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Asymptotic stability of stationary waves for symmetric hyperbolic-parabolic system in half space

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Abstract: In the present paper, we consider a large-time behavior of solutions to the symmetric hyperbolic-parabolic system in the half space. Our main concern is to show existence and asymptotic stability of the stationary solution (boundary layer solution) under the situation where all of characteristics are non-positive. We firstly prove the existence of the stationary solution by assuming that a boundary strength is sufficiently small. Especially, in the case where one eigenvalue of Jacobian matrix appeared in a stationary problem becomes zero, we assume that the characteristics field corresponding to the zero eigenvalue is genuine nonlinear in order to show the existence of a degenerate stationary solution with the aid of a center manifold theory. We next prove that a nondegenerate stationary solution is time asymptotically stable with a small initial perturbation. The key to proof is to derive the uniform a priori estimates by using the energy method. To obtain a priori estimates, we use the energy method in half space developed by Matsumura and Nishida as well as the stability condition of Shizuta–Kawashima type.

1 Introduction

This article is a survey of the paper [10] on large-time behavior of solutions to a system of viscous conservation laws over one-dimensional half space $\mathbb{R}_+ := (0, \infty)$,

$$f^0(u)_t + f(u)_x = (G(u)u_x)_x, \quad x \in \mathbb{R}_+, \ t > 0. \tag{1.1}$$

Here $u = u(t,x)$ is an unknown m-vector function taking values in an open convex set $\mathcal{O} \in \mathbb{R}^m$; $f^0(u)$ and $f(u)$ are smooth m-vector functions; $G(u)$ is a smooth $m \times m$ real matrix function. We assume that $f^0(u)$ has no singularity, i.e., det $D_u f^0(u) \neq 0$ holds for $u \in \mathcal{O}$. It is also assumed that $G(u)$ is a non-negative matrix given by a form

$$G(u) = \begin{pmatrix} 0 & 0 \\ 0 & G_2(u) \end{pmatrix},$$

where $G_2(u)$ is an $m_2 \times m_2$ real matrix function and uniformly positive definite for $u \in \mathcal{O}$, where m_2 is a positive integer satisfying $0 < m_2 < m$. Thus the system (1.1) consists of m_1-hyperbolic equations and m_2-parabolic equations where $m_1 := m - m_2$.

We assume that the system (1.1) has a strictly convex entropy $\eta = \eta(z)$ satisfying

(i) $\eta(z)$ is a strictly convex scalar function, i.e., the Hessian matrix $D_z^2\eta(z)$ is positive definite for $z \in f^0(\mathcal{O})$.

(ii) There exists a smooth scalar function $q(u)$ (entropy flux) such that $D_u q(u) = D_z\eta(f^0(u))D_u f(u)$.

(iii) The matrix $B(u) := {}^T D_u f^0(u) D_z^2\eta(f^0(u))G(u)$ is real symmetric and non-negative.

Then the system (1.1) is deduced to the symmetric system

$$A^0(u)u_t + A(u)u_x = B(u)u_{xx} + g(u, u_x), \tag{1.2}$$

where $A^0(u)$ is a real symmetric and positive matrix, $A(u)$ is a real symmetric matrix, $B(u)$ is a real symmetric and non-negative matrix, and $g(u, u_x)$ is non-linear terms satisfying $|g(u, u_x)| \leq C|u_x|^2$. Moreover, under suitable conditions as in [4], the system (1.2) is rewritten to the decomposed form

$$A_1^0(u)v_t + A_{11}(u)v_x + A_{12}(u)w_x = g_1(u, w_x), \tag{1.3a}$$

$$A_2^0(u)w_t + A_{21}(u)v_x + A_{22}(u)w_x = B_2(u)w_{xx} + g_2(u, u_x), \tag{1.3b}$$

where v and w are unknown m_1- and m_2-vector functions respectively, given by $u = {}^T(v, w)$. In the system (1.3), $A_1^0(u)$ and $A_2^0(u)$ are real symmetric and positive matrices; $A_{ij}(u)$ $(i, j = 1, 2)$ are real matrices satisfying

$$A(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u) \end{pmatrix},$$

and $A(u)$ is symmetric, i.e., $A_{11}(u)$ and $A_{22}(u)$ are symmetric and $A_{21}(u) = {}^T A_{12}(u)$; $B_2(u)$ is a real symmetric positive matrix; $g_1(u, w_x)$ and $g_2(u, u_x)$ are non-linear terms. For system (1.3), we put the following condition.

[A1] The matrix $A_{11}(u)$ is negative and $A(u)$ is non-positive for $u \in \mathcal{O}$.

We prescribe the initial and boundary conditions for (1.2) as

$$u(0, x) = u_0(x) = {}^T(v_0, w_0)(x), \tag{1.4}$$

$$w(t, 0) = w_b, \tag{1.5}$$

where $w_b \in \mathbb{R}^{m_2}$ is a constant. Notice that the problem (1.3)–(1.5) is well-posed since the boundary condition for v is not necessary due to the condition $A_{11}(u) < 0$. We assume that a spatial asymptotic state of the initial data is a constant:

$$\lim_{x\to\infty} u_0(x) = u_+ = {}^T(v_+, w_+), \quad \text{i.e.,} \quad \lim_{x\to\infty} (v_0, w_0)(x) = (v_+, w_+).$$

Related results. For the heat-conductive model of compressible viscous gases in $\mathbb{R}^3$, Matsumura and Nishida in [7] show the asymptotic stability of a constant state (or a stationary solution corresponding to an external potential force) and establish a technical energy method. For the system (1.1) in the full space $\mathbb{R}^n$, Umeda, Kawashima and Shizuta in [14] consider a sufficient condition, introduced in Section 3 as the condition [K], which guarantee a dissipative structure of the system (1.1) and show the asymptotic stability of the constant state. Shizuta and Kawashima in [13] show an equivalence of the condition [K] and the condition [SK] introduced in Section 3.

For a barotropic model of compressible viscous gases in half space, Kawashima, Nishibata and Zhu in [6] consider an outflow problem, where a negative Dirichlet data for the velocity is imposed, and show the existence and the asymptotic stability of boundary layer solutions. The generalization of this problem to a multi-dimensional half space $\mathbb{R}^n_+ = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ is considered by Kagei and Kawashima in [2]. For the heat-conductive model, Kawashima, Nakamura, Nishibata and Zhu [5] prove the existence and the asymptotic stability of boundary layer solutions for the outflow problem. For the inflow problem, the barotropic model is considered in [9] and the heat-conductive model is considered in [1, 11, 12].

The main purpose of the present paper is to show the existence and the asymptotic stability of boundary layer solutions in the half space $\mathbb{R}_+$ which covers the results for the outflow problem [5, 6].

Notations. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}_+)$ denotes a standard Lebesgue space over $\mathbb{R}_+$ equipped with a norm $\|\cdot\|_{L^p}$. For a non-negative integer s, $H^s(\mathbb{R}_+)$ denotes an s-th order Sobolev space over $\mathbb{R}_+$ in the L^2 sense with a norm $\|\cdot\|_{H^s}$. Notice that $H^0(\mathbb{R}_+) = L^2(\mathbb{R}_+)$ and $\|\cdot\|_{H^0} = \|\cdot\|_{L^2}$. For a function $f = f(u)$, $D_u f(u)$ denotes a Fréchet derivative of f with respect to u. Especially, in the case of $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $f(u) = (f_1, \ldots, f_m)(u) \in \mathbb{R}^m$, the Fréchet derivative $D_u f = (\frac{\partial f_i}{\partial u_j})_{ij}$ is an $m \times n$ matrix.

2 Stationary solution

The stationary solution $\tilde{u}(x) = {}^T(\tilde{v}, \tilde{w})(x)$ is defined as a solution to (1.1) independent of t. Thus $\tilde{u} = {}^T(\tilde{v}, \tilde{w})$ satisfies equations

$$f(\tilde{u})_x = (G(\tilde{u})\tilde{u}_x)_x, \quad \text{i.e.,} \quad \left\{ \begin{array}{l} f_1(\tilde{v}, \tilde{w})_x = 0, \\ f_2(\tilde{v}, \tilde{w})_x = (G_2(\tilde{u})\tilde{w}_x)_x, \end{array} \right. \tag{2.1}$$

where $f = {}^T(f_1, f_2)$. The boundary conditions are prescribed as

$$\tilde{w}(0) = w_b, \quad \lim_{x \to \infty} \tilde{u}(x) = u_+. \tag{2.2}$$

Integrating the first equation in (2.1) over (x, ∞), we have

$$f_1(\tilde{v}, \tilde{w}) = f_1(v_+, w_+).$$

We solve this equation with respect to $\tilde{v}$ by using the implicit function theorem. To do this, we assume

[A2] $\det D_v f_1(v_+, w_+) \neq 0$.

Then there exists $V = V(\tilde{w})$ satisfying $f_1(V(\tilde{w}), \tilde{w}) = f_1(v_+, w_+)$ and $V(w_+) = v_+$. Let $\mu_j(w)$ $(j = 1, \ldots, m_2)$ be eigenvalues of the matrix $\tilde{A}(w) := G_2(u_+)^{-1} D_w H(w)$, where $H(w) := f_2(V(w), w)$, and let $r_j(w)$ be corresponding eigenvectors. We assume that the eigenvalues $\mu_j(w)$ are distinct and the first eigenvalue $\mu_1(w_+)$ is non-positive. Namely, We assume

[A3] Eigenvalues of $\tilde{A}(w)$ are distinct, i.e., $\mu_1(w) > \mu_2(w) > \cdots > \mu_{m_2}(w)$.

[A4] $\mu_1(w_+) \le 0$.

Under the above assumptions, we solve the boundary value problem (2.1) and (2.2).

Theorem 2.1. *Assume that* [A2]–[A4] *hold and that* $\delta := |w_b - w_+|$ *is sufficiently small.*

(i) (Non-degenerate case) *For the case of* $\mu_1(w_+) < 0$, *there exists a unique smooth solution* $\tilde{u}(x)$ *to* (2.1) *and* (2.2) *satisfying*

$$|\partial_x^k(\tilde{u}(x) - u_+)| \le Ce^{-cx} \quad \text{for} \quad k = 0, 1, \ldots.$$

(ii) (Degenerate case) *For the case of* $\mu_1(w_+) = 0$, *there exists a certain region* $\mathcal{M} \subset \mathbb{R}^{m_2}$ *such that if* $w_b \in \mathcal{M}$ *and* $D_w\mu_1(w_+) \cdot r_1(w_+) \neq 0$, *then there exists a unique smooth solution* $\tilde{u}(x)$ *satisfying*

$$|\partial_x^k(\tilde{u}(x) - u_+)| \le C\frac{\delta^{k+1}}{(1+\delta x)^{k+1}} + Ce^{-cx} \quad \text{for} \quad k = 0, 1, \ldots.$$

Proof. Integrating the second equation in (2.1) over (x, ∞) and substituting $\tilde{v} = V(\tilde{w})$ in the resultant equation, we have

$$\tilde{w}_x = G_2(\tilde{w})^{-1}(f_2(V(\tilde{w}), \tilde{w}) - f_2(v_+, w_+)) \tag{2.3}$$

$$= \tilde{A}(w_+)(\tilde{w} - w_+) + \frac{1}{2}G_2(w_+)^{-1}D_w^2 H(\tilde{w})(\tilde{w} - w_+)^2 + O(|\tilde{w}|^3). \tag{2.4}$$

The non-degenerate case can be proved easily because the condition $\mu_1(w_+) < 0$ yields that the equilibrium w_+ of the system (2.3) is asymptotically stable. In the degenerate case, we diagonalize the system (2.4) by employing a new unknown function $\tilde{z}(x) = (\tilde{z}_1, \ldots, \tilde{z}_{m_2})(x)$ defined by

$$\tilde{z} := P^{-1}(\tilde{w} - w_+), \quad P := (r_1(w_+), \ldots, r_{m_2}(w_+)).$$

We have the equation for $\tilde{z}$ as

$$\tilde{z}_{1x} = h_1(\tilde{z}),$$
$$\tilde{z}_{kx} = \mu_k(w_+)\tilde{z}_k + h_k(\tilde{z}) \quad \text{for} \quad k = 2, \ldots, m_2,$$

where $h_k(\tilde{z})$ is a nonlinear term. By a straightforward computation, we see that h_1 satisfies

$$h_1(\tilde{z}) = \frac{1}{2}D_w\mu_1(w_+) \cdot r_1(w_+)\tilde{z}_1^2 + O(|\tilde{z}|^3).$$

Therefore, using the fact that $\mu_k(w_+) < 0$ $(k = 2, \ldots, m_2)$ and the center manifold theorem, we obtain the result in Theorem 2.1-(ii). □

3 Stability of stationary solution

In this section, we summarize the stability result of the non-degenerate stationary solution, of which existence is shown in Theorem 1-(i). We also show a brief outline of a proof of a priori estimates. To do this, we have to assume a condition which guarantee a dissipative structure of the system. This kind of dissipative structure was studied mainly by Kawashima in 1980's, and the following condition was imposed in [3, 14].

[K] There exists an $m \times m$ real matrix K such that $KA^0(u_+)$ is skew-symmetric and $[KA(u_+)] + B(u_+)$ is positive definite, where $[A] := (A + {}^T\!A)/2$ is a symmetric part of a matrix A.

Shizuta and Kawashima in [13] prove the equivalence of the condition [K] and the following condition [SK].

[SK] Let $\lambda A^0(u_+)\phi = A(u_+)\phi$ and $B(u_+)\phi = 0$ for $\lambda \in \mathbb{R}$ and $\phi \in \mathbb{R}^m$. Then $\phi = 0$.

Kawashima proved the asymptotic stability of a constant state for the full space problem under the condition [K] (or [SK]) in his doctor thesis [3]. The main purpose of the present paper is to show the asymptotic stability of the boundary layer solution in half space under the condition [SK].

Theorem 3.1. *Let $\tilde{u}(x)$ be a non-degenerate stationary solution shown in Theorem 1-(i). Assume that the condition* [SK] *(or* [K]*) holds. Then there exists a positive constant ε_1 such that if*

$$\|u_0 - \tilde{u}\|_{H^2} + \delta \leq \varepsilon_1,$$

the problem (1.3), (1.4) *and* (1.5) *has a unique solution $u(t,x)$ globally in time satisfying*

$$u - \tilde{u} \in C([0,\infty), H^2(\mathbb{R}_+)).$$

Moreover the solution u converges to the stationary solution $\tilde{u}$:

$$\lim_{t\to\infty} \|u(t) - \tilde{u}\|_{L^\infty} = 0.$$

The crucial point of a proof of Theorem 3.1 is to obtain a uniform a priori estimate of a perturbation from the stationary solution. Let $(\varphi, \psi) := (v, w) - (\tilde{v}, \tilde{w})$ be a perturbation from the stationary solution. Then we have the equation for (φ, ψ) as

$$A_1^0(u)\varphi_t + A_{11}(u)\varphi_x + A_{12}(u)\psi_x = \tilde{g}_1, \tag{3.1a}$$
$$A_2^0(u)\psi_t + A_{21}(u)\varphi_x + A_{22}(u)\psi_x = B_2(u)\psi_{xx} + \tilde{g}_2, \tag{3.1b}$$

where $\tilde{g}_1$ and $\tilde{g}_2$ are non-linear terms. The initial and the boundary conditions are prescribed as

$$(\varphi, \psi)(0, x) = (\varphi_0, \psi_0) := (v_0, w_0) - (\tilde{v}, \tilde{w}), \tag{3.2}$$
$$\psi(t, 0) = 0. \tag{3.3}$$

To summarize the a priori estimate, we define an energy norm $N(t)$ and a dissipative norm $D(t)$ by

$$N(t) := \sup_{0\le\tau\le t} \|(\varphi,\psi)(\tau)\|_{H^2},$$
$$D(t)^2 := \int_0^t \left(\|\varphi_x(\tau)\|_{H^1}^2 + \|\psi_x(\tau)\|_{H^2}^2\right) d\tau.$$

Proposition 3.2. *Let $(\varphi,\psi) \in C([0,T];H^2(\mathbb{R}_+))$ be a solution to (3.1)–(3.3) for a certain $T > 0$. Then there exists a positive constant ε_1 such that if $N(t) + \delta \le \varepsilon_1$, the solution satisfies*

$$\|(\varphi,\psi)(t)\|_{H^2}^2 + \int_0^t \left(\|\varphi_x(\tau)\|_{H^1}^2 + \|\psi_x(\tau)\|_{H^2}^2\right) d\tau \le C\|(\varphi_0,\psi_0)\|_{H^2}^2. \tag{3.4}$$

Proof. the proof of Proposition 3.2 is divided into several steps. In this paper, we only show the brief derivation of estimates for the solution up to first order derivatives. The estimate for the second order estimate can be obtained similarly.

Step 1. Firstly, we obtain a lower order estimate of (φ,ψ):

$$\|(\varphi,\psi)(t)\|_{L^2}^2 + \int_0^t \left(\|\psi_x(\tau)\|_{L^2}^2 + |\varphi(\tau,0)|^2\right) d\tau \le C\|(\varphi_0,\psi_0)\|_{L^2}^2 + C\delta\int_0^t \|\varphi_x(\tau)\|_{L^2}^2\, d\tau. \tag{3.5}$$

To get the estimate (3.5), we employ an energy form $\mathcal{E}$ defined by

$$\mathcal{E} := \eta(f^0(u)) - \eta(f^0(\tilde{u})) - D_z\eta(f^0(\tilde{u}))(f^0(u) - f^0(\tilde{u})).$$

Note that, if $N(t)$ is sufficiently small, the energy form $\mathcal{E}$ is equivalent to $|(\varphi,\psi)|^2$ because the Hessian matrix $D_z^2\eta$ is positive. From a direct computation, we see that $\mathcal{E}$ satisfies

$$\mathcal{E}_t + \mathcal{F}_x + \langle B_2(u)\psi_x, \psi_x\rangle = \mathcal{B}_x + \mathcal{R}, \tag{3.6}$$
$$\mathcal{F} := q(u) - q(\tilde{u}) - D_z\eta(f^0(\tilde{u}))(f(u) - f(\tilde{u})),$$
$$\mathcal{B} := \left(D_z\eta(f^0(u)) - D_z\eta(f^0(\tilde{u}))\right)(G(u)u_x - G(\tilde{u})\tilde{u}_x),$$

where $\mathcal{R}$ is a remainder term satisfying $|\mathcal{R}| \le C|\tilde{u}_x|(|(\varphi,\psi)|^2 + |(\varphi,\psi)||(\varphi_x,\psi_x)|)$. Integrating (3.6) over $(0,T)\times\mathbb{R}_+$ and using the assumption $A_{11} < 0$ in **[A1]**, we get the estimate (3.5). Notice that we also utilize the Poincaré type inequality to control remainder terms $\mathcal{R}$.

Step 2. Next we obtain estimates for first order derivatives φ_x. Namely we get

$$\|\varphi_x(t)\|_{L^2}^2 + \int_0^t \left(\|{}^TA_{12}\varphi_x(\tau)\|_{L^2}^2 + |\varphi_x(\tau,0)|^2\right) d\tau \le C\|(\varphi_0,\psi_0)\|_{H^1}^2 + \int_0^t \left(\varepsilon\|\varphi_x(\tau)\|_{L^2}^2 + C_\varepsilon\|\psi_x(\tau)\|_{L^2}^2\right) d\tau + CN(t)D(t)^2 \tag{3.7}$$

by using Matsumura–Nishida's energy method in half space developed in [8], where ε is an arbitrary positive constant and C_ε is a positive constant depending on ε. Precisely, we compute $\langle \partial_x(3.1\text{a}), \varphi_x \rangle + \langle B_2^{-1}(3.1\text{b}), {}^T A_{12}\varphi_x \rangle$ and integrate the resultant equality to get (3.7).

Step 3. Next we obtain the estimate for ψ_x:

$$\|\psi_x(t)\|_{L^2}^2 + \int_0^t \|\psi_{xx}(\tau)\|_{L^2}^2\, d\tau \\ \le C\|(\varphi_0, \psi_0)\|_{H^1}^2 + C\int_0^t \|{}^T A_{12}\varphi_x(\tau)\|_{L^2}^2\, d\tau + CN(t)D(t)^2,$$

which can be obtained by computing an inner product $\langle (3.1\text{b}), -\psi_{xx} \rangle$.

Step 4. Finally, we obtain the dissipative estimate for φ_x by using the condition [K] as follows:

$$\int_0^t \|\varphi_x(\tau)\|_{L^2}^2\, d\tau \le C\|(\varphi_0, \psi_0)\|_{H^1}^2 + C\|(\varphi_x, \psi_x)(t)\|_{L^2}^2 \\ + C\int_0^t \left(\|\psi_{xx}(\tau)\|_{L^2}^2 + |\varphi_x(\tau, 0)|^2\right) d\tau + CN(t)D(t)^2. \quad (3.8)$$

Combining the estimates from Step 1 to Step 4, we obtain the estimate up to the first derivatives as

$$\|(\varphi, \psi)(t)\|_{H^1}^2 + \int_0^t \left(\|\varphi_x(\tau)\|_{L^2}^2 + \|\psi_x(\tau)\|_{H^1}^2\right) d\tau \le C\|(\varphi_0, \psi_0)\|_{H^1}^2 + CN(t)D(t)^2.$$

By a similar computation, we get the estimate for the second order derivatives. Combining these estimates, we obtain the desired estimate (3.4). □

References

[1] F. Huang, J. Li, and X. Shi, *Asymptotic behavior of solutions to the full compressible Navier-Stokes equations in the half space*, Commun. Math. Sci. **8** (2010), no. 3, 639–654.

[2] Y. Kagei and S. Kawashima, *Stability of planar stationary solutions to the compressible Navier-Stokes equation on the half space*, Comm. Math. Phys. **266** (2006), no. 2, 401–430.

[3] S. Kawashima, *Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics*, Doctoral Thesis, Kyoto Univ. (1984).

[4] S. Kawashima, *Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications*, Proc. Roy. Soc. Edinburgh Sect. A **106** (1987), no. 1-2, 169–194.

[5] S. Kawashima, T. Nakamura, S. Nishibata, and P. Zhu, *Stationary waves to viscous heat-conductive gases in half-space: existence, stability and convergence rate*, Math. Models Methods Appl. Sci. **20** (2010), no. 12, 2201–2235.

[6] S. Kawashima, S. Nishibata, and P. Zhu, *Asymptotic stability of the stationary solution to the compressible Navier-Stokes equations in the half space*, Comm. Math. Phys. **240** (2003), no. 3, 483–500.

[7] A. Matsumura and T. Nishida, *The initial value problem for the equations of motion of viscous and heat-conductive gases*, J. Math. Kyoto Univ. **20** (1980), no. 1, 67–104.

[8] A. Matsumura and T. Nishida, *Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids*, Comm. Math. Phys. **89** (1983), no. 4, 445–464.

[9] A. Matsumura and K. Nishihara, *Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas*, Comm. Math. Phys. **222** (2001), no. 3, 449–474.

[10] T. Nakamura and S. Nishibata, *Existence and asymtotic stability of boundary layer solutions for symmetric hyperbolic-parabolic systems*, preprint.

[11] T. Nakamura and S. Nishibata, *Stationary wave to inflow problem in half line for viscous heat-conductive gas*, to appear in Journal of Hyperbolic Differential Equations.

[12] X. Qin and Y. Wang, *Stability of wave patterns to the inflow problem of full compressible Navier-Stokes equations*, SIAM J. Math. Anal. **41** (2009), no. 5, 2057–2087.

[13] Y. Shizuta and S. Kawashima, *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*, Hokkaido Math. J. **14** (1985), no. 2, 249–275.

[14] T. Umeda, S. Kawashima, and Y. Shizuta, *On the decay of solutions to the linearized equations of electromagnetofluid dynamics*, Japan J. Appl. Math. **1** (1984), no. 2, 435–457.