Whitney preserving maps onto dendrites

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Abstract

This note is a survey on Whitney preserving maps. In particular we introduce next results.

(1) Let $X$ be a continuum such that $X$ contains a dense arc component and let $D$ be a dendrite with a closed set of branch points. If $f : X \rightarrow D$ is a Whitney preserving map, then $f$ is a homeomorphism.

(2) For each dendrite $D'$ with a dense set of branch points there exist a continuum $X'$ containing a dense arc component and a Whitney preserving map $f' : X' \rightarrow D'$ such that $f'$ is not a homeomorphism.

1 Introduction

In this note, all spaces are separable metrizable spaces and maps are continuous. We denote the interval $[0, 1]$ by $I$. A compact metric space is called a compactum and continuum means a connected compactum. If $X$ is a continuum $C(X)$ denotes the space of all subcontinua of $X$ with the topology generated by the Hausdorff metric.

In this note we study maps called Whitney preserving maps. If $f : X \rightarrow Y$ is a map between continua, then define a map $\hat{f} : C(X) \rightarrow C(Y)$ by $\hat{f}(A) = f(A)$ for each $A \in C(X)$. A map $f : X \rightarrow Y$ is called a Whitney preserving map if there exist Whitney maps (see p105 of [12]) $\mu : C(X) \rightarrow I$ and $\nu : C(Y) \rightarrow I$ such that for each $s \in [0, \mu(X)]$, $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(t)$ for some $t \in [0, \nu(Y)]$. In this case, we say that $f$ is $\mu, \nu$-Whitney preserving. Let $f : X \rightarrow Y$ be a $\mu, \nu$-Whitney preserving map. Then it is easy to see that if $s, s' \in [0, \mu(X)]$ and $t, t' \in [0, \nu(Y)]$ satisfy $s \leq s'$, $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(t)$ and $\hat{f}(\mu^{-1}(s')) = \nu^{-1}(t')$, then $t \leq t'$.

The notion of a Whitney preserving map is introduced by Espinoza (see [1] and [2]). In this article we study these maps.

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1 AMS Subject Classification: Primary 54C05, 54C10; Secondary 54F15, 54F45.
2 Key words and phrases: Whitney preserving map. terminal continuum, hyperspace.
2 Whitney preserving maps onto dendrites

At first we give an example of a Whitney preserving map.

**Example 2.1** (Example 2 of [1]) let $f: [0, \pi] \to S^1$ be a map defined by $f(t) = e^{4ti}$. Then $f$ is Whitney preserving. But $f$ is not a homeomorphism.

In [1] Espinoza proved the following result.

**Theorem 2.2** (Theorem 16 of [1]) Let $X$ be a continuum such that $X$ contains a dense arc component. If $f: X \to I$ is a Whitney preserving map, then $f$ is a homeomorphism.

A Peano continuum is called a dendrite if it contains no simple closed curve. Let $D$ be a dendrite. A point $e \in D$ is called an end point of $D$ if $D \setminus \{e\}$ is connected. A point $b \in D$ is called a branch point of $D$ if there exists a neighbourhood $U$ of $b$ such that for each neighbourhood $V$ of $b$ with $V \subset U$, $|\text{Bd}(V)| \geq 3$. We denote the set of all end points in $D$ by $E(D)$. Also we denote the set of all branch points of $D$ by $B(D)$.

Recently the author proved the next theorem ([9], see also [8]).

**Theorem 2.3** Let $X$ be a continuum such that $X$ contains a dense arc component and let $D$ be a dendrite with the closed set of branch points. Then a map $f: X \to D$ is a Whitney preserving map if and only if $f$ is a homeomorphism.

**Corollary 2.4** Let $X$ be a continuum such that $X$ contains a dense arc component and let $T$ be a tree. Then a map $f: X \to T$ is a Whitney preserving map if and only if $f$ is a homeomorphism.

Generally, Theorem 2.3 does not hold when $D$ is a graph by Example 2.1.

**Remark.** For every 1-dimensional continuum $M$ there exists a 1-dimensional continuum $\hat{M}$ (other than $M$) such that there is a Whitney preserving map $f: \hat{M} \to M$ by Theorem 2.9 of [2].

It is natural to ask that whether Theorem 2.3 holds when $D$ is any dendrite. In fact, this does not hold.

If $X$ and $Y$ be compacta, then $C(X, Y)$ denotes the set of all continuous maps from $X$ to $Y$ endowed with sup metric. Also $S(X, Y)$ denotes the set of all surjective maps in $C(X, Y)$. If $v, w \in X$, then we denote the set $\{f \in C(X, Y) | f(v) = f(w)\}$ by $C_{(v,w)}(X, Y)$. Also we denote the set $\{f \in$
$S(X, Y)|f(v) = f(w)|$ by $S_{(v, w)}(X, Y)$. It is easy to see that $C_{(v, w)}(X, Y)$ and $S_{(v, w)}(X, Y)$ are closed subsets of $C(X, Y)$. Let $N \subset X$. Then we denote the set $\{f \in C(X, Y)|f^{-1}(f(x)) = \{x\}$ for each $x \in N\}$ by $A_N(X, Y)$. If $N$ is a one point set $\{a\}$, then we denote the set $A_N(X, Y)$ by $A_a(X, Y)$. Let $x \in X$ and $r > 0$. Then we denote the set $\{f \in C(X, Y)|\text{diam } f^{-1}(f(x)) < r\}$ by $A_{x, r}(X, Y)$.

Finally, we denote the identity map on a space $S$ by $id_S$.

A surjective map $e$ from $I$ onto a graph $G$ is called an Eulerian path if $e$ satisfies; (i) $e(0) = e(1)$, (ii) $|\{y \in G|e^{-1}(y) \text{ is nondegenerate }\}| < \infty$ and (iii) each fiber of $e$ is finite. In [3] Espinoza and Illanes proved the next result.

**Theorem 2.5 ([3])** For each graph $G$ which admits an Eulerian path, there exist a continuum $X_G$ containing a dense arc component and a Whitney preserving map $f : X_G \to G$ such that $f$ is not a homeomorphism.

In [9] the author showed that this result holds when $G$ is a superdendrite. A dendrite $D$ is called a superdendrite if $E(D)$ is dense in $D$. It is known that a dendrite $D$ is a superdendrite if and only if $B(D)$ is dense in $D$.

**Lemma 2.6 ([9])** Let $X$ be a compactum and let $D$ be a superdendrite. If $v, w$ and $a$ are points in $X$ such that $a \notin \{v, w\}$, then $C_{(v, w)}(X, D) \cap A_a(X, D)$ is a dense $G_\delta$-subset in $C_{(v, w)}(X, D)$.

**Lemma 2.7 ([9])** Let $X$ be a nondegenerate continuum and let $D$ be a superdendrite. If $v, w$ and $a$ are points in $X$ such that $a \notin \{v, w\}$, then $S_{(v, w)}(X, D) \cap A_a(X, D)$ is a dense $G_\delta$-subset in $S_{(v, w)}(X, D)$.

By Lemma 2.7 and Baire Category Theorem, we get the next corollary.

**Corollary 2.8 ([9])** Let $X$ be a nondegenerate continuum, $N$ a countable subset of $X$ and $D$ a superdendrite. If $v, w$ are points in $X$ such that $N \cap \{v, w\} = \emptyset$, then $S_{(v, w)}(X, D) \cap A_N(X, D)$ is a dense $G_\delta$-subset in $S_{(v, w)}(X, D)$.

By using Corollary 2.8 and arguments in [3], we can prove the following result.

**Theorem 2.9 ([9])** For each superdendrite $D$, there exist a continuum $X_D$ containing a dense arc component and a Whitney preserving map $f : X_D \to D$ such that $f$ is not a homeomorphism.

Recently the author generalized this result.
Theorem 2.10 ([10]) For each 1-dimensional locally connected continuum without free arcs $P$, there exist a continuum $X_P$ containing a dense arc component and a Whitney preserving map $f : X_P \to P$ such that $f$ is not a homeomorphism.

Theorem 2.11 ([10]) For each $n \geq 2$ and an $n$-dimensional manifold $M$, there exist a continuum $X_M$ containing a dense arc component and a Whitney preserving map $f : X_M \to M$ such that $f$ is not a homeomorphism.

3 Other topics related to Whitney preserving maps

A subcontinuum $T$ of a continuum $X$ is terminal, if every subcontinuum of $X$ which intersects both $T$ and its complement must contain $T$.

Now we give a notation. If $f : X \to Y$ is a map, let $A_f = \{f^{-1}(y) | y \in Y\}$ and $A'_f = \{C | C$ is a component of a fiber of $f\}$.

Let $f : X \to Y$ be a Whitney preserving map. Then $A_f$ need not be a continuous decomposition of $X$. For example let $f : [0, \pi] \to S^1$ be a map defined by $f(t) = e^{4ti}$. Then $f$ is Whitney preserving (cf. Example 2 of [1]). But $f$ is not an open map.

In [7] the author proved next results.

Proposition 3.1 ([7]) Let $f : X \to Y$ be a $\mu, \nu$-Whitney preserving map. Then $A'_f$ is a continuous decomposition of $X$ and each element of $A'_f$ is terminal in $X$.

A map $f : X \to Y$ between continua is called an atomic map if $f^{-1}(f(A)) = A$ for each $A \in C(X)$ such that $f(A)$ is nondegenerate. It is known that a map $f$ of a continuum $X$ onto a continuum $Y$ is atomic if and only if every fiber of $f$ is a terminal continuum of $X$.

A map $f : X \to Y$ between compacta is called a Krasinkiewicz map if any continuum in $X$ either contains a component of a fiber of $f$ or is contained in a fiber of $f$ (cf. [6]). These maps are related to Whitney preserving maps.

Proposition 3.2 ([7]) Let $f : X \to Y$ be a map such that $A'_f$ does not contain a one point set. Then the following conditions are equivalent.

(1) $A'_f$ is a continuous decomposition of $X$ and each element of $A'_f$ is terminal in $X$.

(2) $A'_f$ is a continuous decomposition of $X$ and $f$ is a Krasinkiewicz map.

By using Proposition 3.2 the author proved next results.
Theorem 3.3 ([8]) Let $X$ be a continuum such that $X$ contains a dense arc component. If $f : X \to f(X)$ is a Whitney preserving map such that $f$ is not a constant map, then $f$ is a light map.

Theorem 3.4 ([7]) Let $X, Y$ be continua and let $f : X \to Y$ be a monotone map such that $f^{-1}(y)$ is a nondegenerate continuum in $X$. Then the following conditions are equivalent.

1. $f$ is an open map and each fiber of $f$ is terminal in $X$.
2. $f$ is an open Krasinkiewicz map.
3. $f$ is a Whitney preserving map.

As an application of Theorem 3.4 we obtain next results.

Theorem 3.5 ([8]) There exists a 1-dimensional continuum $T \subset I^2$, a Whitney map $\mu : C(T) \to I$ and $s_0, s_1 \in I$ such that

1. $0 < s_0 < s_1 < \mu(T)$,
2. $\dim \mu^{-1}(s) = 1$ for each $s \in [0, s_0)$,
3. $\dim \mu^{-1}(s_0) = 2$, and
4. $\dim \mu^{-1}(s) = \infty$ for each $s \in (s_0, s_1]$.

Theorem 3.6 ([8]) There exists a 1-dimensional continuum $T \subset I^2$ such that

1. $\dim C(T) = \infty$, and
2. for each Whitney map $w : C(T) \to I$ there exists $a_0 \in (0, w(T))$ such that $\dim w^{-1}(s) = 1$ for each $s \in [0, a_0]$.

At last we give some results related to Whitney preserving maps.

Proposition 3.7 ([8]) Let $f : X \to Y$ be a monotone $\mu, \nu$-Whitney preserving map and let $s_0 = \max \{s \in I | \hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)\}$. Then $\hat{f}|_{\mu^{-1}([s_0, \mu(X)])} : \mu^{-1}([s_0, \mu(X)]) \to C(Y)$ is a homeomorphism. Hence $\mu^{-1}(s)$ is homeomorphic to $f(\mu^{-1}(s))$ for each $s \in [s_0, \mu(X)]$.

A topological property $P$ is said to be a Whitney property provided that if a continuum $X$ has property $P$, so does $\mu^{-1}(t)$ for each Whitney map $\mu$ for $C(X)$ and for each $t \in [0, \mu(X)]$. As a corollary of Proposition 3.7 we get the next result.

Corollary 3.8 ([8]) Let $f : X \to Y$ be a monotone Whitney preserving map. If $X$ has a topological property $P$ which is a Whitney property, then so does $Y$. 

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References


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