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On local connectivity of boundaries of CAT(0) spaces

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We introduce (non-)local connectivity of boundaries of CAT(0) spaces and hyperbolic CAT(0) spaces.

Definitions and basic properties of CAT(0) spaces, hyperbolic spaces and their boundaries are found in [3], [10] and [11].

A metric space X is said to be ***proper*** if every closed metric ball is compact. A group G is called a ***CAT(0) group*** if G acts geometrically (i.e. properly and cocompactly by isometries) on some CAT(0) space. It is known that a CAT(0) space on which a CAT(0) group acts geometrically is proper. A boundary ∂X of a CAT(0) space X on which a CAT(0) group G acts geometrically is called a ***boundary*** of the CAT(0) group G. It is known that in general a CAT(0) group G does not determine its boundary [5]. If G is a hyperbolic group then G determines its boundary up to homeomorphisms (cf. [3], [10] and [11]).

The following problems are open.

Problem. When is a boundary of a CAT(0) group (non-)locally connected?

Problem. If G is a hyperbolic CAT(0) group whose boundary is connected then is the boundary locally connected?

Problem. For a CAT(0) group G and CAT(0) spaces X and Y on which G acts geometrically, is it the case that the boundary ∂X is locally connected if and only if the boundary ∂Y is locally connected?

There is a research on (local) n-connectivity of boundaries of hyperbolic Coxeter groups by A. N. Dranishnikov in [8], and there are some research on (non-)local connectivity of boundaries of CAT(0) groups and Coxeter groups by M. Mihalik, K. Ruane and S. Tschantz in [17] and [18].

The purpose of this paper is to introduce sufficient conditions of

(i) a hyperbolic CAT(0) group whose boundary is locally n-connected by using reflections, and
(ii) a CAT(0) space whose boundary is non-locally connected by using a hyperbolic isometry and a reflection.

Local n-connectivity of boundaries of hyperbolic CAT(0) spaces

We define a *reflection* of a geodesic space as follows: An isometry r of a geodesic space X is called a *reflection* of X, if

(1) r^2 is the identity of X,
(2) $X \setminus F_r$ has exactly two convex connected components X_r^+ and X_r^- and
(3) $rX_r^+ = X_r^-$,

where F_r is the fixed-points set of r. We note that "reflections" in this paper need not satisfy the condition (4) $\operatorname{Int} F_r = \emptyset$ in [15].

Theorem 1. *Suppose that a group G acts geometrically (i.e. properly and cocompactly by isometries) on a hyperbolic CAT(0) space X. If*

(1) *there exist some reflections $r_1, \dots, r_n \in G$ of X such that $G = \langle r_1, \dots, r_n \rangle$ and*
(2) *the boundary ∂X of X is n-connected,*

then the boundary ∂X is locally n-connected.

Corollary 2. *Suppose that a hyperbolic Coxeter group W acts geometrically on a hyperbolic CAT(0) space X. If the boundary ∂X of X is n-connected then ∂X is locally n-connected.*

From [8], we also obtain a corollary.

Corollary 3. *Let (W, S) be a hyperbolic Coxeter system and let $L = L(W, S)$ be the nerve of the Coxeter system (W, S). For any hyperbolic CAT(0) space X on which the hyperbolic Coxeter group W acts geometrically, the following statements are equivalent:*

(i) *L is connected and $L - \sigma$ is connected for any simplex σ of L,*
(ii) *$\check{H}^0(\partial X) = 0$ where $\check{H}^*$ denote the reduced Čech cohomology,*

(iii) *the boundary* ∂X *of* X *is connected, and*

(iv) *the boundary* ∂X *of* X *is locally connected.*

Here the following problems are open.

Problem. If G is a hyperbolic CAT(0) group whose boundary is n-connected then is the boundary locally n-connected?

Problem. For a non-elementary hyperbolic Coxeter group W on which acts geometrically on a CAT(0) space X, is it the case that the following statements are equivalent?

(i) $\check{H}^i(\partial X) = 0$ for any $0 \le i \le n$,

(ii) L is n-connected and $L - \sigma$ is n-connected for any simplex σ of L,

(iii) the boundary ∂X of X is n-connected, and

(iv) the boundary ∂X of X is locally n-connected.

Non-local connectivity of boundaries of CAT(0) spaces

Let X be a proper CAT(0) space and let g be an isometry of X. The *translation length* of g is the number $|g| := \inf\{d(x, gx) \mid x \in X\}$, and the *minimal set* of g is defined as $\mathrm{Min}(g) = \{x \in X \mid d(x, gx) = |g|\}$. An isometry g of X is said to be *hyperbolic*, if $\mathrm{Min}(g) \neq \emptyset$ and $|g| > 0$ (cf. [3, p.229]). For a hyperbolic isometry g of a proper CAT(0) space X, g^∞ is the limit point of the boundary ∂X to which the sequence $\{g^i x_0\}_i$ converges, where x_0 is a point of X. Here we note that the limit point g^∞ is not depend on the point x_0.

A CAT(0) space X is said to be *almost geodesically complete*, if there exists a constant $M > 0$ such that for each pair of points $x, y \in X$, there is a geodesic ray $\zeta : [0, \infty) \to X$ such that $\zeta(0) = x$ and ζ passes within M of y. In [9, Corollary 3], R. Geoghegan and P. Ontaneda have proved that every non-compact cocompact proper CAT(0) space is almost geodesically complete. Here a CAT(0) space X is said to be *cocompact*, if some group acts cocompactly by isometries on X.

On non-local connectivity of CAT(0) spaces, we obtained the following.

Theorem 4. *Let* X *be a proper and almost geodesically complete CAT(0) space, let* g *be a hyperbolic isometry of* X *and let* r *be a reflection of* X*. If*

(1) $g^\infty \notin \partial F_r$,
(2) $g(\partial F_r) \subset \partial F_r$ *and*
(3) $\mathrm{Min}(g) \cap F_r = \emptyset$,

then the boundary ∂X of X is non-locally connected.

Here we note that the action of the group G on the CAT(0) space X in Theorem 4 need not be proper and cocompact.

The conditions in Theorem 4 are rather technical. We introduce some remarks.

First, every CAT(0) space on which some group acts geometrically (i.e. properly and cocompactly by isometries) is proper ([3, p.132]) and *almost geodesically complete* ([9], [20]).

Also, in [22], Ruane has proved that $\partial\,\mathrm{Min}(g)$ is the fixed-points set of g in ∂X, i.e.,

$$\partial\,\mathrm{Min}(g) = \{\alpha \in \partial X \mid g\alpha = \alpha\}.$$

Hence, for example, if $\partial F_r \subset \partial\,\mathrm{Min}(g)$ then $g(\partial F_r) = \partial F_r$ and the condition (2) in Theorem 4 holds.

As an example of CAT(0) spaces on which some reflections act, there is the Davis complex of a Coxeter system. A Coxeter system (W, S) determines the *Davis complex* $\Sigma(W, S)$ which is a CAT(0) space ([6], [19]). Then the Coxeter group W acts geometrically on $\Sigma(W, S)$ and each $s \in S$ is a reflection of $\Sigma(W, S)$.

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