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Fundamental solutions of diffusion equations related to certain Dirichlet forms and the quasi-geostrophic equation

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1 Introduction

We consider the non-local diffusion equations in the presence of a drift term

$$\partial_t\theta + A_K(t)\theta + v\cdot\nabla\theta = 0, \qquad t>0, \quad x\in\mathbb{R}^d, \tag{1.1}$$

where $d\geq 2$ and $A_K(t)$ is a linear operator formally defined by

$$\big(A_K(t)f\big)(x) = P.V.\int_{\mathbb{R}^d}(f(x)-f(y))K(t,x,y)\,\mathrm{d}y. \tag{1.2}$$

Here $K(t,x,y)$ is a positive function satisfying $K(t,x,y) = K(t,y,x)$ and $v(t,x) = (v_1(t,x),\cdots,v_d(t,x))$ is a vector field in $\mathbb{R}^d$ satisfying the divergence free condition, $\nabla\cdot v(t) = 0$. In particular, $A_K(t)$ will be supposed to possess a diffusion effect like $(-\Delta)^{\alpha/2}$ for some $\alpha\in(0,2)$. Note that in the case $A_K(t) = (-\Delta)^{\alpha/2}$ with $\alpha\in(0,2)$ the kernel K is given by $K(t,x,y) = C_{d,\alpha}|x-y|^{-d-\alpha}$ for some positive constant $C_{d,\alpha}$. The aim of this note is to show the existence and the continuity of fundamental solutions for (1.1) under less regularity assumptions on K and v.

When there is no drift term (i.e., $v = 0$) this problem appears in the theory of Dirichlet forms of jump type. For the diffusion operator $A_K(t)$ defined by (1.2) the associated Dirichlet form is

$$\mathcal{E}_K^{(t)}(f,g) = \frac{1}{2}\iint_{\mathbb{R}^{2d}} [f][g](x,y)K(t,x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad [f](x,y) = f(x) - f(y), \tag{1.3}$$

and it has been investigated mainly from the probabilistic approach [6, 16, 17, 3, 1, 13, 2]. On the other hand, in recent years the case with a drift term has also attracted much attention, especially in the field of fluid mechanics, mathematical finance, biology, and so on. For example, many works have been done for the two-dimensional dissipative quasi-geostrophic equations (QG), where $A_K(t) = (-\Delta)^{\alpha/2}$ and the drift term is a nonlinear term such that v is given in terms of θ via the Riesz transform; [7]. For such nonlinear problems it is crucial to obtain detailed informations of solutions under less regularity conditions on v.

In [4, 12] fundamental solutions were constructed when $A_K(t) = (-\Delta)^{\alpha/2}$ with $\alpha \in (1,2)$ and v belongs to a suitable Kato class without assuming the divergence free condition. In this case the diffusion term is the leading term and they showed two-sided heat kernel estimates by using perturbation arguments. However, despite of the increasing interest, there seems to be still few works on fundamental solutions for $\alpha \in (0,1]$. In such cases the drift term formally becomes the leading term and is no longer regarded as a simple perturbation of $A_K(t)$, which causes difficulties in the study of (1.1). For example, so far little seem to be known about the uniqueness of weak solutions for such cases and this makes even the semigroup property of fundamental solutions nontrivial.

To state our main results we give the precise assumptions on the kernel K and the velocity v. We assume that there are $\alpha \in (0,2)$ and $C_0 > 0$ such that

$$\begin{aligned} K(t,x,y) &= K(t,y,x) && \text{for } a.e.\ (t,x,y) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d, && (1.4)\\ \operatorname{ess.sup}_{t>0,x\in\mathbb{R}^d}\int_{|x-y|\le M} |x-y|^2 K(t,x,y)\,\mathrm{d}y &\le C_0 M^{2-\alpha} && \text{for each } M \in (0,\infty), && (1.5)\\ \operatorname{ess.inf}_{t>0,x,y\in\mathbb{R}^d}|x-y|^{d+\alpha}K(t,x,y) &\ge C_0^{-1}. && && (1.6) \end{aligned}$$

Following the conventions in (QG), we will call the case $\alpha \in (1,2)$ subcritical, the case $\alpha = 1$ critical, and the case $\alpha \in (0,1)$ supercritical. We note that if K satisfies

$$C^{-1}|x-y|^{-d-\alpha} \le K(t,x,y) \le C|x-y|^{-d-\alpha}, \tag{1.7}$$

then (1.5) and (1.6) are satisfied. The condition (1.6) guarantees the diffusion effect like $(-\Delta)^{\alpha/2}$. Taking this in mind, we assume that v belongs to a class of functions which is invariant under the scaling

$$v(t,x) \longmapsto r^{1-1/\alpha}v(rt, r^{1/\alpha}x), \quad r > 0. \tag{1.8}$$

This scaling is natural in the following sense: if $\theta(t,x)$ is a solution to (1.1) with $A_K(t) = (-\Delta)^{\alpha/2}$ and $v = v(t,x)$ then the rescaled function $\theta(rt, r^{1/\alpha}x)$ satisfies (1.1) with $A_K(t) =$

$(-\Delta)^{\alpha/2}$ and $v = r^{1-1/\alpha}v(rt, r^{1/\alpha}x)$, instead of $v(t,x)$. Heuristically it is essential that v belongs to a function space which is invariant with respect to (1.8), in order to ensure a smoothing effect by $(-\Delta)^{\alpha/2}$; for example, see [19, 22, 20] and references there in for second order parabolic equations and [5, 14, 21] for the fractional diffusion equations.

To describe the regularity assumption on v let us introduce the Campanato spaces; see [9].

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^d) = \{f \in L^q_{loc}(\mathbb{R}^d) \mid \|f\|_{\mathcal{L}^{p,\lambda}} = \sup_B \left(R^{-\lambda}\int_B |f(x) - \fint_B f|^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty\}. \tag{1.9}$$

Here the supremum is taken over all balls $B = B_R(y)$ (the ball with radius $R > 0$ centered at $y \in \mathbb{R}^d$), and $|B|$ is the volume of the ball B. We will sometimes write B_R for $B_R(0)$ for simplicity of notations. The value $\fint_B f$ is defined by

$$\fint_B f = \frac{1}{|B|}\int_B f(x)dx. \tag{1.10}$$

Then we have

$$\begin{aligned} L_{\mathrm{w}}^{\frac{pd}{d-\lambda}}(\mathbb{R}^d) &\hookrightarrow \mathcal{L}^{p,\lambda}(\mathbb{R}^d) \quad \text{if} \quad 0 < \lambda < d, && (1.11)\\ \mathcal{L}^{p,\lambda}(\mathbb{R}^d) &= BMO(\mathbb{R}^d) \quad \text{if} \quad \lambda = d, && (1.12)\\ \mathcal{L}^{p,\lambda}(\mathbb{R}^d) &= \dot{C}^{\frac{\lambda-d}{p}}(\mathbb{R}^d) \quad \text{if} \quad d < \lambda \le d+p. && (1.13) \end{aligned}$$

Here $L^p_{\mathrm{w}}(\mathbb{R}^d)$ is the weak L^p space and $\dot{C}^\beta(\mathbb{R}^d)$, $\beta \in (0,1]$, is the homogeneous Hölder space of the order β.
Next we introduce the Morrey type spaces of $\mathcal{L}^{p,\lambda}$-valued functions.

$$\begin{aligned} L^{p,\lambda_1}(0,\infty;\mathcal{L}^{q,\lambda_2}(\mathbb{R}^d)) &= \{f \in L^p_{loc}(0,\infty;\mathcal{L}^{q,\lambda_2}(\mathbb{R}^d)) \mid \\ \|f\|_{L^{p,\lambda_1}(0,\infty;\mathcal{L}^{q,\lambda_2}(\mathbb{R}^d))} &= \sup_{t>0}\sup_{0<s<t} \left((t-s)^{-\lambda_1}\int_s^t \|f(\tau)\|^p_{\mathcal{L}^{q,\lambda_2}}\,\mathrm{d}\tau\right)^{\frac{1}{p}} < \infty\}. \end{aligned} \tag{1.14}$$

For $1 \le p, q \le \infty$ let $L^q_{loc}(0,\infty;L^p_{loc}(\mathbb{R}^d))$ be the class of functions defined by

$$L^q_{loc}(0,\infty;L^p_{loc}(\mathbb{R}^d)) = \{f \in L^1_{loc}((0,\infty)\times\mathbb{R}^d) \mid \|f\|_{L^q(0,R;L^p(B_R))} < \infty \quad \text{for all} \quad R > 0\}. \tag{1.15}$$

When $K(t,x,y)$ satisfies (1.5) and (1.6) for some $\alpha \in (0,2)$ the velocity v is assumed to satisfy the following two conditions:

(C1) there are $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$ and $1 < q \le \infty$ such that

$$v \in L^{1,\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}}\left(0,\infty;(\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d\right) \cap L^q_{loc}\left(0,\infty;(L^{p_\lambda}_{loc}(\mathbb{R}^d))^d\right), \tag{1.16}$$

where $p_\lambda = 1$ if $\lambda \in [2d/\alpha - d, d]$ and $p_\lambda = \infty$ if $\lambda \in (d, 2d/\alpha + d]$.

(C2) $\nabla \cdot v(t) = 0$ for a.e. $t > 0$ in the sense of distributions.

Remark 1.1 The space $L^{1,\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}}(0,\infty;\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))$ is invariant under the scaling (1.8). One of the advantages to use the Campanato spaces (1.9) is that for some exponents (p,λ) they contain functions growing at spatial infinity. In particular, the case $\lambda = 2d/\alpha + d$ in **(C1)** allows v to grow at most linearly as $|x| \to \infty$. We also note that the condition $v \in L^{1,\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}}(0,\infty;(\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d)$ includes the case

$$|t-t_0|^{\frac{\lambda}{2d}+\frac{1}{2}-\frac{1}{\alpha}}v(t) \in L^\infty(0,\infty;(\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d) \qquad \text{for some } t_0 \in [0,\infty). \tag{1.17}$$

Under the divergence free condition **(C2)** the drift term becomes skew-symmetric with respect to the usual $L^2(\mathbb{R}^d)$ inner product, and hence, the adjoint equation for (1.1) takes the same form as (1.1). This additional structure is essentially used in constructing fundamental solutions under weak regularity condition **(C1)**. The divergence free condition sometimes plays important roles also in the second order parabolic equations with singular drifts; [19, 22, 20].

For simplicity of notations we will introduce the seminorm

$$\|v\|_{X_\lambda} = \|v\|_{L^{1,\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}}(0,\infty;\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))}. \tag{1.18}$$

For $T>0$ and $x \in \mathbb{R}^d$ we also set

$$\|v\|_{Y^{q,\lambda}_{T,x}} = \|v\|_{L^q(0,T;L^{p_\lambda}(B_1(x)))}, \tag{1.19}$$

where p_λ is as in **(C1)**.

Let $T > s \geq 0$. A function $\theta \in L^\infty(s,T;L^2(\mathbb{R}^d))$ is said to be a weak solution to (1.1) for $t \in [s,T)$ with initial data θ_s at $t=s$ if θ satisfies

$$\int_s^T \mathcal{E}^{(t)}_K(\theta(t),\theta(t))\,\mathrm{d}t < \infty, \tag{1.20}$$

and

$$\int_s^T \Big(-<\theta(t),\partial_t\varphi(t)> + \mathcal{E}^{(t)}_K(\theta(t),\varphi(t)) - <\theta(t), v(t)\cdot\nabla\varphi(t)>\Big)\,\mathrm{d}t = <\theta_s,\varphi(s)> \tag{1.21}$$

for all $\varphi \in C_0^\infty([s,T)\times\mathbb{R}^d)$, where $<\cdot,\cdot>$ is the usual L^2-L^2 pairing in $\mathbb{R}^d$. Then a measurable function $P_{K,v}(t,x;s,y)$ on $\{(t,s,x,y) \mid t>s\geq 0,\ x,y\in\mathbb{R}^d\}$ is said to be a fundamental solution to (1.1) if for each $T>s\geq 0$ and $f \in L^2(\mathbb{R}^d)$ the function

$$(P_{K,v}f)(t,s,x) := \int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)f(y)\,\mathrm{d}y, \tag{1.22}$$

is a weak solution to (1.1) for $t \in [s,T)$ with initial data f at $t=s$.

The main result of this note is the existence of fundamental solutions for (1.1).

Theorem 1.2 *Suppose that (1.4) - (1.6) and* **(C1)** - **(C2)** *hold. Then there exists a fundamental solution $P_{K,v}(t,x;s,y)$ for (1.1) satisfying the following properties.*

$$\int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)\,\mathrm{d}x = \int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)\,\mathrm{d}y = 1, \tag{1.23}$$

$$0 \le P_{K,v}(t,x;s,y) \le C(t-s)^{-\frac{d}{\alpha}}, \tag{1.24}$$

$$P_{K,v}(t,x;s,y) = \int_{\mathbb{R}^d} P_{K,v}(t,x;\tau,z)P_{K,v}(\tau,z;s,y)\,\mathrm{d}z, \qquad t > \tau > s \ge 0, \tag{1.25}$$

$$|P_{K,v}(t,x_1;s,y_1) - P_{K,v}(t,x_2;s,y_2)| \le \frac{C'\big(|x_1-x_2|^\beta + |y_1-y_2|^\beta\big)}{(t-s)^c}, \tag{1.26}$$

and for $T \ge t_i > s_i \ge 0$, $i = 1,2$,

$$|P_{K,v}(t_1,x;s_1,y) - P_{K,v}(t_2,x;s_2,y)| \le \frac{C_{T,x}|t_1-t_2|^{\beta'} + C_{T,y}|s_1-s_2|^{\beta'}}{(\min\{t_1-s_1,t_2-s_2\})^{c'}}. \tag{1.27}$$

Here the positive constant C depends only on d, α, and C_0, the positive constants C', c, β depend only on d, α, C_0, λ, and $\|v\|_{X_\lambda}$, the positive constant $C_{T,x}$ (or $C_{T,y}$) depends only on T, d, α, C_0, λ, q, $\|v\|_{X_\lambda}$, and $\|v\|_{Y^{q,\lambda}_{T,x}}$ (or $\|v\|_{Y^{q,\lambda}_{T,y}}$), and the positive constants c', β' depend only on d, α, C_0, λ, q, $\|v\|_{X_\lambda}$.

Remark 1.3 In the proof of Theorem 1.2 we will also show that

$$(P_{K,v}f)(\cdot,s,\cdot) \in C([s,\infty);L^p(\mathbb{R}^d)) \quad \text{ if } f \in L^p(\mathbb{R}^d),\ 1 \le p < \infty, \tag{1.28}$$

and the energy inequality

$$\|\theta(t)\|_{L^2}^2 + 2\int_s^t \mathcal{E}_K^{(\tau)}\big(\theta(\tau),\theta(\tau)\big)\,\mathrm{d}\tau \le \|f\|_{L^2}^2, \tag{1.29}$$

for $t > s \ge 0$ if $f \in L^2(\mathbb{R}^d)$.

The estimates (1.26) and (1.27) show the Hölder continuity of the fundamental solution, where the Hölder exponents and the constant C' are estimated uniformly in time and space, while the constants $C_{T,x}$ and $C_{T,y}$ can be larger as $|x|$ and $|y|$ increase, if v grows at $|x| \to \infty$. We note that for some class of (K,v) and solutions the Hölder continuity is obtained in [5, 14] for the critical case and also in [8, 21] for the supercritical case. In [5, 14, 8] the case $A_K(t) = (-\Delta)^{\alpha/2}$ and $v \in L^\infty(0,\infty;(\mathcal{L}^{2d/\alpha,2d/\alpha-d})^d)$ was treated under the condition **(C2)**, and [21] dealt with the case (1.7) and $v \in (C^{1-\alpha}((0,\infty)\times\mathbb{R}^d))^d$ but without **(C2)**.

In order to prove Theorem 1.2 it is important to obtain the a priori estimates for fundamental solutions of the approximate equations which are, roughly speaking, of the form $\partial_t\theta + \delta(-\Delta)^{\tilde{\alpha}/2}\theta + A_{\tilde{K}}(t)\theta + \tilde{v}\cdot\nabla\theta = 0$ with $\delta > 0$. Here $\tilde{\alpha} \in (1,2)$, and $\tilde{K}$ and $\tilde{v}$ are suitable mollifications of K and v. It is more or less well known that the unique existence of fundamental solutions holds for such mollified equations due to the fact that the leading term is the extra diffusion term. So our main step is to prove the a priori (equi-)continuity estimates of the fundamental solutions. For the purpose we will use the Nash-type arguments by Komatsu [16, 17] where he studied the non-local diffusion equations without the drift term. As in [18, 16, 17], the arguments consist of four steps; the moment bound, the relative entropy bound, the overlap estimate, and the iteration estimate. However, due to the presence of the nonsmooth drift term, it seems to be difficult to obtain these estimates. In order to overcome the difficulty, we will derive these estimates in time-dependent coordinates along the trajectory determined by a local average of v, instead of the usual coordinates $\mathbb{R}^d$. Although similar coordinates were used in [5, 8, 14], we have to choose the appropriate trajectory in each step carefully. We note that, in fact, the arguments in [18, 16, 17] highly rely on the scaling property of (1.1), while the above approximation does not preserve such property. As a result, for example, it is difficult to get the equi-continuity estimates for solutions to the approximate equations, which causes another technicality in taking the limit and showing the desired estimate rigorously.

Theorem 1.2 has an application for the global regularity of solutions to (QG) in the critical case, as in [15, 5, 14]. Indeed, our result gives the alternative approach to this problem, based on the Nash-type arguments for fundamental solutions. In particular, it should be noted that different from [5] we need not study extension problems to use special property of the fractional Laplacian.

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