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$*$-PARANORMAL OPERATORS AND RELATED TOPICS

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Abstract A bounded linear operator T on a Hilbert space is said to be normaloid if the operator norm $\|T\|$ of T equals to the spectral radius $r(T)=\sup\{|z| \mid z\in\sigma(T)\}$ of T where $\sigma(T)$ denotes the spectrum of T, and said to be $*$-paranormal if $\|T^{*}x\|^{2}\leq\|T^{2}x\|\|x\|$ for all $x\in \mathcal{H}$. Also we say that T belongs to the class $\mathfrak{P}(n)$ for an $n\in \mathbb{N}$ if $\|Tx\|^{n}\leq\|T^{n}x\|\|x\|^{n-1}$ for all $x\in \mathcal{H}$. An operator T in $\mathfrak{P}(n)$ is called n-paranormal.

In this talk, we introduce that, for every $*$-paranormal operator T, (1) T is isoloid (2) the Riesz idempotent E_{λ} of T w. r. t. any isolated point λ of $\sigma(T)$ is self-adjoint with the property that $E_{\lambda}\mathcal{H}=\ker(T-\lambda)=\ker(T-\lambda)^{*}$ (3) Weyl's theorem holds for T and (4) T satisfies the spectral property (I) hence T has the single valued extension property (SVEP) and Bishop's property (β). We also show some parallel results for the class $\mathfrak{P}(n)$.

1. Isoloidness of $\mathfrak{P}(n)$ operators.

There are many important classes of Hilbert space normaloid operators. For examples normal, subnormal, hyponormal, paranormal are famous and important classes. We study two claases of operators, $*$-paranormal and the class $\mathfrak{P}(n)$ in this paper. These operators are generalization of paranormal and expected to have same properties as paranormal operators, e.g.,

Isoloidness, self-adjointness of Riesz idempotent w. r. t. a non-zero isolated point of the spectrum, Weyl's theorem, SVEP and Bishop's priperty (β).

To show that $*$-paranormal and $\mathfrak{P}(n)$ operators have these properties we need some lemmas.

The following are well-known:

Facts 1.

For a $*$-paranormal T

1. $\|T\|=r(T)$, 2. $T\in\mathfrak{P}(3)$, 3. $\ker(T-\lambda)\subset\ker(T-\lambda)*$

4. The restriction T to its invariant subspace is also $*$-paranormal

Facts 2.

For an operator $T \in \mathfrak{P}(n)$

1. $\|T\| = r(T)$, 2. The restriction T to its invariant subspace is also $*$-paranormal

3. If $T \in \mathfrak{P}(2)$ is invertible then $T^{-1} \in \mathfrak{P}(2)$

To prove that every $T \in \mathfrak{P}(2)$ is isoloid, i.e., any isolated point of $\sigma(T)$ is an eigen value of T, we use the facts T and T^{-1} are normaloid and any restriction of T to its invariant subspace is in $\mathfrak{P}(2)$ and hence normaloid. From these facts, we have if $T \in \mathfrak{P}(2)$ and $\sigma(T) \subset S^1 := \{z \in \mathbb{C} \,|\, |z| = 1\}$ then T is unitary. And it is easy to see that $TE = \lambda E$ for any isolated point λ of $\sigma(T)$ and the Riesz idempotent E w. r. t. λ and hence $\lambda \in \sigma_p(T)$. However, it is not necessarily true that T^{-1} is normaloid for $\mathfrak{P}(n)$ $(n \geq 3)$. The next theorem show that every $T \in \mathfrak{P}(n)$ is isoloid.

we say that a class of operators has property A if every operator T in the class such as $\sigma(T) \subset S^1$ is unitary.

Proposition. Paranormal, i.e., the class $\mathfrak{P}(2)$, has property A.

Though the inverse of an invertible paranormal operator is always paranormal hence it is normaloid, however, it is known that there is an example of invertible $*$-paranormal whose inverse is not normaloid. The following theorem implies the classes $\mathfrak{P}(n)$ have property A and hence they are normaloid.

Theorem 1. If $T \in \mathfrak{P}(n)$ is invertible then

$$\|T\| \leq r(T^{-1})^{\frac{n(n-1)}{2}} \times r(T)^{\frac{(n-2)(n+1)}{2}}.$$

If $T \in \mathfrak{P}(n)$ satisfies $\sigma(T) \subset S^1$ then T is unitary.

Corollary 1. Every $T \in \mathfrak{P}(n)$ is isoloid.

2. Self-adjointness of Riesz idempotent for $\mathfrak{P}(n)$ operators.

For a $*$-paranormal operator T and an isolated point λ of its spectrum the Riesz idempotent $E = \dfrac{1}{2\pi i}\displaystyle\int_{\partial D_\lambda} (z - T)^{-1} dz$ satisfies $E_\lambda \mathcal{H} \subset \ker(T - \lambda)$, where D_λ is a closed disk with center λ and small enough radius r such as $D_\lambda \cap \sigma(T) = \{\lambda\}$. Since $\ker(T - \lambda) \subset E_\lambda \mathcal{H}$ is trivial and $\ker(T - \lambda)$ reduces T we have the following theorem.

Theorem 2. If T is $*$-paranormal and λ is an isolated point of $\sigma(T)$ then the Riesz idempotent E_λ w. r. t. λ is self-adjoint and

$$E_\lambda \mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*.$$

Since eigen space of a $\mathfrak{P}(n)$ operator is not necessarily reducing, similarly result does not hold for the class $\mathfrak{P}(n)$ without some additional conditions. Let $n \in \mathbb{N}, \lambda \in \mathbb{C}$. The polynomial

$$F_{n,\lambda}(z) := -(n-1)\lambda^{n-1} + \lambda^{n-2}z + \lambda^{n-3}z^2 \cdots + \lambda z^{n-2} + z^{n-1}$$

is important to study the class $\mathfrak{P}(n)$.

Theorem 2′. Let $T \in \mathfrak{P}(n)$ for an $n \geq 3$, λ be a non-zero isolated point of $\in \sigma_p(T)$. Put $T = \begin{pmatrix} \lambda & S \\ 0 & A \end{pmatrix}$ on $\mathcal{H} = \ker(T-\lambda) \oplus (\ker(T-\lambda))^\perp$. Then $S(\lambda^{n-1} + \lambda^{n-2}A + \cdots + \lambda A^{n-2} + A^{n-1}) = n\lambda^{n-1}S$. In particular, if $\sigma(T) \cap \{z \in \mathbb{C} \,|\, F_{n,\lambda}(z) = 0\} = \{\lambda\}$, then the Riesz idempotent E_λ with respect to λ is self-adjoint which satisfies

$$E_\lambda \mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*.$$

3. Weyl's theorem, SVEP and Bishop's property (β).

We say that Weyl's theorem holds for an operator $T \in \mathcal{B}(\mathcal{H})$, or T satisfies Weyl's theorem, iff

$$\sigma(T) \setminus w(T) = \pi_{00}(T),$$

where $w(T) := \{z \in \mathbb{C} \,|\, T - z \text{ is not a Fredholm operator with index } 0\}$ (Weyl spectrum of T) and $\pi_{00}(T) := \{z \in \sigma_p(T) \,|\, z \text{ is isolated in } \sigma(T) \text{ and } \dim\ker(T-z) < \infty\}$.

Uchiyama, 2006

If $T \in \mathfrak{P}(2)$ then T satisfies Weyl's theorem.

By using Riesz idempotent and Fredholm theory, we extend this result to the case where $\mathfrak{P}(n)$ $(n \geq 3)$ as follows.

Theorem 3. If $T \in \mathfrak{P}(n)$ then T satisfies Weyl's theorem.

An operator $T\mathcal{B}(\mathcal{H})$ is said to have SVEP at $\lambda \in \mathbb{C}$ if for any open nbd $\mathcal{U}$ of λ and analytic function $f : \mathcal{U} \to \mathcal{H}$ the zero function is the only analytic solution of the equation

$(T-z)f(z) = 0$ for all $z \in \mathcal{U}$. And T is said to have SVEP iff T has SVEP at any $\lambda \in \mathbb{C}$. Also, T is said to have Bishop's property (β) iff for any open set $\mathcal{D} \subset \mathbb{C}$ and any sequence of analytic functions $f_n : \mathcal{D} \to \mathcal{H}$ such as $\|(T-z)f_n(z)\| \to 0$ uniformly on every compact subset of $\mathcal{D}$ then $f_n \to 0$ uniformly on every compact subset of $\mathcal{D}$.

The following theorem implies every $T \in \mathfrak{P}(n)$ has SVEP.

Theorem 4. If $T \in \mathfrak{P}(n)$ and $\lambda, \mu \in \mathbb{C}$ $(\lambda \neq \mu)$ then $\ker(T-\lambda) \perp \ker(T-\mu)$.

If $f : \mathcal{U} \to \mathcal{H}$ satisfies $(T-z)f(z) \equiv 0$ then $\langle f(z), f(w) \rangle = 0$ if $z \not= w$. Hence

$$\|f(z)\|^2 = \lim_{w \to z} \langle f(z), f(w) \rangle = 0, \quad f \equiv 0.$$

To prove that every $T \in \mathfrak{P}(n)$ has Bishop's property (β), we use the following theorem.

Uchiyama-Tanahashi, 2009

If T has one of the following properties then T has SVEP and Bishop's property (β).

Property (I) : If $\lambda \in \sigma_a(T)$ is arbitrary and $\{x_n\} \subset \mathcal{H}$ is an arbitrary bdd sequence with $\|(T-\lambda)x_n\| \to 0$ then $\|(T-\lambda)^* x_n\| \to 0$.

Property (I′) : If $\lambda \in \sigma_a(T) \setminus \{0\}$ is arbitrary and $\{x_n\} \subset \mathcal{H}$ is an arbitrary bdd sequence with $\|(T-\lambda)x_n\| \to 0$ then $\|(T-\lambda)^* x_n\| \to 0$.

Property (II) : If $\lambda, \mu \in \sigma_a(T)$ $(\lambda \neq \mu)$ are arbitrary and $\{x_n\}, \{y_n\} \subset \mathcal{H}$ are arbitrary bdd sequences with $\|(T-\lambda)x_n\| \to 0$, $\|(T-\mu)y_n\| \to 0$ then $\langle x_n, y_n \rangle \to 0$.

If $T \in \mathfrak{P}(2)$ then T has the property (II), so it has SVEP and Bishop's property (β).

For $*$-paranormal operators and $\mathfrak{P}(n)$ operators we have the following.

Theorem 5. If T is $*$-paranormal then T has the property (I).
If $T \in \mathfrak{P}(n)$ then T has the property (II).
Hence, if T is $*$-paranormal or $T \in \mathfrak{P}(n)$ then T has SVEP and Bishop's property (β).