1]^{\frac{1+r}{\mathfrak{q}[1]}}.$$

Remark 2.1. We remark that quite similar definitions to $\mathbb{C}_{A,B}[n]$ and $\mathfrak{q}[n]$ are given in [18] and related results are discussed in [18], [23], [24], [35] and etc.

§3 Basic results associated with $\mathbb{C}_{A,B}[n]$ and $\mathfrak{q}[n]$

Theorem 3.1. *Let $A \gg B$ and $r_1, r_2, \dots, r_n \geq 0$ for a natural number n. Then the following inequality holds,*

$$A^{r_1+r_2\dots+r_n} = \mathbb{C}_{A,A}[n]^{\frac{r_1+r_2\dots+r_n}{\mathfrak{q}[n]}} \geq \mathbb{C}_{A,B}[n]^{\frac{r_1+r_2\dots+r_n}{\mathfrak{q}[n]}} \tag{3.1}$$

for $p_1, p_2, \dots, p_n$ satisfying

$$p_j \geq \frac{r_1+r_2+\dots+r_{j-1}}{\mathfrak{q}[j-1]} \quad \text{for } j = 1, 2, \dots, n \quad (\ r_0 = 0 \text{ and } \mathfrak{q}[0] = 1\ ),$$

(3.2)

that is,

$$p_1 \geq 0,\ p_2 \geq \frac{r_1}{p_1+r_1},\ p_3 \geq \frac{r_1+r_2}{(p_1+r_1)p_2+r_2}, \dots,\ p_n \geq \frac{r_1+r_2+\dots+r_{n-1}}{\mathfrak{q}[n-1]},$$

where $\mathbb{C}_{A,B}[n]$ is defined in (2.1) and $\mathfrak{q}[n]$ is defined in (2.4).

Corollary 3.2. *Let $A \gg B$ and $r_1, r_2, r_3 \geq 0$. Then*

(i) $$A^{r_1+r_2+r_3} \geq \{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{r_1+r_2+r_3}{[(p_1+r_1)p_2+r_2]p_3+r_3}}.$$

holds for $p_2 \geq \frac{r_1}{p_1+r_1}$ and $p_3 \geq \frac{r_1+r_2}{(p_1+r_1)p_2+r_2}$.

(ii) $$A^{r_1+r_2} \geq \{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{\frac{r_1+r_2}{(p_1+r_1)p_2+r_2}}$$

holds for $p_1 \geq 0$ and $p_2 \geq \frac{r_1}{p_1+r_1}$.

Theorem 3.3. *Let $A \geq B \geq 0$ and $r_1, r_2, \dots, r_n \geq 0$ for a natural number n. Then the following inequality holds,*

$$A^{1+r_1+r_2\dots+r_n} = \mathbb{C}_{A,A}[n]^{\frac{1+r_1+r_2\dots+r_n}{\mathfrak{q}[n]}} \geq \mathbb{C}_{A,B}[n]^{\frac{1+r_1+r_2\dots+r_n}{\mathfrak{q}[n]}} \tag{3.6}$$

for $p_1, p_2, \dots, p_n$ satisfying

$$p_j \geq \frac{1+r_1+r_2+\dots+r_{j-1}}{\mathfrak{q}[j-1]} \quad \text{for } j = 1, 2, \dots, n \quad (\ r_0 = 0 \text{ and } \mathfrak{q}[0] = 1), \tag{3.7}$$

that is,

$$p_1 \geq 1,\ p_2 \geq \frac{1+r_1}{p_1+r_1},\ p_3 \geq \frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}, \dots,\ p_n \geq \frac{1+r_1+r_2+\dots+r_{n-1}}{\mathfrak{q}[n-1]}.$$

Corollary 3.4 *Let $A \geq B \geq 0$ and $r_1, r_2, r_3 \geq 0$. Then*

(i)
$$A^{1+r_1+r_2+r_3} \geq \{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{1+r_1+r_2+r_3}{[(p_1+r_1)p_2+r_2]p_3+r_3}}.$$

holds for $p_1 \geq 1$, $p_2 \geq \frac{1+r_1}{p_1+r_1}$ and $p_3 \geq \frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}$.

(ii)
$$A^{1+r_1+r_2} \geq \{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{\frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}}$$

holds for $p_1 \geq 1$ and $p_2 \geq \frac{1+r_1}{p_1+r_1}$.

Remark 3.2. We remark that Theorem 3.3 is a parallel result to Theorem 3.1 and also Corollary 3.4 is a parallel one to Corollary 3.2, and Theorem 3.1 is usually obtained from Theorem 3.3 by applying Uchiyama's nice technique [32] after proving Theorem 3.3.

Although many results on the chaotic order ($A \gg B$) have been derived from the corresponding results on the usual order ($A \geq B \geq 0$) by applying Uchiyama's nice method, we shall show Corollary 5.4 on the usual order ($A \geq B \geq 0$), which is a further extension of Theorem 3.3, *by using the corresponding result Corollary 5.2 on the chaotic order ($A \gg B$)* at the end of §5.

§4 Monotonicity property on operator functions $\mathfrak{F}_k(p_k, r_k) = A^{\frac{-r_k}{2}}\mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+r_2+...+r_k}{\mathfrak{q}[k]}}A^{\frac{-r_k}{2}}$

Theorem 4.1. *Let $A \gg B$ and $r_1, r_2, ..., r_n \geq 0$ for a natural number n. For any fixed $\delta \geq 0$, let $p_1, p_2, ..., p_n$ be satisfied by*

$$p_j \geq \frac{\delta+r_1+r_2+...+r_{j-1}}{\mathfrak{q}[j-1]} \quad \text{for } j = 1, 2, ..., n, \tag{4.1}$$

that is,

$$p_1 \geq \delta,\ p_2 \geq \frac{\delta+r_1}{p_1+r_1}, ...,\ p_k \geq \frac{\delta+r_1+r_2+...+r_{k-1}}{\mathfrak{q}[k-1]}, ...,\ p_n \geq \frac{\delta+r_1+r_2+...+r_{n-1}}{\mathfrak{q}[n-1]}.$$

The operator function $\mathfrak{F}_k(p_k, r_k)$ for any natural number k such that $1 \leq k \leq n$ is defined by

$$\mathfrak{F}_k(p_k, r_k) = A^{\frac{-r_k}{2}}\mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+r_2+...+r_k}{\mathfrak{q}[k]}}A^{\frac{-r_k}{2}}. \tag{4.2}$$

Then the following inequality holds:

$$A^{\frac{r_{k-1}}{2}}\mathfrak{F}_{k-1}(p_{k-1}, r_{k-1})A^{\frac{r_{k-1}}{2}} \geq \mathfrak{F}_k(p_k, r_k) \qquad (\ \mathfrak{F}_0(p_0, r_0) = B^{\delta})$$

(4.3)

for every natural number k such that $1 \leq k \leq n$.

Remark 4.1. We shall give an alternative proof of Theorem 4.1 in Remark 6.1 via Theorem 6.1 at the end of §6.

§5 Order preserving operator inequalities via operator functions in §4

We shall give order preserving operator inequalities as an application of Theorem 4.1.

Theorem 5.1. *Let $A \gg B$ and $r_1, r_2, \ldots, r_n \geq 0$ for a natural number n. Then the following inequalities hold for any fixed $\delta \geq 0$:*

$$B^{\delta} \geq A^{\frac{-r_1}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{\delta+r_1}{p_1+r_1}}A^{\frac{-r_1}{2}}$$

$$\geq A^{\frac{-(r_1+r_2)}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{\frac{\delta+r_1+r_2}{(p_1+r_1)p_2+r_2}}A^{\frac{-(r_1+r_2)}{2}}$$

$$\geq A^{\frac{-(r_1+r_2+r_3)}{2}}\{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{\delta+r_1+r_2+r_3}{\{(p_1+r_1)p_2+r_2\}p_3+r_3}}A^{\frac{-(r_1+r_2+r_3)}{2}}$$

....

$$\geq A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}}\mathbb{C}_{A,B}[n]^{\frac{\delta+r_1+r_2+\ldots+r_n}{\mathfrak{q}[n]}}A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}} \tag{5.1}$$

for $p_1, p_2, \ldots, p_n$ satisfying

$$p_j \geq \frac{\delta+r_1+r_2+\ldots+r_{j-1}}{\mathfrak{q}[j-1]} \quad \text{for } j = 1, 2, \ldots, n, \tag{4.1}$$

that is,

$$p_1 \geq \delta,\ p_2 \geq \frac{\delta+r_1}{p_1+r_1}, \ldots,\ p_k \geq \frac{\delta+r_1+r_2+\ldots+r_{k-1}}{\mathfrak{q}[k-1]}, \ldots,\ p_n \geq \frac{\delta+r_1+r_2+\ldots+r_{n-1}}{\mathfrak{q}[n-1]}.$$

where $\mathbb{C}_{A,B}[n]$ is defined in (2.1) and $\mathfrak{q}[n]$ is defined in (2.4).

Corollary 5.2. *Let $A \gg B$ and $r_1, r_2, \ldots, r_n \geq 0$ for a natural number n. Then the following* (i) *and* (ii) *hold.*

(i)
$$B \geq A^{\frac{-r_1}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}A^{\frac{-r_1}{2}}$$

$$\geq A^{\frac{-(r_1+r_2)}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{\frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}}A^{\frac{-(r_1+r_2)}{2}}$$

$$\geq A^{\frac{-(r_1+r_2+r_3)}{2}}\{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{1+r_1+r_2+r_3}{\{(p_1+r_1)p_2+r_2\}p_3+r_3}}A^{\frac{-(r_1+r_2+r_3)}{2}}$$

....

$$\geq A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}}\mathbb{C}_{A,B}[n]^{\frac{1+r_1+r_2+\ldots+r_n}{\mathfrak{q}[n]}}A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}} \tag{5.2}$$

holds for $p_1, p_2, \ldots, p_n$ satisfying (3.7), *that is,*

$$p_1 \geq 1,\ p_2 \geq \frac{1+r_1}{p_1+r_1},\ p_3 \geq \frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}, \ldots,\ p_n \geq \frac{1+r_1+r_2+\ldots+r_{n-1}}{q[n-1]};$$

(ii) (5.2) *holds for $p_1, p_2, \ldots, p_n \geq 1$,*

where $\mathbb{C}_{A,B}[n]$ is defined in (2.1) and $\mathfrak{q}[n]$ is defined in (2.4).

Corollary 5.3. *Let $A \gg B$ and $r_1, r_2, \ldots, r_n \geq 0$ for a natural number n. Then*

$$\begin{aligned}
I &\geq A^{\frac{-r_1}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}}A^{\frac{-r_1}{2}} \\
&\geq A^{\frac{-(r_1+r_2)}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{\frac{r_1+r_2}{(p_1+r_1)p_2+r_2}}A^{\frac{-(r_1+r_2)}{2}} \\
&\geq A^{\frac{-(r_1+r_2+r_3)}{2}}\{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{r_1+r_2+r_3}{\{(p_1+r_1)p_2+r_2\}p_3+r_3}}A^{\frac{-(r_1+r_2+r_3)}{2}} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\geq A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}}\mathbb{C}_{A,B}[n]^{\frac{r_1+r_2+\ldots+r_n}{\mathfrak{q}[n]}}A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}} \qquad (5.3)
\end{aligned}$$

holds for $p_1, p_2, \ldots, p_n$ satisfying (3.2), *that is,*

$$p_1 \geq 0,\ p_2 \geq \frac{r_1}{p_1+r_1},\ p_3 \geq \frac{r_1+r_2}{(p_1+r_1)p_2+r_2}, \ldots\ldots, p_n \geq \frac{r_1+r_2+\ldots+r_{n-1}}{q[n-1]}.$$

Corollary 5.4. *Let $A \geq B \geq 0$ and $r_1, r_2, \ldots, r_n \geq 0$ for a natural number n. Then*

$$\begin{aligned}
A \geq B &\geq A^{\frac{-r_1}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}A^{\frac{-r_1}{2}} \\
&\geq A^{\frac{-(r_1+r_2)}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{\frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}}A^{\frac{-(r_1+r_2)}{2}} \\
&\geq A^{\frac{-(r_1+r_2+r_3)}{2}}\{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{1+r_1+r_2+r_3}{\{(p_1+r_1)p_2+r_2\}p_3+r_3}}A^{\frac{-(r_1+r_2+r_3)}{2}} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\geq A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}}\mathbb{C}_{A,B}[n]^{\frac{1+r_1+r_2+\ldots+r_n}{\mathfrak{q}[n]}}A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}} \qquad (5.4)
\end{aligned}$$

holds for $p_1, p_2, \ldots, p_n$ satisfying (3.7), *that is,*

$$p_1 \geq 1,\ p_2 \geq \frac{1+r_1}{p_1+r_1},\ p_3 \geq \frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}, \ldots\ldots, p_n \geq \frac{1+r_1+r_2+\ldots+r_{n-1}}{q[n-1]},$$

where $\mathbb{C}_{A,B}[n]$ is defined in (2.1) *and $\mathfrak{q}[n]$ is defined in* (2.4).

Remark 5.1. Corollary 5.2 is a further extension of [25], [17], [20], [34] and Theorem FKN-2 in [9]. Corollary 5.3 is more precise estimation than Corollary 3.2.

We would like to emphasize that Corollary 5.4 is a further extension of Theorem 3.3 since (5.4) *easily implies* (3.6) *in Theorem 3.3 and moreover the essential part of* (5.4) *in Corollary 5.4 on the usual order* ($A \geq B \geq 0$) *is derived from Corollary 5.2 on the chaotic order* ($A \gg B$).

§6 Further extensions of Theorem B and Theorem C

Further extensions of Theorem B and Theorem C are given by using the operator function $\mathfrak{F}_n(p_n, r_n) = A^{\frac{-r_n}{2}}\mathbb{C}_{A,B}[n]^{\frac{\delta+r_1+r_2...+r_n}{q[n]}}A^{\frac{-r_n}{2}}$ in §4.

Theorem 6.1. *Let $A \gg B$ and $r_1, r_2, ..., r_n \geq 0$ for a natural number n. For any fixed $\delta \geq 0$, let $p_1, p_2, ..., p_n$ be satisfied by*

$$p_j \geq \frac{\delta+r_1+r_2+...+r_{j-1}}{q[j-1]} \quad \text{for } j = 1, 2, ..., n, \tag{4.1}$$

that is,

$$p_1 \geq \delta,\ p_2 \geq \frac{\delta+r_1}{p_1+r_1}, ...,\ p_k \geq \frac{\delta+r_1+r_2+...+r_{k-1}}{q[k-1]}, ...,\ p_n \geq \frac{\delta+r_1+r_2+...+r_{n-1}}{q[n-1]}.$$

Then

$$\mathfrak{F}_n(p_n, r_n) = A^{\frac{-r_n}{2}}\mathbb{C}_{A,B}[n]^{\frac{\delta+r_1+r_2+...+r_n}{q[n]}}A^{\frac{-r_n}{2}} \tag{6.1}$$

is a decreasing function of both $r_n \geq 0$ and p_n which satisfies

$$p_n \geq \frac{\delta+r_1+r_2+...+r_{n-1}}{q[n-1]}. \tag{6.2}$$

Corollary 6.2. *Let $A \gg B$ and $r_1, r_2, ..., r_n \geq 0$ and also $p_1, p_2, ..., p_n \geq 1$ for a natural number n. Then*

$$\mathfrak{F}_n(p_n, r_n) = A^{\frac{-r_n}{2}}\mathbb{C}_{A,B}[n]^{\frac{1+r_1+r_2+...+r_n}{q[n]}}A^{\frac{-r_n}{2}} \tag{6.1'}$$

is a decreasing function of both $r_n \geq 0$ and $p_n \geq 1$.

Remark 6.1. *There is an alternative proof of Theorem 4.1 via Theorem 6.1.*

Remark 6.2. Theorem 6.1 is a further extensions of (ii) in Theorem C. In fact, (ii) of Theorem C is just Theorem 6.1 in the case $n = 1$. Moreover Theorem 6.1 is a further extension of Theorem B since the hypothesis $A \gg B$ in Theorem 6.1 is weaker than the hypothesis $A \geq B \geq 0$ in Theorem B.

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