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THE MARTIN AND ROYDEN COMPACTIFICATIONS OF TÔKI COVERING SURFACES

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1. Introduction

For an open (i.e., noncompact) Riemann surface R a compact Hausdorff space R^* containing R as its dense subspace is said to be a *compactification* of R. To develope the function theory and also the potential theory efficiently on open Riemann surfaces it is innevitable in many instances to have an *ideal boundary* $R^* \setminus R$ of R. To be able to react properly to various situations depending upon the direction of the proposed study on Riemann surfaces we need to select an appropriate compactification R^* of R and usually one of 7 typical compactifications mentioned later is chosen. In selecting some suitable ideal boundary in such an occasion primarily important recognotion is the relation among compactifications of R. We say that two compactifications of R are related or there is a relation between these two compactifications if there is a projection (i.e., a continuous mapping fixing R pointwise) from one of these two compactifications to another, and otherwise, we say these two compactifications are not related or there is no relation between these two compactifications. A complete list of relations among 7 main compactifications of any given open Riemann surface is given in the item with heading "Ideal Boundaries" in the second edition [5] of Iwanami Dictionary of Mathematics edited by the Mathematical Society of Japan (cf. also [6]) and this item, slightly revised but with essentially identical content, is transplanted to the fourth edition [7] of the above dictionary divided into two subitems "Ideal Boundaries" and "Compactifications by Function Families" included in a large item with heading "Analysis on Riemann Surfaces". This dictionary is probably the only place where such a list of relations among 7 compactifications mentioned above can be found. Now we focus on one particular spot in the list: the situation between Martin compactifications and Royden compactifications. The conclusion there is that these two compactifications are not related. Actually the present author was in charge of the above item "Ideal Boundaries" newly prepared at that time for the second edition. He knew the above conclusion and its proof at that time too although the result was not publicized in any form. Nevertheless he dared to include the above conclusion in the manuscript due to his earnest desire to make the list complete. Except for this part relations among 7 compactifications were either trivial or published in the book or paper forms by several authors. He continued to have the intention to publicize the proof of the above conclusion without realizing it by now. The primary purpose of this paper is, thus finally, using this precious opportunity fully to achieve this intention: we will prove that Martin and Royden compactifications are not related.

To make our motivation clearer we explain some general properties of compactifications of Riemann surfaces related to potential theory such as Dirichlet problem, harmonic measures and capacities on the ideal boundaries. Especially the comparison of various potential theoretic notions for two compactifications is very important and gives many applications to function theoretic discussions. Keeping these view points mentioned above in mind we will give a bird's-eye view of the 7 main ideal boundaries frequently used. The Alexandroff, Stoïlow (i.e., Kerékjártó-Stoïlow), and Čech compactifications are of purely topological in nature. The first is just a one point compactification and the smallest among these 7 compactifications. The third is the largest in the sense that other 6 compactifications are some quotient spaces of this. Viewing each ideal boundary component as one point we obtain the Stoïlow compactification. The rest 4 compactifications are of potential theoretic in nature. The Martin (resp. Kuramochi) compactification is merely the metric completion of R with with respect to the metric induced by the relative Green kernel called Martin kernel (resp. the Neumann kernel). These are the most adequate to develope the kernel potential theory. The Royden compactification was given birth to by Royden in 1953. These determine and also are determined by the quasiconformal structures of open Riemann surfaces. The supporting idea of this compactification is the Dirichlet principle. The Wiener compactification is the newest among 7 compactifications mainly developed by Japanese people such as Kusunoki, S. Mori, and K. Hayashi around 1961 on one hand and the term Wiener compactification was coined by Constantinescu-Cornea as well as they contributed in the essential way to clarify important properties of it such as its connection with the Martin compactification on the other hand. Classical function theory is of course discussed on Riemann surfaces which are subregions of the complex sphere $\hat{\mathbb{C}}$. The most natural compactification of such R is the closure $\overline{R}$ of R in $\hat{\mathbb{C}}$, which is occasionally called the Euclidean compactification of R. The relation of each of 7 compactifications of the plane region R to its Euclidean compactification $\overline{R}$ is of course an important theme to investigate in view of the applications of compactification theory to the classical function theory. The most frequently considered is about the relation of the Martin compactification to the Euclidean compactification not only for plane regions R but also for higher dimensional Euclidean space case as well. Since the Martin and Kuramochi compactifications are identical with the Euclidean compactifications for e.g. smooth plane regions R, one often says that these two compactifications are the most prefarable. However, in general, the Martin (resp. Kuramochi) compactifications and the Euclidean ones of plane regions are not related while each of the other 5 are always related to Euclidean one. Nevertheless these charming topics will not be touched even slightly in this paper.

2. Compactifications and ideal boundaries

For a given topological space X we denote by, as usual, $C(X)$ the totality of real valued continuous functions on X, i.e., continuous mappings of X to $\mathbb{R}$, the real number field, and set $C_b(X) := \{f \in C(X) : \sup_X |f| < +\infty\}$. We also denote by $C(X : \widehat{\mathbb{R}})$ the totality of continuous mappings of X to $\widehat{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, the extended real line. Take an open (i.e., noncompact) Riemann surface R. In addition to $C(R)$, $C_b(R)$, and $C(R; \widehat{\mathbb{R}})$, we consider the function space $H(R)$, the linear space of real valued harmonic functions on R, which plays the central role in the sequel. A compact Hausdorff space R^* is said to be a *compactification* of R if R^* contains R as its dense subspace and $\gamma := R^* \setminus R$ is an *ideal boundary* of R relative to the compactification R^*. It is seen that γ is compact, or equivalently, R is open in R^*. Consider two compactifications R_1^* and R_2^* of an open Riemann surface R. We say that R_1^* *lies over* R_2^*, or, R_1^* is *greater than* R_2^*, if the identity mapping $I : R \to R$ can be continued to a unique continuous mapping $I^* : R_1^* \to R_2^*$. In such a case we write as $R_1^* \to R_2^*$ to indicate the fact R_1^* lying over R_2^*. The negation of $R_1^* \to R_2^*$, i.e., R_1^* does not lie over R_2^*, is indicated by $R_1^* \not\to R_2^*$. If either $R_1^* \to R_2^*$ or $R_2^* \to R_1^*$, then we say that R_1^* is *related* to R_2^*, or equivalently, there is a *relation* between R_1^* and R_2^*.

It is not always the case that if the boundary of a subregion S of a compactification R^* relative to R^* is contained in R, then $S \setminus \gamma$ is still connected so that $S \setminus \gamma$ is still a subregion of R^*. But if this is always the case for a compactification R^* for every choice of subregion S in R^*, then the compactification R^* is said to be of *Stoïlow type.* This naming comes from the fact that the Stoïlow compactification of an open Riemann surface R explained later is the smallest compactification among those of R of Stoïlow type and every compactification of R lying over the Stoïlow compactification of R is of Stoïlow type. A *potential* p on an open Riemann surface R is a nonnegative superharmonic function on R, the graetest harmonic minorant of which is zero on R. The subset δ of γ consisting of points ζ in γ satisfying

$$\liminf_{z \in R, z \to \zeta} p(z) = 0 \tag{2.1}$$

for every potential p on R is referred to as the *harmonic boundary* of R relative to the compactification R^* of R. The subset δ of γ is a compact subset of R^*.

For any compact subset K of $\gamma \setminus \delta$ we can find a potential p_K on R such that $p_K(R) \subset \mathbb{R}^+ := \{t \in \mathbb{R} : t \geqq 0\}$ and

$$\lim_{z \in R, z \to \zeta} p_K(z) = +\infty \tag{2.2}$$

for every $\zeta \in K$. In the case $\delta \neq \emptyset$, whether p_K can be chosen in $H(R)$ is not only important but also quite interesting problem (cf. [11]). This function p_K is an important auxiliary function in studying various properties concerning the harmonic boundary δ, i.e., p_K is conveniently used to show many important roles played by δ. As an illustration of its use we prove the maximum principle in its most simple form: let $u \in H(R)$ be bounded from above on R and suppose there is a constant $C \in \mathbb{R}$ such that $\limsup_{z \in R, z \to \zeta} u(z) \leqq C$ for every $\zeta \in \delta$, then $u \leqq C$ on R. In fact, for any number $\varepsilon > 0$ the compactness of δ assures the existence of a compact subset K of γ such that $\limsup_{z \in R, z \to \zeta} u(z) \leqq C + \varepsilon$ for every $\zeta \in \gamma \setminus K$. Then pick the above auxiliary function p_K and consider the subharmonic function $s_n := u - (C + \varepsilon) - (1/n)p_K$ on R for every fixed $n \in \mathbb{N}$, the set of positive integers, which is bounded from above along with u. Observe that $\limsup_{z \in R, z \to \zeta} s_n(z) \leqq 0$ for every $\zeta \in \gamma$, which assures that $s_n \leqq 0$ on R by the usual maximum principle, i.e., $u(z) \leqq C + \varepsilon + (1/n)p_K(z)$ for every $z \in R$, $\varepsilon > 0$, and $n \in \mathbb{N}$. On letting $\varepsilon \downarrow 0$ and then $n \uparrow +\infty$, we deduce $u \leqq C$ on R.

An open Riemann surface R is referred to as being hyperbolic (resp. parabolic) if R carries (resp. does not carry) the Green kernel on R and we denote by $\mathcal{O}_G$ the class of parabolic Riemann surfaces R. Then suppose $R \notin \mathcal{O}_G$ and consider real valued function φ on γ. We can formulate the usual PWB (i.e., Perron-Wiener-Brelot) procedure to solve the Dirichlet problem on R for the boundary $\gamma = R^* \setminus R$. If the PWB solution H_φ^R is found for a boundary data φ on γ, then as usual φ is said to be resolutive. It can happen that every $\varphi \in C(\gamma)$ is resolutive and in such a case the compactification R^* of R is said to be *resolutive.* In this case a point $\zeta \in \gamma$ is said to be *regular* if $\lim_{z \in R, z \to \zeta} H_\varphi^R(z) = \varphi(\zeta)$ holds for every $\varphi \in C(\gamma)$. It is seen that the set of regular points in γ is contained in the harmonic boundary δ and actually these two sets coincide with each other when considered in the Wiener and Royden compactifications mentioned later. The typical example of resolutive compactification is, as mentioned above, the Wiener compactification and, moreover, the Wiener compactification is the largest compactification among every resolutive compactifications and a compactification R^* is resolutive if and only if the Wiener compactification lies over R^*.

Suppose $R \notin \mathcal{O}_G$ and fix a point $a \in R$ and a compactification R^* of R. For a compact subset $K \subset \gamma = R^* \setminus R$ we now define the *harmonic measure* $\mathrm{hm}(K) = \mathrm{hm}_{R^*}(K)$ evaluated at the reference point $a \in R$ as follows. Let S be the family of nonnegative superharmonic functions s on R such that there exists an open neighborhood U_s of K in R^* with $s|U_s \cap R \geqq 1$. Then we set

$$\text{(2.3)} \qquad \mathrm{hm}(K) = \mathrm{hm}_a(K) = \mathrm{hm}_{R^*,a}(K) := \inf_{s\in S} s(a),$$

which is the harmonic measure of $K \subset \gamma$ inducing a regular Borel measure hm on γ called the harmonic measure on γ. In case R is resolutive, the functional $\varphi \to H^R_\varphi$ is linear and continuous on $C(\gamma)$ so that it is a regular Borel measure on γ, which is seen to be the above harmonic measure hm on γ:

$$\text{(2.4)} \qquad H^R_\varphi(a) = \int_\gamma \varphi d\mathrm{hm}$$

for every $\varphi \in C(\gamma)$. Now consider two compactifications R^*_j $(j = 1, 2)$ of a general hyperbolic Riemann surface R. Suppose R^*_1 lies over R^*_2 and let $\pi : R^*_1 \to R^*_2$ be the projection, i.e., π is the continuous mapping of R^*_1 onto R^*_2 such that $\pi|R = id.$ (i.e., the identity mapping of R onto itself). Then we obtain the following implication:

$$\text{(2.5)} \qquad \mathrm{hm}_{R^*_1}(K) > 0 \Longrightarrow \mathrm{hm}_{R^*_2}(\pi(K)) > 0$$

for every compact subset $K \subset \gamma_1 := R^*_1 \setminus R$, or equivalently,

$$\text{(2.6)} \qquad \mathrm{hm}_{R^*_2}(K) = 0 \Longrightarrow \mathrm{hm}_{R^*_1}(\pi^{-1}(K)) = 0$$

for every compact subset $K \subset \gamma_2 := R^*_2 \setminus R$. Besides the measurement the harmonic measure $\mathrm{hm}(K)$ of a compact subset $K \subset \gamma$ we also have another measurement the capacity $\mathrm{cap}(K)$ for K. This time we take a closed parametric disc $\overline{V}$ of any open Riemann surface R and consider a compactification R^* of R and a compact subset $K \subset \gamma := R^* \setminus R$. Recall tht a Dirichlet function f on R is a real valued function belonging to the local Sobolev space $W^{1,2}_{\mathrm{loc}}(R)$ such that the Dirichlet integral $D(f; R) := \int_R df \wedge *df$ of f taken over R is finite and the Dirichlet space $L^{1,2}(R)$ is the totality of Dirichlet functions on R. Let F be a family of continuous Dirichlet functions $f \in L^{1,2}(R) \cap C(R)$ such that there is an open neighborhood U_f of K in R^* with $f|U_f \cap R \geqq 1$ and $f|\overline{V} = 0$. Then we call

$$\text{(2.7)} \qquad \mathrm{cap}(K) = \mathrm{cap}_{\overline{V}}(K) = \mathrm{cap}_{R^*,\overline{V}}(K) := \inf_{f\in F} D(f; R)$$

the *capacity* (or more precisely *variational* 2-*capacity*) of K. We also have the counter pasrts of (2.5) and (2.6) for capacities as well. There is a constant $\kappa \in [1, +\infty)$ such that

$$\text{(2.8)} \qquad \mathrm{hm}(K) \leqq \kappa \cdot \mathrm{cap}(K)$$

for every compact subset $K \subset \gamma = R^* \setminus R$. However we will not use the capacity in this paper for our later purposes.

3. Q compactifications

Take an arbitrary open Riemann surface R and consider any subclass $Q = Q(R)$ of $C(R)$. We denote by R^*_Q the unique compactification of R satisfying the following two more additional conditions: first, each $f \in Q$ is continued to R^*_Q so as to be a $\widehat{\mathbb{R}}$ valued continuous function on R^*_Q which is denoted also by the same letter f; second, Q separates points in the ideal boundary $\gamma_Q := R^*_Q \setminus R$ of R relative to R^*_Q, i.e., for any two distinct points ζ_1 and ζ_2 in γ_Q there is an $f \in Q$ such that $f(\zeta_1) \neq f(\zeta_2)$. This compactification R^*_Q of R is referred to as the Q *compactification* of R and γ_Q the Q *ideal boundary*, or rather simply, Q *boundary* of R. These Q compactifications are nothing special in the class of compactifications as being compactifications. Actually any compactification R^* is a Q compactification for a suitable family Q. In fact, we only have to choose as a Q the set $Q = \{f|R : f \in C(R^*; \widehat{\mathbb{R}}), f(R) \subset \mathbb{R}\}$ so as to have $R^* = R^*_Q$. Let $\{Q\}$ be the family of every subset $Q \subset C(R)$ and $\{R^*\}$ be the family of every compactification R^* of R. Then the mapping $\psi : \{Q\} \to \{R^*\}$ defined by $\psi(Q) = R^*_Q$ is surjective as we saw above but not injective at all and actually $\psi^{-1}(R^*_Q)$ contains infinitely many subfamilies of $C(R)$. In fact, let $C_0^\infty(R)$ be as usual the family of every C^∞ function on R with compact support in R. Then clearly $\psi(Q \cup C_0^\infty) = \psi(Q)$, which implies the above statement. Therefore it is always essentially important to choose "nice subfamily" Q to investigate a given compactification R^* of R from the view point of the Q compactification $R^* = R^*_Q$. If Q forms some normed ring (i.e., a commutative real Banach algebra with multiplicative unit), then R^*_Q can be grasped as the maximal ideal space of Q and various applications of the corpus of abundant knowledges in that theory of functional analysis are well expected. Thus the choice of Q with algebra structure is a candidate of "nice families". The notion of Q compactification was introduced by Constantinescu-Cornea in their celebrated monograph [2]. As for the existence of R^*_Q for any given Q, there can be many constructions but the one proposed by Constantinescu-Cornea may be the simplest. We set $I_f = \widehat{\mathbb{R}}$ for every $f \in Q$ and consider the topological product

$$\prod_{f \in Q \cup C_0^\infty} I_f, \tag{3.1}$$

which is a compact Hausdorff space due to the Tychonoff theorem since $I_f = \widehat{\mathbb{R}} = [-\infty, +\infty]$ is compact. The mapping ϕ of R to the above product space (3.1) given by

$$\phi(\zeta) := \prod_{f \in Q \cup C_0^\infty} f(\zeta) \tag{3.2}$$

for every $\zeta \in R$. Then it is seen that $R^*_Q := \overline{\phi(R)}$, the closure of the set $\phi(R)$ in the above space (3.1) is the required Q compactification of R. By specifying Q we will obtain the following 7 typical Q compactifications of any given open Riemann surface R. They are:

$$
(3.3) \qquad \left\{
\begin{array}{ll}
\text{Čech compactification} & R^*_{\mathcal{C}} \\
\text{Wiener compactification} & R^*_{\mathcal{W}} \\
\text{Royden compactification} & R^*_{\mathcal{R}} \\
\text{Martin compactification} & R^*_{\mathcal{M}} \\
\text{Kuramochi compactification} & R^*_{\mathcal{K}} \\
\text{Stoïlow compactification} & R^*_{\mathcal{S}} \\
\text{Alexandroff compactification} & R^*_{\mathcal{A}}
\end{array}
\right.
$$

We will explain the above 7 compactifications one by one about their definitions, fields for which they are conveniently or efficiently made use of, important or well known results obtained due to the use of them, and the like.

3.1. The Čech compactification. Set $\mathcal{C} = \mathcal{C}(R) = C(R)$ so that $\mathcal{C}$ is the largest subfamily of $C(R)$ which is nothing but the total family $C(R)$ itself. We call each function $f \in \mathcal{C}$ a *Čech function* on R so that f is a Čech function on R if and only if f is continuous on R. Then the $\mathcal{C}$ compactification (i.e., Q compactification with Q=$\mathcal{C}$) $R^*_{\mathcal{C}}$ is called as the *Čech compactification* of R. The Čech compactification $R^*_{\mathcal{C}}$ of any R is always of Stoïlow type but $R^*_{\mathcal{C}}$ of any $R \notin \mathcal{O}_G$ is not resolutive. This latter fact is explained to come from that the *Čech boundary* $\gamma_{\mathcal{C}} := R^*_{\mathcal{C}} \setminus R$ contains too many points, which suffocate the full developement of the potential theory. It is very convenient that any continuous function on R is automatically continuous on $R^*_{\mathcal{C}}$ and especially at each point of the Čech boundary $\gamma_{\mathcal{C}}$. Full use of this convenient fact produced a very simple proof for the following Riemann surface version of the Evans-Selberg theorem: an open Riemann surface R is parabolic, i.e., $R \in \mathcal{O}_G$, if and only if there exists an Evans-Selberg potential $E(z,\zeta)$ on R, i.e., $E(\cdot,\zeta) \in H(R \setminus \{\zeta\})$; $E(z,\zeta) - \log|z-\zeta| = \mathcal{O}(1)$ $(z \to \zeta)$; for any $n \in \mathbb{N}$ there is a compact subset K_n of R such that $E(\cdot,\zeta) > n$ on $R \setminus K_n$ (see [12]). The following characterization of $\gamma_{\mathcal{C}}$ in $R^*_{\mathcal{C}}$ is useful: a point $\zeta \in R^*_{\mathcal{C}}$ belongs to R (resp. $\gamma_{\mathcal{C}}$) if and only if the first countability axiom is valid (resp. does not valid) at ζ. In particular the Čech compactification $R^*_{\mathcal{C}}$ of any open Riemann surface is not metrizable.

3.2. The Wiener compactification. We introduce the function family $\mathcal{W} = \mathcal{W}(R) \subset C(R)$ consisting of functions f on R called *Wiener functions* on R. To begin with, if an open Riemann surface $R \in \mathcal{O}_G$, then we set $\mathcal{W}(R) = C(R)$. Next suppose that $R \notin \mathcal{O}_G$. Then a function $f \in \mathcal{W}(R)$ if and only if a continuous function f on R satsfies the following two conditions: first, $|f|$ has a superharmonic majorant s_f on R, i.e. there is a nonnegative superharmonic function s_f on R such that $|f| \leqq s_f$ on R; second, there exists a unique $h_f^R \in H(R)$ such that

$$(3.2.1) \qquad h_f^R = \lim_{n\to\infty} H_f^{R_n}$$

locally uniformly on R for every regular exhaustion $(R_n)_{n\in\mathbb{N}}$ of R, where $H_f^{R_n}$ indicates the harmonic function on R_n with boundary values $f|\partial R_n$ on ∂R_n. The above second property is said to be the harmonizability of f on R, so that, the second condition may be rephrased as that f is *harmonizable* on R. A function $f \in \mathcal{W}(R)$ is said to be a *Wiener potential* on R if $h_f^R \equiv 0$ on R. The fact $h_f^R = 0$ on R is equivalent to the existence of genuine potential p_f on R such that $|f| \leqq p_f$ on R. We denote by $\mathcal{W}_0(R)$ the subset of $\mathcal{W}(R)$ consisting of Wiener potentials on R so that $W_0(R) = \{f \in \mathcal{W}(R) : h_f^R = 0\} = \{f \in \mathcal{W}(R) : |f| \leqq p_f : \text{a potential on } R\}$. Then the following direct sum decomposition is a generalized *Riesz decomposition theorem* with $HP(R)$ the linear subspace generated by $H(R)^+ = \{u \in H(R) : u \geqq 0 \text{ on } R\}$ (i.e., $HP(R) = H(R)^+ \ominus H(R)^+$):

$$(3.2.2) \qquad \mathcal{W}(R) = HP(R) \oplus \mathcal{W}_0(R).$$

Then the $\mathcal{W}$ compactification $R^*_{\mathcal{W}}$ is referred to as the *Wiener compactification* of R. By the very definition of families of $\mathcal{W}$ and $\mathcal{C}$, $R^*_{\mathcal{W}} = R^*_{\mathcal{C}}$ as topological spaces if $R \in \mathcal{O}_G$. For hyperbolic Riemann surface R, $R^*_{\mathcal{W}}$ is not only of Stoïlow type but also resolutive. Actually $R^*_{\mathcal{W}}$ is the largest resolutive compactification of R, i.e., if $R^*_{\mathcal{W}} \to R^*$, then the compactification R^* of R is resolutive. Thus the potential theory can be sufficiently and freely discussed on any compactification R^* of R over which $R^*_{\mathcal{W}}$ lies. Similarly to the case of $R^*_{\mathcal{C}}$, a point $\zeta \in R^*_{\mathcal{W}}$ belongs to the *Wiener boundary* $\gamma_{\mathcal{W}} := R^*_{\mathcal{W}} \setminus R$ if and only if $\{\zeta\}$ is a G_δ set, i.e., $\{\zeta\}$ is the intersection of countable number of open neighborhoods of ζ, or equivalently the first countability axiom is valid at ζ. Hence in particular the Wiener compactification $R^*_{\mathcal{W}}$ of R is not metrizable. The *Wiener harmonic boundary* $\delta = \delta_{\mathcal{W}}$ is seen to be identical with

$$(3.2.3) \qquad \delta_{\mathcal{W}} = \{\zeta \in \gamma_{\mathcal{W}} : f(\zeta) = 0 \text{ for every } f \in \mathcal{W}_0(R)\}.$$

From this, it follows that $\delta_{\mathcal{W}}$ is identical with the set of regular points in $\gamma_{\mathcal{W}}$. It is also seen that $R \in \mathcal{O}_G$ if and only if $\delta_{\mathcal{W}} = \emptyset$. This compactification is particularly convenient to study the class $HB(R) := \{u \in H(R) : \sup_R |u| < +\infty\}$ of bounded harmonic functions on R in the frame of the classification theory of Riemann surfaces. In the remarkable study of harmonic and holomorphic Hardy spaces of exponent $1 \leqq p \leqq \infty$ on Riemann surfaces in the UCLA thesis [3] of M. Hayashi, we can find a powerful role played by the Wiener compactifications. The basic idea of this compactification $R^*_{\mathcal{W}}$ comes from the Wiener procedure to solve the Dirichlet problem and also the harmonic version of normal family argument.

3.3. The Royden compactification. We set $\mathcal{R} = \mathcal{R}(R) := L^{1,2}(R) \cap C(R)$ and let us call each $f \in \mathcal{R}$ a *Royden function* on R. The Royden-Brelot decomposition theorem says that if $R \in \mathcal{O}_G$, then $\mathcal{R}(R) = \overline{C_0^\infty(R)}$ in the sense that for every $f \in \mathcal{R}(R)$ there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_0^\infty(R)$ such that $(f_n)_{n\in\mathbb{N}}$ is locally bounded on R, $f_n \to f$ *a.e.* in every parametric disc on R, and $D(f - f_n; R) \to 0$ $(n \to \infty)$; if $R \notin \mathcal{O}_G$, then $\mathcal{R}(R) \subset \mathcal{W}(R)$ and if we set $\mathcal{R}_0(R) := \mathcal{R}(R) \cap \mathcal{W}_0(R)$, then the Royden-Brelot decomposition is

$$\mathcal{R}(R) = HD(R) \oplus \mathcal{R}_0, \tag{3.3.1}$$

where $HD(R) := \{u \in H(R) : D(u; R) < \infty\}$. Actually (3.3.1) is a special case of (3.2.2) so that if $f \in \mathcal{R}(R)$ is decomposed as $f = h_f + g_f$ in (3.2.2), then h_f originally in $HP(R)$ belongs to $HD(R)$ if $f \in \mathcal{R}(R)$ and $g_f \in \mathcal{R}_0 := \mathcal{R}(R) \cap \mathcal{W}_0(R)$. In particular we have the *Dirichlet principle*

$$D(f; R) = D(h_f; R) + D(g_f; R), \quad D(h_f, g_f; R) := \int_R dh_f \wedge *dg_f = 0. \tag{3.3.2}$$

Similar to $\mathcal{W}_0(R)$, we have $\mathcal{R}_0(R) = \{f \in \mathcal{R}(R) : h_f = 0\} = \{f \in \mathcal{R}(R) : |f| \leqq p_f :$ a potential on $R\}$. Each function $f \in \mathcal{R}_0(R)$ is referred to as a *Royden potential*, or rather in more frequently used term *Dirichlet potential* on R. Moreover the following characterization is extremely important:

$$\mathcal{R}_0(R) = \overline{C_0^\infty(R)} \tag{3.3.3}$$

in the sense that each function $f \in \mathcal{R}_0(R)$ is the limit of a locally bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $C_0^\infty(R)$ in two ways: $f_n \to f$ *a.e.* in each parametric disc of R; $D(f - f_n; R) \to 0$. The $\mathcal{R}$ conpactification $R^*_{\mathcal{R}}$ is referred to as the *Royden compactification* of R and $\gamma_{\mathcal{R}} := R^*_{\mathcal{R}} \setminus R$ the *Royden boundary* (Royden ideal boundary) of R. The *Royden harmonic boundary* $\delta_{\mathcal{R}}$ is seen to be identical with

$$\delta_{\mathcal{R}} = \{\zeta \in \gamma_{\mathcal{R}} : f(\zeta) = 0 \text{ for every } f \in \mathcal{W}_0(R)\}. \tag{3.3.4}$$

From this it follows that $\delta_{\mathcal{R}}$ is identical with the set of regular points in $\gamma_{\mathcal{R}}$ as was so in the case of Wiener harmonic boundary. Similarly it is seen that $R \in \mathcal{O}_G$ if and only if $\delta_{\mathcal{R}} = \emptyset$. The Royden compactification $R^*_{\mathcal{R}}$ of any open Riemann surface R is of Stoïlow type and the Royden compactification $R^*_{\mathcal{R}}$ of hyperbolic Riemann surface R is resolutive as in Wiener case. Similar to $R^*_{\mathcal{C}}$ and $R^*_{\mathcal{W}}$, a point ζ in $R^*_{\mathcal{R}}$ belongs to the Royden boundary $\gamma_{\mathcal{R}}$ if and only if the first countability axiom is valid at ζ. In particular like $R^*_{\mathcal{C}}$ and $R^*_{\mathcal{W}}$, the Royden compactification $R^*_{\mathcal{R}}$ is not metrizable.

This compactification $R^*_{\mathcal{R}}$ is conveniently used in the classification problems related to the class $HD(R)$. Especially quasiconformal mappings behave nicely on $R^*_{\mathcal{R}}$ and as a result the quasiconformal invariance of certain null classes of Riemann surfaces can be very easily derived: the class $\mathcal{O}_G$ and $\mathcal{O}_{HD}$ are quasiconformally invariant, i.e., if there is a quasiconformal mapping of a Riemann surface R_1 onto another R_2, then $R_1 \in \mathcal{O}_G$ (resp. $\mathcal{O}_{HD}$) if and only if $R_2 \in \mathcal{O}_G$ (resp. $\mathcal{O}_{HD}$), where $\mathcal{O}_{HD}$ is the family of Riemann surfaces R such that $HD(R) = \mathbb{R}$. These comes from the following conceptually important and also a beautiful result [8] (though its proof is very simple and easy technically): two Riemann surfaces are quasiconformally equivalent (i.e., there is a quasiconformal mapping from one Riemann surface onto another Riemann surface) if and only if their Royden compactifications are homeomorphic.

3.4. The Martin compactification. Take any regular subregion R_0 of an open Riemann surface R. Then the relative boundary $\partial(R \setminus R_0)$ of the open subset $R \setminus \overline{R}_0$ of R is identical with ∂R_0 as sets. First take an $f \in C(\partial R_0)^+$, the class of nonnegative functions in $C(\partial R_0)$, and set $H_f^{R\setminus\overline{R}_0} := \inf u$, where the infimum is taken with respect to $u \in H(R \setminus \overline{R}_0) \cap C(R \setminus R_0)$ such that $u \geqq 0$ on $R \setminus \overline{R}_0$ and $u|\partial R_0 = f$. Then for any $f \in C(\partial R_0)$ we define $H_f^{R\setminus\overline{R}_0} := H_{f^+}^{R\setminus\overline{R}_0} - H_{f^-}^{R\setminus\overline{R}_0}$ where $f = f^+ - f^-$ with f^+ (rsp.f^-) the positive (resp. negative) part of f. Then $H_f^{R\setminus\overline{R}_0}$ is the standard solution of the Dirichlet problem for the "outer region" $R \setminus \overline{R}_0$ with the boundary data f on ∂R_0. We call an $f \in C(R)$ a *Martin function* on R if there exists a regular subregion R_f of R such that

$$f = H_f^{R\setminus\overline{R}_f} / H_1^{R\setminus\overline{R}_f} \tag{3.4.1}$$

on $R \setminus \overline{R}_f$. We denote by $\mathcal{M}(R)$ the totality of Martin functions on R. The *Martin compactification* $R^*_{\mathcal{M}}$ of R is defined as the $\mathcal{M}$ compactification of R and $\gamma_{\mathcal{M}} := R^*_{\mathcal{M}} \setminus R$ is the *Martin boundary* of R. The Martin compactification $R^*_{\mathcal{M}}$ is resolutive and of Stoïlow type. When $R \in \mathcal{O}_G$, it is usual to fix a closed parametric disc $\overline{V} = \{|z| \leqq 1\}$ in R whose interior is the open unit disc $V = \{|z| < 1\}$ to replace R by $R \setminus \overline{V} \notin \mathcal{O}_G$. Then it is seen that

$$(R \setminus \overline{V})^*_{\mathcal{M}} = R^*_{\mathcal{M}} \setminus V \tag{3.4.2}$$

so that the Martin boundary $\gamma_{\mathcal{M}}(R \setminus \overline{V})$ of $R \setminus \overline{V}$ consists of the union of the Martin boundary $\gamma_{\mathcal{M}}(R)$ of R and the circle ∂V which is the border of the partly bordered surface $R \setminus V$, i.e.,

$$\gamma_{\mathcal{M}}(R \setminus \overline{V}) = \gamma_{\mathcal{M}}(R) \cup \partial V. \tag{3.4.3}$$

In view of these two relations (3.4.2) and (3.4.3) above, we can restrict ourselves to the case $R \notin \mathcal{O}_G$ in studying many properties of the Martin compactification

$R^*_{\mathcal{M}}$ of R. When $R \notin O_G$, we have the Green kernel $G(z,w) = G(z,w;R)$ on R. Fix a point $a \in R$ and consider the *Martin kernel* on R as a function on $R \times R$ given by

$$K(z,w) = G(z,w)/G(a,w). \tag{3.4.4}$$

Fix a countable dense subset $\{z_n : n \in \mathbb{N}\}$ in R and consider the metric d on R given by

$$d(w_1,w_2) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} \cdot \frac{|K(z_n,w_1) - K(z_n,w_2)|}{1 + |K(z_n,w_1) - K(z_n,w_2)|} \tag{3.4.5}$$

for any couple (w_1,w_2) of points w_1 and w_2 in R. Then (R,d) is a metric space homeomorphic to R with the original Riemann surface topology of R and the completion of this metric space is seen to be homeomorphic to $R^*_{\mathcal{M}}$. Then the Martin kernel $K(z,\zeta)$ is extended to $(z,\zeta) \in R \times R^*_{\mathcal{M}}$ continuously and the metric $d(\zeta_1,\zeta_2)$ to $(\zeta_1,\zeta_2) \in R^*_{\mathcal{M}} \times R^*_{\mathcal{M}}$ in a natural fashion and (3.4.5) is still valid even for any couple (w_1,w_2) of points w_1 and w_2 in $R^*_{\mathcal{M}}$. Thus we see that the Martin compactification $R^*_{\mathcal{M}}$ is metrizable. For the extension of the potential and the function theory on the unit disc $\mathbb{D}$ to general hyperbolic Riemann surface R, the Martin compactification $R^*_{\mathcal{M}}$ is considered to be the most adequate since $\mathbb{D}^*_{\mathcal{M}} = \overline{\mathbb{D}}$, the closure in the complex plane $\widehat{\mathbb{C}}$ or the Euclidean compactification so to speak. We will give two examples testifying the above statement, especially one for later use. The first is the Poisson representation. We denote by E the set of points $\zeta \in \gamma_{\mathcal{M}}$ such that the Martin kernel $K(\cdot,\zeta)$ is an extreme point of the convex set $\{u \in H(R)^+ : u(a) = 1\}$. Then E is a G_δ subset of $\gamma_{\mathcal{M}}$ and there is a bijective correspondence $u \leftrightarrow \mu$ between the space $\{u \in H(R)^+ : u(a) = 1\}$ and the space $M(E)^+$ of probability measures μ on E given by

$$u = \int_E K(\cdot,\zeta)d\mu(\zeta). \tag{3.4.6}$$

This is the Martin representation theorem of positive harmonic functions which is the generalization of the Poisson representation on $\mathbb{D}$, to obtain which was the motivation for Martin to introduce the compactification now bearing his name. The second is the Fatou theorem and the F. and M. Riesz theorem concerning the boundary behaviors of bounded holomorphic functions on $\mathbb{D}$. Let f be a bounded holomorphic function on a hyperbolic Riemann surface R. Using the fine limit $\hat{f}(\zeta)$ of f at $\zeta \in \gamma_{\mathcal{M}}$ and the harmonic measure $\mathrm{hm}_{\mathcal{M}}$ on the Martin boundary $\gamma_{\mathcal{M}}$ of R, the following results are stated:

THE FATOU THEOREM: *For any bounded holomorphic function f on R, the fine* *limit* $\hat{f}$ *exists* $\mathrm{hm}_{\mathcal{M}}$ *a.e. on* $\gamma_{\mathcal{M}}$;

THE F. AND M. RIESZ THEOREM: *For any bounded holomorphic function* f *on* R, *the condition* $\mathrm{hm}_{\mathcal{M}}(\{\zeta \in \gamma_{\mathcal{M}} : \hat{f}(\zeta) = c\}) > 0$ *for a constant* c *implies that* $f \equiv c$ *on* R.

Many authors like Naïm, Kuramochi, Doob, and Constantinescu-Cornea published results from which the above results follow. As the compactification $R^*_{\mathcal{M}}$ itself the following result of Constantinescu-Cornea [2] is very important (cf. §4 below):

$$\mathcal{M}(R) \subset \mathcal{W}(R) \tag{3.4.7}$$

for any open Riemann surface R, and, actually, every continuous function on the Martin compactification $R^*_{\mathcal{M}}$ of R is a Wienèr function on R (cf. §4 below).

3.5. The Kuramochi compactification. Let R_0 be a regular subregion of an open Riemann surface R. For any $f \in L^{1,2}(R) \cap C(R)$ we consider the class F of functions $u \in L^{1,2}(R \setminus \overline{R}_0) \cap C(R \setminus R_0)$ such that $u|\partial R_0 = f$ and $u \in H(R \setminus \overline{R}_0)$. Then the extremal problem $\inf_{u \in F} D(u; R \setminus \overline{R}_0)$ can be solved by a unique function $f^D = f^D_{R \setminus \overline{R}_0} \in F$ such that

$$\inf_{u \in F} D(u; R \setminus \overline{R}_0) = D(f^D_{R \setminus \overline{R}_0}; R \setminus \overline{R}_0). \tag{3.5.1}$$

We say that a function $f \in L^{1,2}(R) \cap C(R)$ is a *Kuramochi function* on R if there exists a regular subregion R_f of R such that $f = f^D_{R \setminus \overline{R}_f}$ on $R \setminus \overline{R}_f$. We denote by $\mathcal{K}(R)$ the totality of Kuramochi functions on R. The $\mathcal{K}$ compactification $R^*_{\mathcal{K}}$ is referred to as the *Kuramochi compactification* of R. The set $\gamma_{\mathcal{K}} := R^*_{\mathcal{K}} \setminus R$ is the *Kuramochi boundary* of R. We fix a closed parametric disc $\overline{V} = \{|z| \leqq 1\}$ in R whose interior is a parametric disc $V = \{|z| < 1\}$. Then consider the kernel function $N(z, w)$ on $R \setminus \overline{V}$ determined as follows. First,

$$-\Delta N(\cdot, w) = \mathrm{Dirac}_w, \tag{3.5.2}$$

where Dirac_w is the Dirac measure supported at $w \in R \setminus \overline{V}$, and $N(\cdot, w)|\partial V = 0$. Second, for any regular subregion R_0 containing $\overline{V} \cup \{w\}$, $(N(\cdot, w))^D_{R \setminus \overline{R}_0} = N(\cdot, w)$ on $R \setminus \overline{R}_0$. Usually $N(z, w)$ is referred to as the (relative)*Neumann kernel* on $R \setminus \overline{R}_0$, or *Kuramochi kernel* on $R \setminus \overline{R}_0$. As far as formality concerns, the Kuramochi compactifications resembles the Martin compactification and these two compactifications share most basic properties. For example, $R^*_{\mathcal{K}}$ is resolutive and of Stoïlow type like $R^*_{\mathcal{M}}$. Using Kuramochi kernel instead of Martin kernel, a similar argument used for the Martin comppactification assures that the Kuramochi compactifications is metrizable. Also the completion of R with the metric given by the Kuramochi kernel instead of Martin kernel produces the Kuramochi compactification of R exactly similaly as in the case of Martin compactification. Like $\mathbb{D}^*_{\mathcal{M}} = \overline{\mathbb{D}}$, we also have $\mathbb{D}^*_{\mathcal{K}} = \overline{\mathbb{D}}$. The fact that the Beurling theorem related to the boundary behavior of Dirichlet finite holomorphic functions on $\mathbb{D}$ concerning $\partial\mathbb{D}$ is generalized to those on hyperbolic Riemann surface R concerning $\gamma_{\mathcal{K}}$ by Kuramochi is an example of adequateness of $R^*_{\mathcal{K}}$ for the study of Dirichlet finite holomorphic and harmonic functions on hyperbolic Riemann surfaces R.

3.6. The Stoïlow compactification. Let R_0 be a regular subregion of an open Riemann surface R. Then $R \setminus \overline{R}_0$ consists of a finite number of relatively noncompact components. A function $f \in C(R)$ is a *Stoïlow function*, by definition, if ther exists a regular subregion R_f of R such that, when $R \setminus \overline{R}_f$ consists of some finite number m of relatively noncompact components $S_1, S_2, \cdots, S_m$, i.e., $R \setminus \overline{R_f} = \cup_{1 \leqq j \leqq m} S_j$, then there exists m constants $c_1, c_2, \cdots, c_m$ with $f|S_j = c_j$ $(1 \leqq j \leqq m)$. We denote by $\mathcal{S}(R)$ the totality of Stoïlow functions on R. Then the $\mathcal{S}$ compactification $R^*_{\mathcal{S}}$ is referred to as the *Stoïlow compactification* of R and $\mathcal{S}$ ideal boundary $\gamma_{\mathcal{S}} := R^*_{\mathcal{S}} \setminus R$ of R the *Stoïlow boundary* of R. Occasionally the Stoïlow compactification $R^*_{\mathcal{S}}$ is also called the Kerékjártó-Stoïlow compactification. To show the traditional topological approach to $R^*_{\mathcal{S}}$, take a regular exhaustion $(R_n)_{n \in \mathbb{N}}$ of R and let $K_1^{(n)}, K_2^{(n)}, \cdots, K_{N(n)}^{(n)}$ be relatively noncompact components of $R \setminus \overline{R}_n$. A *determining sequence* is a sequence $\{K_{i_n}^{(n)}\}$ such that

$$K_{i_1}^{(1)} \supset K_{i_2}^{(2)} \supset \cdots K_{i_n}^{(n)} \supset \cdots .$$

Another regular exhaustion $(R'_n)_{n \in \mathbb{N}}$ gives another determining sequence $\{K'^{(n)}_{i_n}\}$. Two determining sequences $\{K_{i_n}^{(n)}\}$ and $\{K'^{(n)}_{i_n}\}$ are said to be equivalent if for any n there is an m such that $K_{i_n}^{(n)} \supset K'^{(m)}_{i_m}$ and conversely. An equivalence class of determining sequence is called a *Stoïlow end.* Then fix a regular exhaustion $(R_n)_{n \in \mathbb{N}}$ of R. The totality of determining sequences correrspونds in a bijective manner to the totality of ends of R. Let a determining sequence $\{K_{i_n}^{(n)}\}$ give a Stoïlow end $e\{K_{i_n}^{(n)}\}$. Let R^* be the union of R and the family of ends $e\{K_{i_n}^{(n)}\}$. Denote by $E(K_{i_n}^{(n)})$ the totality of ends which correspond to determining sequences containing $K_{i_n}^{(n)}$. For the base of the neighborhood system at $e\{K_{i_n}^{(n)}\}$ choose $K_{i_n}^{(n)} \cup E(K_{i_n}^{(n)})$. Then it is seen that $R^* = R^*_{\mathcal{S}}$. From the view point of the above introduction of $R^*_{\mathcal{S}}$, a point in $\gamma_{\mathcal{S}} = R^*_{\mathcal{S}} \setminus R$ is often referred to as a Stoïlow *ideal boundary component.* The Stoïlow compactification $R^*_{\mathcal{S}}$ is resolutive and, of course, of Stoïlow type and actually the smallest compactification of Stoïlow type, i.e., any compactification lying over $R^*_{\mathcal{S}}$ is of Stoïlow type. Clearly $R^*_{\mathcal{S}}$ is metrizable. This compactification has long been used not only as ideal boundaries for Riemann surfaces but also for the plane regions besides their proper Euclidean boundaries even in the classical function theory. It is particularly convenient for the study related to the analytic continuations such as Iversen property, the Gross property, the Stoïlow theory of covering surfaces, and so forth (cf. [10], [11], etc.).

3.7. The Alexandroff compactification. Let R be an open Riemann surface and $f \in C(R)$ such that there exists a regular subregion R_f of R satisfying the condition that $f|R \setminus \overline{R}_f$ is a constant function c_f on $R \setminus R_f$. Such an f is called an *Alexandroff function* on R and the totality of Alexandroff functions on R is denoted by $\mathcal{A}(R)$. Then $\mathcal{A}$ compactification $R^*_{\mathcal{A}}$ is the *Alexandroff compactification* of R. The usual topological introduction of $R^*_{\mathcal{A}}$ is as follows. Consider an element $\infty = \infty_R$ and let $R^* := R \cup \{\infty\}$. In addition to the original topology on R, let the family of sets of the form $\{\infty\} \cup (R \setminus K)$ with K compact subsets of R be a base of neighborhood system at ∞. Then it is seen that $R^* = R \cup \{\infty\}$ is a compact Hausdorff space and, in fact, $R^* = R^*_{\mathcal{A}}$. Hence the Alexandroff compactification $R^*_{\mathcal{A}}$ of R is also called the one point compactification of R since the *Alexandroff boundary* $\gamma_{\mathcal{A}} := R^*_{\mathcal{A}} \setminus R = \{\infty\}$ consisting only one point $\infty = \infty_R$, which is often called the Alexandroff point or the the point at infinity. Hence, for example, the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C}^*_{\mathcal{A}} = \mathbb{C} \cup \{\infty\}$. The Alexandroff compactification $R^*_{\mathcal{A}}$ is resolutive but in general not of Stoïlow type. Clearly it is metrizable. Every compactification R^* of R lies over $R^*_{\mathcal{A}}$. It is so crude that no particular instances of its convenient and essential use in the function theory and the potential theory on Riemann surfaces can hardly be found except its predicative use, e.g. we can conveniently use it in the statement of the definition of Evans-Selberg potential $E(z,\zeta)$ on an open Riemann surface R with the negative pole at $\zeta \in R : E(\cdot,\zeta) \in H(R \setminus \{\zeta\})$ with $\lim_{z\to\zeta} E(z,\zeta) = -\infty$ and $\lim_{z\to\infty_R} E(z,\zeta) = +\infty$.

We repeat: the Čech compactification $R^*_{\mathcal{C}}$ is not resolutive for any open Riemann surface R but all of the rest 6 compactifications are all resolutive among which $R^*_{\mathcal{W}}$ is the largest; the Alexandroff compactification $R^*_{\mathcal{A}}$ is in general not of Stoïlow type but all of the rest 6 compactifications are of Stoïlow type among which $R^*_{\mathcal{S}}$ is the smallest; the Čech compactification $R^*_{\mathcal{C}}$, the Wiener compactification $R^*_{\mathcal{W}}$, and the Royden compactification $R^*_{\mathcal{R}}$ are not metrizable for any open Riemann surface R but the rest 4 compactifications aer all metrizable.

4. Relations among 7 compactifications

Consider two Q compactifications $R^*_{Q_i}$ $(i = 1,2)$ of an open Riemann surface R and we observe that if $Q_1(R) \supset Q_2(R)$, then $R^*_{Q_1}$ lies over $R^*_{Q_2}$: $R^*_{Q_1} \to R^*_{Q_2}$. In fact, choose any generalized sequence $(z_\alpha)_{\alpha\in A}$ with A, a directed net, in R converging to an arbitrarily given point $\xi \in \gamma_Q$ in $R^*_{Q_1}$ and we are to find an $\eta \in \gamma_{Q_2}$ such that $(z_\alpha)_{\alpha\in A}$ converges to η in γ_{Q_2} in $R^*_{Q_2}$. Contrary to the assertion assume that there are two distinct points η' and η'' in γ_{Q_2} and two subsequences $(z_{\alpha'})_{\alpha'\in A'}$ and $(z_{\alpha''})_{\alpha''\in A''}$ of $(z_\alpha)_{\alpha\in A}$ such that $z_{\alpha'}\to\eta'$ and $z_{\alpha''}\to\eta''$ in $R^*_{Q_2}$. Take an $f\in Q_2(R)$ with $f(\eta')\neq f(\eta'')$. In view of $Q_2(R)\subset Q_1(R)$ the function f also belongs to $Q_1(R)$ and hence f is also continuous on $R^*_{Q_1}$. Then $(f(z_\alpha))_{\alpha\in A}\to f(\xi)$ while its two subsequences $(f(z_{\alpha'}))_{\alpha'\in A'}$ and $(f(z_{\alpha''}))_{\alpha''\in A''}$ converge to $f(\eta')$ and $f(\eta'')$. This contradiction proves our assertion. Then we can exhibit the following strings of inclusion relations:

$$\begin{cases} \mathcal{C}(R)\supset\mathcal{W}(R)\supset\mathcal{R}(R)\supset\mathcal{K}(R)\supset\mathcal{S}(R)\supset\mathcal{A}(R) \\ \qquad\qquad\quad \mathcal{W}(R)\supset\mathcal{M}(R)\supset\mathcal{S}(R) \end{cases}\tag{4.1}$$

All inclusions except two are either just trivial or very easy to derive from definitions of relevent families. Two exceptions here are $\mathcal{W}(R)\supset\mathcal{R}(R)$ and $\mathcal{W}(R)\supset\mathcal{M}(R)$. We give proofs for these two inclusion relations. To show the inclusion $\mathcal{W}(R)\supset\mathcal{R}(R)$, take an arbitrary $f\in\mathcal{R}(R)$. By the Royden-Brelot decomposition (3.3.1) of f, we have $f=h+g$ on R, where $h\in HD(R):=\{u\in H(R):D(u;R)<+\infty\}$ and g a Dirichlet potential on R so that there exists a genuine potential p on R with $|g|\leqq p$ on R. Let $h=h^+-h^-$ be the Jordan decomposition of h and set $u=h^++h^-\in HD(R)^+$. Then $s:=u+p$ is a nonnegative superharmonic function on R. Then $|f|\leqq|h|+|g|\leqq u+p=s$, i.e., $|f|\leqq s$ on R so that the first condition for f to be in $\mathcal{W}(R)$ is satisfied. Let $(R_n)_{n\in\mathbb{N}}$ be any regular exhaustion of R. We have

$$H_f^{R_n}=H_h^{R_n}+H_g^{R_n}=h+H_g^{R_n}$$

on R and $\left|H_g^{R_n}\right|\leqq H_{|g|}^{R_n}\leqq H_p^{R_n}$ on R. Since $p\geqq 0$ is superharmonic, the comparison principle implies that $0\leqq H_p^{R_{n+1}}\leqq H_p^{R_n}$ on R_n and $0\leqq\lim_{n\to\infty}H_p^{R_n}\leqq p$ on R, i.e., the nonnegative harmonic function $\lim_{n\to\infty}H_p^{R_n}$ on R is dominated by the potential p on R so that $\lim_{n\to\infty}H_p^{R_n}=0$ on R. Therefore $\lim_{n\to\infty}H_f^{R_n}=h$ on R, which shows the second condition, the essential requirment of the harmonizability of f for f to belong to $\mathcal{W}(R)$, is satisfied. Next we show the rest of two inclusion relations: $\mathcal{W}(R)\supset\mathcal{M}(R)$. More generally we show that $\mathcal{W}(R)\supset C(R^*_{\mathcal{M}})$. Then take any $f\in C(R^*_{\mathcal{M}})$ and we are to show that $f\in\mathcal{W}(R)$ and, for the purpose, we may assume that $0\leqq f\leqq 1$. For each $n\in\mathbb{N}$, let $E_i:=\{\zeta\in E:(i-1/2)/n\leqq f(\zeta)\leqq(i+1/2)/n\}$ (cf. (3.4.6)) and $F_i:=\{z\in R:f(z)\leqq(i-1)/n \text{ or } f(z)\geqq(i+1)/n\}$ for $i=0,1,2,\cdots,n$. We set

$$w_i:=\int_{E_i}K(\cdot,\zeta)d\mathrm{hm}(\zeta)$$

on R. Since $R^*_{\mathcal{M}}\setminus\overline{F}_i$ forms a neighborhood of each $\zeta\in E_i\subset E$, the balayaged function $(K(\cdot,\zeta))_{F_i}$ of $K(\cdot,\zeta)$ relative to F_i is a potential, which is the characterizing condition for $K(\cdot,\zeta)$ $(\zeta \in \gamma_{\mathcal{M}})$ to belong to the class of extreme points of the convex set $\{u \in H(R)^+ : u(a) = 1\}$. Thus, from the identity

$$(w_i)_{F_i} = \int_{E_i} (K(\cdot,\zeta))_{F_i} d\mathrm{hm}(\zeta),$$

it follows that $(w_i)_{F_i}$ is a potential. Hence we obtain

$$\frac{i-1}{n}(w_i - (w_i)_{F_i}) \leqq f w_i \leqq \frac{i+1}{n} w_i + (w_i)_{F_i} \quad (i = 0, 1, \cdots, n)$$

on R except for a polar subset of R. Summing up the above inequalities for $i = 0$ to n, we obtain, by observing $\sum_{i=0}^{n} f w_i = f$, that

$$\sum_{i=1}^{n} \frac{i-1}{n}(w_i - (w_i)_{F_i}) \leqq f \leqq \sum_{i=0}^{n} \frac{i+1}{n} w_i + \sum_{i=0}^{n} (w_i)_{F_i}$$

on R. Take a regular exhaustion $(R_m)_{m\in\mathbb{N}}$ of R. Then we see that

$$\sum_{i=1}^{n} \frac{i-1}{n}(w_i) - H_p^{R_m} \leqq H_f^{R_m} \leqq \sum_{i=0}^{n} \frac{i+1}{n} w_i + H_q^{R_m}$$

on R_m, where

$$p = \sum_{i=1}^{n} (i-1)(w_i)_{F_i}/n \qquad \text{and} \qquad q = \sum_{i=0}^{n} (w_i)_{F_i}$$

are potentials on R. Thus we have, on letting $m \to \infty$ in the above inequalities,

$$\sum_{i=1}^{n} \frac{i-1}{n}(w_i) \leqq \liminf_{m\to\infty} H_f^{R_m} \leqq \limsup_{m\to\infty} H_f^{R_m} \leqq \sum_{i=0}^{n} \frac{i+1}{n} w_i$$

on R, which implies that

$$0 \leqq \limsup_{m\to\infty} H_f^{R_m} - \liminf_{m\to\infty} H_f^{R_m}$$

$$\leqq \sum_{i=0}^{n} \frac{i+1}{n} w_i - \sum_{i=1}^{n} \frac{i-1}{n} w_i \leqq \frac{2}{n} \sum_{i=0}^{n} w_i = \frac{2}{n}$$

on R. On letting $n \uparrow \infty$, we conclude that $\liminf_{n\to\infty} H_f^{R_m} = \limsup_{n\to\infty} H_f^{R_m}$ on R so that the local uniform convergence of the sequence $(H_f^{R_m})_{m\in\mathbb{N}}$ on R is deduced, i.e., we have shown that $f \in \mathcal{W}(R)$.

Based upon the table (4.1) of the inclusion relations among the 7 function subclasses $Q(R)$ of $C(R)$ determining the 7 particular Q compactifications R_Q^* we can obtain the following table of "lying over" relations among 7 particular Q compactifications as follows:

(4.2)
$$\begin{tikzcd}
R^*_{\mathcal{C}} \arrow[rr] & & R^*_{\mathcal{W}} \arrow[rr] \arrow[drrr] & & R^*_{\mathcal{R}} \arrow[rr] \arrow[dr, shift left, "/" marking] & & R^*_{\mathcal{K}} \arrow[rr] \arrow[dl, shift left, "/" marking] & & R^*_{\mathcal{S}} \arrow[rr] & & R^*_{\mathcal{A}} \\
& & & & & R^*_{\mathcal{M}} \arrow[ul, shift left, "/" marking] \arrow[ur, shift left, "/" marking] \arrow[urrr] & & & & &
\end{tikzcd}$$

Once more we recall that $R^*_{Q_1} \rightarrow R^*_{Q_2}$ means that $R^*_{Q_1}$ lies over $R^*_{Q_2}$ and its negation $R^*_{Q_1} \not\rightarrow R^*_{Q_2}$ is also denoted by $R^*_{Q_1} \mathrel{-\!\!\!+\!\!\!\rightarrow} R^*_{Q_2}$. However, in the table (4.2) $R^*_{Q_1} \rightarrow R^*_{Q_2}$ means that $R^*_{Q_1}$ lies over $R^*_{Q_2}$ for *every* choice of an open Riemann surface R and $R^*_{Q_1} \mathrel{-\!\!\!+\!\!\!\rightarrow} R^*_{Q_2}$ means that there exists *some* open Riemann surface R for which $R^*_{Q_1}$ does not lie over $R^*_{Q_2}$. We have not yet discussed about

$$R^*_{\mathcal{R}} \mathrel{\substack{-\!\!\!+\!\!\!\rightarrow \\ \leftarrow\!\!\!+\!\!\!-}} R^*_{\mathcal{M}} \tag{4.3}$$

in the table (4.2), i.e., there is an open Riemann surface R for which $R^*_{\mathcal{R}} \rightarrow R^*_{\mathcal{M}}$ does not hold and also there is an open Riemann surface R for which $R^*_{\mathcal{M}} \rightarrow R^*_{\mathcal{R}}$ does not hold so that, in short, there is no relation between $R^*_{\mathcal{R}}$ and $R^*_{\mathcal{M}}$ in general, and about

$$R^*_{\mathcal{K}} \mathrel{\substack{-\!\!\!+\!\!\!\rightarrow \\ \leftarrow\!\!\!+\!\!\!-}} R^*_{\mathcal{M}} \tag{4.4}$$

also in the table (4.2), i.e., there is an open Riemann surface R for which $R^*_{\mathcal{K}} \rightarrow R^*_{\mathcal{M}}$ is not valid and also there exists an R for which $R^*_{\mathcal{M}} \rightarrow R^*_{\mathcal{K}}$ is not the case so that, in short, $R^*_{\mathcal{K}}$ and $R^*_{\mathcal{M}}$ are not related in general. As far as the relations among 7 particular Q compactifications are concerned, the table (4.2) is the final complete conclusion. What are left here are "no relation" relations (4.3) and (4.4). Bibliographically speaking the relation (4.4), i.e., the following part

(4.5)
$$\begin{tikzcd}
R^*_{\mathcal{R}} \arrow[rr] & & R^*_{\mathcal{K}} \arrow[dl, shift left, "/" marking] \\
& R^*_{\mathcal{M}} \arrow[ur, shift left, "/" marking] &
\end{tikzcd}$$

in the table (4.2), was established by Kuramochi in his paper [4]. Especially the part of his proof for $R^*_{\mathcal{M}} \not\rightarrow R^*_{\mathcal{K}}$ is quite elaborate and extremely intricate but the surface R he constructed to show $R^*_{\mathcal{M}} \not\rightarrow R^*_{\mathcal{K}}$ is *planar.* As our central object of this paper we will prove (4.3), i.e., the part

(4.6)
$$\begin{tikzcd}
R^*_{\mathcal{R}} \arrow[rr] \arrow[dr, shift left, "/" marking] & & R^*_{\mathcal{K}} \\
& R^*_{\mathcal{M}} \arrow[ul, shift left, "/" marking] &
\end{tikzcd}$$

in the table (4.2). As far as the present author is aware of, no proof for this part has been in publicized print, which may justifies his intention. We start from proving $R^*_{\mathcal{M}} \not\to R^*_{\mathcal{R}}$ first. For the purpose we choose the following approach.

THEOREM 4.7. *Any compactification R^* of any open Riemann surface R is not metrizable if R^* lies over the Royden compactification $R^*_{\mathcal{R}}$ of R.*

PROOF: Let I be the identity mapping of R onto itself and I^* be the extension of I which is the continuous mapping of R^* onto $R^*_{\mathcal{R}}$, i.e., I^* is the projection of R^* onto $R^*_{\mathcal{R}}$. Contrary to the assertion assume that R^* is metrizable. Let d be a metric on R^* such that the metric space (R, d) is homeomorphic to R with the original Riemann surface topology. Choose arbitrarily but then fix a $\xi \in \gamma := R^* \setminus R$ and let $\eta := I^*(\xi) \in \gamma_{\mathcal{R}} := R^*_{\mathcal{R}} \setminus R$. Take a sequence $(z_n)_{n\in\mathbb{N}}$ in R such that $z_n \neq z_m$ $(n \neq m)$ and $d(z_n, \xi) \to 0$ $(n \to \infty)$. Fix a closed parametric disc $\overline{U}_n = \{|z| \leqq 1\}$ $(z(z_n) = 0)$ at z_n with its interior $U_n := \{|z| < 1\}$ on R for each $n \in \mathbb{N}$. We can assume that $\overline{U}_n \cap \overline{U}_m = \emptyset$ $(n \neq m)$. We use the notation ρU_n $(1 > \rho > 0)$ and $\rho\overline{U}_n$ for the subset of U_n and $\overline{U}_n$ given by $\{|z| < \rho\}$ and $\{|z| \leqq \rho\}$, respectively. Fix a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $0 < \varepsilon_n < 1$ $(n \in \mathbb{N})$ and $\sum_{n\in\mathbb{N}} \varepsilon_n \leqq 1$. For each $n \in \mathbb{N}$, choose $0 < a_n < 1$ so small that that the diameter $\overline{d}(a_n\overline{U}_n) := \sup\{d(z', z'') : z', z'' \in a_n\overline{U}_n\} < \varepsilon_n$ and $0 < b_n < a_n$ further so small as to make $\log(a_n/b_n) > 1/\varepsilon_n$. Consider the annulus $A_n := a_nU_n \setminus b_n\overline{U}_n$ and the closed disc $\overline{B}_n := b_n\overline{U}_n$ in R for each $n \in \mathbb{N}$. By the choice of the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, we have $\overline{A}_n \cap \overline{A}_m = \emptyset$ $(n \neq m)$ and

$$\text{(4.8)} \qquad \operatorname{mod} A_n := \log \frac{a_n}{b_n} > \frac{1}{\varepsilon_n} \quad (n \in \mathbb{N}).$$

We choose and fix one more point w_n in the outer boundary of A, i.e., $w_n \in \partial(a_nU_n)$ for each $n \in \mathbb{N}$. Then $d(\xi, w_n) \leqq d(\xi, z_n) + d(z_n, w_n) < d(\xi, z_n) + \varepsilon_n \to 0$ as $n \to \infty$. Hence we have

$$\text{(4.9)} \qquad \lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = \xi$$

on R^*. We define a function $f \in C(R)$ as follows: $f|A_n = H^{A_n}_{\varphi_n}$ for each $n \in \mathbb{N}$, where φ_n is the boundary function on ∂A_n with $f|\partial(a_nU_n) = 0$ and $f|\partial(b_nU_n) = 1$; $f|\overline{B}_n = 1$ for each $n \in \mathbb{N}$; and $f|R \setminus \cup_{n\in\mathbb{N}}(A_n \cup \overline{B}_n) \equiv 0$. Observe that $f|A_n = H^{A_n}_{\varphi_n}$ is the harmonic measure function ω_n of the inner boundary of the annulus A_n relative to the annulus A_n so that, by (4.8), we see that

$$D(f; A_n) = D(\omega_n; A_n) = \frac{2\pi}{\operatorname{mod} A_n} < 2\pi\varepsilon_n \quad (n \in \mathbb{N}).$$

Therefore we have

$$D(f; R) = \sum_{n\in\mathbb{N}} \left(D(f; A_n) + D(f; \overline{B}_n)\right) + D\left(f; R \setminus \cup_{n\in\mathbb{N}}(A_n \cup \overline{B}_n)\right)$$

$$= \sum_{n\in\mathbb{N}} D(f; A_n) < 2\pi \sum_{n\in\mathbb{N}} \varepsilon_n = 2\pi,$$

and therefore $f \in L^{1,2}(R)$. Since $f \in C(R)$, we conclude that $f \in \mathcal{R}(R) = C(R) \cap L^{1,2}(R)$ so that f is continuous on $R^*_{\mathcal{R}}$. Since $I^* : R^* \to R^*_{\mathcal{R}}$ is continuous and $f : R^*_{\mathcal{R}} \to \mathbb{R}$ is also continuous, we see that the function $g := f \circ I^* : R^* \to \mathbb{R}$ is continuous. By (4.9) we see that $g(z_n) \to g(\xi)$ $(n \to \infty)$ and $g(w_n) \to g(\xi)$ $(n \to \infty)$. But $z_n \in \overline{B}_n$ shows that $g(z_n) = f(z_n) = 1$ and $w_n \notin A_n \cup \overline{B}_n$ shows that $g(w_n) = f(w_n) = 0$. From the former $g(\xi)$ must be 1 while from the latter $g(\xi)$ must be 0, a contradiction. □

Corollary to Theorem 4.7. *The Martin compactification $R^*_{\mathcal{M}}$ of any open Riemann surface R does not lie over the Royden compactification $R^*_{\mathcal{R}}$ of R, i.e., $R^*_{\mathcal{M}} \not\twoheadrightarrow R^*_{\mathcal{R}}$ for every open Riemann surface R.*

Thus we have finished the proof of a part of (4.7): $R^*_{\mathcal{R}} \not\twoheadleftarrow R^*_{\mathcal{M}}$. Since we have the Kuramochi result (4.4) and in particular $R^*_{\mathcal{K}} \not\twoheadleftarrow R^*_{\mathcal{M}}$, we can infer as follows. Suppose $R^*_{\mathcal{R}} \leftarrow R^*_{\mathcal{M}}$ for every R. Then (4.2) shows $R^*_{\mathcal{R}} \to R^*_{\mathcal{K}}$ and by the transitivity of the relation "lying over" we would have to conclude that $R^*_{\mathcal{K}} \leftarrow R^*_{\mathcal{M}}$ for every R, contradicting the Kuramochi result $R^*_{\mathcal{K}} \not\twoheadleftarrow R^*_{\mathcal{M}}$. However, there is a wide gap here. Our result stated above is $R^*_{\mathcal{M}} \not\twoheadrightarrow R^*_{\mathcal{R}}$ for *every* open Riemann surface R. On the other hand the result derived from the Kuramochi result explained above is that $R^*_{\mathcal{M}} \not\twoheadrightarrow R^*_{\mathcal{R}}$ for *some* open Riemann surface R so that there might exist the case $R^*_{\mathcal{M}} \to R^*_{\mathcal{R}}$ by a suitable choice of R, which can never happen is the assertion of our Corollary above. However the following observation falls into the different category from the above remark. We will complete (4.3) by showing $R^*_{\mathcal{R}} \not\twoheadrightarrow R^*_{\mathcal{R}}$ for *some* R in the next section 5. Now if $R^*_{\mathcal{K}} \to R^*_{\mathcal{M}}$ is the case for every R, then by using $R^*_{\mathcal{R}} \to R^*_{\mathcal{K}}$ for every R in (4.2), we would have to conclude that $R^*_{\mathcal{R}} \to R^*_{\mathcal{M}}$ for every R contradicting $R^*_{\mathcal{R}} \not\twoheadrightarrow R^*_{\mathcal{M}}$ for some R to be shown later in the next section. Therefore, once (4.3) is established, it also takes care of the half of (4.4), and what is left is

$$R^*_{\mathcal{K}} \not\twoheadleftarrow R^*_{\mathcal{M}}$$

for some open Riemann surface R. Although we have a proof for this part given by Kuramochi [4], it is always nice to have another standard proof that can be followed without extraordinary and ridiculous elaboration.

5. Tôki covering surfaces

Recall that we have denoted by $H(R)$ the linear space of real valued harmonic functions on an open Riemann surface R. Two important subspaces of $H(R)$ are (as we have already considered): first,

$$HB(R) := \left\{ u \in H(R) : \sup_R |u| < +\infty \right\},$$

i.e., the subspace of $H(R)$ consisting of bounded functions and B in $HB(R)$ suggests the boundedness, and it forms a Banach space equipped with the supremum norm $\sup_R |u|$; second,

$$HD(R) := \left\{ u \in H(R) : D(u;R) = \int_R du \wedge *du < +\infty \right\},$$

i.e., the subspace of $H(R)$ consisting of Dirichlet finite functions and D in $HD(R)$ suggests the Dirichlet finiteness, and it forms a Banach space, and actually a Hilbert space, with the norm $(u(a)^2 + D(u;R))^{1/2}$ with an arbitrarily fixed reference point $a \in R$. To combine the above two subspaces it is convenient to consider

$$HBD(R) := HB(R) \cap HD(R)$$

and usually the norm $\sup_R |u| + D(u;R)^{1/2}$ is employed to make $HBD(R)$ a Banach space. We denote by $\mathcal{O}_{HX}$ the family of open Riemann surfaces R such that $HX(R) = \mathbb{R}$, i.e., there is no functions in $HX(R)$ other than constant functions, where $X = B$ or D. Together with the class $\mathcal{O}_G$ of parabolic Riemann surfaces we know

$$\mathcal{O}_G < \mathcal{O}_{HB} < \mathcal{O}_{HD}, \tag{5.1}$$

where $<$ stands for the strict inclusion, i.e., $\subset$ and $\neq$. The Virtanen-Royden theorem says that the class $HBD(R)$ is dense in $HD(R)$ in the following sense: there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $HBD(R)$ for any given $u \in HD(R)$ such that $u_n \to u$ $(n \to \infty)$ locally uniformly on R and at the same time $D(u - u_n; R) \to 0$ $(n \to \infty)$. This theorem assures trivially the inclusion $\mathcal{O}_{HB} \subset \mathcal{O}_{HD}$. Once upon a time the question whether this inclusion relation is strict or not was one of central themes in the classification theory of Riemann surfaces in its second evolution following the type problem period. It was Tôki who succeeded in showing that it is certainly strict after some failures by a couple of extremely eminent function theorists at that time (in the sense that their publicized papers claiming the strictness contained certain gaps). Tôki constructed an infinitely sheeted complete covering surface $\tilde{\mathbb{D}}$ of the unit disc $\mathbb{D}$ with $HB(\tilde{\mathbb{D}}) = HB(\mathbb{D}) \circ \pi$ (π being the projection of $\tilde{\mathbb{D}}$ onto $\mathbb{D}$) and thus $HD(\tilde{\mathbb{D}}) = \mathbb{R}$, which proves $\mathcal{O}_{HB} < \mathcal{O}_{HD}$. This gives rise to a notion of Tôki covering surfaces.

For convenience let us run over the definition of covering surfaces and some of its relevant terminology before we go to our main theme of this section (cf. [1]).

A covering Riemann surface or loosely just a covering surface is a triple $(\tilde{R}, R, \pi)$ of two Riemann surfaces $\tilde{R}$ and R and an analytic mapping $\pi : \tilde{R} \to R$. The Riemann surface $\tilde{R}$ itself is again called the *covering surface* of R (of the covering surface $(\tilde{R}, R, \pi)$), R the base surface, and π the projection of the covering surface $(\tilde{R}, R, \pi)$. Pick any point $\tilde{p}$ in $\tilde{R}$ and let $p = \pi(\tilde{p})$. If we choose suitable local parameters z and w at $\tilde{p}$ and p, respectively, then we have $w = z^n$ $(n \in \mathbb{N})$ as a local expression of π at $\tilde{p}$. When $n > 1$, then the point $\tilde{p}$ is referred to a branch point of order $n-1$ and n is called the multiplicity of the branch point $\tilde{p}$. In case the existence of a branch point in $\tilde{R}$ cannot be denied, then $\tilde{R}$ is said to be branched or possibly branched. A covering surface $\tilde{R}$, or more precisely $(\tilde{R}, R, \pi)$, is said to be *complete* if every base point $p \in R$ has a neighborhood V such that every component of $\pi^{-1}(V)$ is compact. The important consequence of $\tilde{R}$ being complete is that $\tilde{R}$ covers each point of R the same number of times, provided that the branch points are counted as many times as their multiplicity indicates. This number may be infinite and in such a case $(\tilde{R}, R, \pi)$ is said to be of infinitely sheeted. This is all for the review of covering surfaces. We now define a Tôki covering surface $\tilde{R}$ of any open Riemann surface R. An open Riemann surface $\tilde{R}$ is said to be a *Tôki covering surface* of a given open Riemann surface R if the following two conditions are fulfilled:

(a) $(\tilde{R}, R, \pi)$ forms a complete covering Riemann surface which is infinitely sheeted and possibly branched;

(b) every bounded harmonic function on $\tilde{R}$ is constant on the fiber $\pi^{-1}(w)$ of any base point $w \in R$.

As far as the notion of Tôki covering surfaces is concerned, the most important and fundamental recognition is its existence.

THEOREM 5.2. *Any open Riemann surface R admits at least one its Tôki covering surface $(\tilde{R}, R, \pi)$.*

The result is established when $R = \mathbb{D}$, the unit disc in the complex plane $\mathbb{C}$, by Tôki [13]. Ameliorating the Tôki construction, the above general case is treated by Nakai-Segawa [9]. Let $(\tilde{R}, R, \pi)$ be an infinitely sheeted covering surface so that the one satisfying above (a). Then it is always the case that

$$HB(\tilde{R}) \supset HB(R) \circ \pi := \{u \circ \pi : u \in HB(R)\}.$$

The subspace $HB(R) \circ \pi$ occupies larger and larger portion of $HB(\tilde{R})$ as each sheet of $\tilde{R}$ is put together closer and closer. The extreme case is

$$HB(\tilde{R}) = HB(R) \circ \pi. \tag{5.3}$$

It is easy to see that this condition is equivalent to the condition (b) above. Thus a covering surface $(\tilde{R}, R, \pi)$ is a Tôki covering surface of R if and only if it is complete, infinitely sheeted, and satisfies (5.3). Then choose any $\tilde{u} \in HBD(\tilde{R})$. In view of (5.3) there is a $u \in HB(R)$ with $\tilde{u} = u \circ \pi$ so that we must conclude that

$$\infty > D(\tilde{u}; \tilde{R}) = D(u; R) \cdot \infty.$$

Hence $D(u; R) = 0$ and u must be a constant, which implies that $\tilde{u} \equiv c$, a constant. Therefore $HBD(\tilde{R}) = \mathbb{R}$. By the Virtanen-Royden theorem that $HBD(\tilde{u})$ is "dense" in $HD(\tilde{R})$, we can conlude the following result.

THEOREM 5.4. *Any Tôki covering surface* $\tilde{R} \in \mathcal{O}_{HD}$, *i.e.*, $HD(\tilde{R}) = \mathbb{R}$.

Hereafter we always use the notation $\tilde{R}$ to denote a Tôki covering surface of an open Riemann surface R unless the contrary is explicitly mentioned.

We proceed to the final one of our main objectives of this paper: the proof of $R^*_{\mathcal{R}} \not\leftrightarrow R^*_{\mathcal{M}}$ for some R. Actually we will achieve our aim by showing $(\tilde{R})^*_{\mathcal{R}} \not\leftrightarrow (\tilde{R})^*_{\mathcal{M}}$ for any Tôki covering surface $\tilde{R}$ of a suitable open Riemann surface. We denote by $A(R)$ the class of all single valued analytic functions on an open Riemann surface. Similar to $HB(R)$ and $HD(R)$ to $H(R)$, we also consider the subspace $AB(R)$ (resp. $AD(R)$) of $A(R)$ consisting of bounded (resp. Dirichlet finite) functions in $A(R)$. Like $\mathcal{O}_{HX}$ we can also consider $\mathcal{O}_{AX}$, i.e., the class of open Riemann surface R for which $AX(R) = \mathbb{C}$, the complex number field, for $X = B$ or D.

THEOREM 5.5. *If R is a hyperbolic Riemann surface satisfying $R \notin \mathcal{O}_{AB}$ (i.e., $AB(R) \setminus \mathbb{C} \neq \emptyset$), then $(\tilde{R})^*_{\mathcal{R}} \not\leftrightarrow (\tilde{R})^*_{\mathcal{M}}$ for any Tôki covering surface $\tilde{R}$ of R.*

PROOF: First we maintain that there is a point ζ_0 in the Royden boundary $\gamma_{\mathcal{R}} := (\tilde{R})^*_{\mathcal{R}} \setminus \tilde{R}$ of $\tilde{R}$ such that

$$\text{hm}_{(\tilde{R})^*_{\mathcal{R}}}(\{\zeta_0\}) = 1. \tag{5.6}$$

To prove (5.6), for the purpose of simplifying notations we set $S := \tilde{R}$, $S^* := S^*_{\mathcal{R}}$, $\gamma := S^* \setminus S$, $\text{hm} = \text{hm}_{S^*}$, and let δ be the Royden harmonic boundary of S so that δ is a compact subset of γ. Since δ is the totality of regular points in γ with respect to the Dirichlet problem, H^S_f has boundary values f on δ for every $f \in C(\delta)$. Choose any $f \in \mathcal{R}(S)$ and let $f = u + g$ be the Royden-Brelot decomposition of f on S with $u \in HD(S)$ and g a Royden potential on S (cf. (2.1)). Since δ is characterized also by the fact every Dirichlet (Royden) potential vanishes on δ and vice versa, we see that $HD(S)|\delta = \mathcal{R}(S)|\delta$, which forms a vector lattice and separates points in δ. Hence in particular the subspace $HBD(S)|\delta$ of $C(\delta)$ is a vector sublattice and separates points in δ, by the Stone-Weierstrass approximation theorem, the uniform closure $\overline{HBD(S)|\delta}$ of $HBD(S)|\delta$ coincides with $C(\delta)$. But, in the present case, $S \in \mathcal{O}_{HD}$ by Theorem 5.4 and a fortiori $C(\delta) = \overline{HBD(S)|\delta} = \mathbb{R}$. This proves that δ consists of only one point ζ_0, say. For every compact $K \subset \gamma \setminus \delta$, using the function p_K in (2.2), we can see that $\mathrm{hm}(K) = 0$. This proves that $\mathrm{hm}(\gamma \setminus \delta) = 0$. Since $\mathrm{hm}(\gamma) = 1$, we can conclude that $\mathrm{hm}(\{\zeta_0\}) = 1$, i.e., (5.6) is deduced. Contrary to our assertion let us make an erroneous assumption that $(\tilde{R})^*_{\mathcal{R}} \to (\tilde{R})^*_{\mathcal{M}}$ and let I^* be the projection of $(\tilde{R})^*_{\mathcal{R}}$ onto $(\tilde{R})^*_{\mathcal{M}}$. Setting $\xi_0 := I^*(\zeta_0) \in \gamma_{\mathcal{M}} := (\tilde{R})^*_{\mathcal{R}} \setminus \tilde{R}$, we see that

$$\mathrm{hm}_{(\tilde{R})^*_{\mathcal{M}}}(\{\xi_0\}) = \mathrm{hm}_{(\tilde{R})^*_{\mathcal{M}}}(I^*(\{\zeta_0\})) > 0 \tag{5.7}$$

along with (5.6) (cf. (2.5)). Clearly (5.3) assures the vadidity of $AB(\tilde{R}) = AB(R) \circ \pi$. Our assumption $R \notin \mathcal{O}_{AB}$ assures that $AB(R)$ and hence $AB(\tilde{R})$ contains other than constants. Hence we can choose a nonconstant bounded holomorphic function f on $\tilde{R}$. Applying the Fatou theorem stated in the subsection 3.4, we can conclude that the fine limit $\hat{f}(\xi_0)$ of f at $\xi_0 \in \gamma_{\mathcal{M}}$ exists. Observe that $\{\xi_0\} \subset \{\xi \in \gamma_{\mathcal{M}} : \hat{f}(\xi) = \hat{f}(\xi_0)\}$ and hence $0 < \mathrm{hm}_{(\tilde{R})^*_{\mathcal{M}}}(\{\xi_0\}) \leqq \mathrm{hm}_{(\tilde{R})^*_{\mathcal{M}}}(\{\xi \in \gamma_{\mathcal{M}} : \hat{f}(\xi) = \hat{f}(\xi_0)\})$. The F. and M. Riesz theorem mentioned in the same subsection 3.4 as above, yields that $f \equiv \hat{f}(\xi_0)$ on $\tilde{R}$, a contradiction. □

A function $u \in H(R)^+$ is said to be *minimal* on a hyperbolic Riemann surface R if $u > 0$ on R and whenever $u \geqq v > 0$ on R holds for some $v \in H(R)^+$ we can find a constant $c_v > 0$ such that $v = c_v u$ on R. It is easily seen that $u \in H(R)^+$ is minimal if and only if $u/u(a)$ is an extreme point of the convex set $\{v \in H(R)^+ : v(a) = 1\}$. In view of the Martin representation theorem (3.4.6), $u \in H(R)^+$ is minimal if and only if $u = u(a)K(\cdot, \zeta_u)$ for a unique point $\zeta_u \in E$. Since

$$1 = \int_E K(\cdot, \zeta) d\mathrm{hm}(\zeta),$$

we see that a $u \in H(R)^+$ is bounded and minimal if and only if $u = u(a)K(\cdot, \zeta_u)$ for a unique $\zeta_u \in E$ with $\mathrm{hm}_{(\tilde{R})^*_{\mathcal{M}}}(\{\zeta_u\}) > 0$. Thus we conclude:

THEOREM 5.8. *A hyperbolic Riemann surface R carries a bounded minimal harmonic function if and only if the Martin boundary of R contains a point of harmonic measure positive.*

The fact that a hyperbolic Riemann surface R carries a bounded minimal harmonic function must be understood to be a quite pathological phenomenon. Therefore, on the contrary, that a hyperbolic Riemann surface R does not carry any bounded minimal harmonic function is just a usual and normal state. Hyperbolic plane regions, hyperbolic Riemann surface of finite genus or even of almost finite genus (cf. [11]) fall in this category. In addition to Theorem 5.5, we give another theorem for $(\tilde{R})^*_{\mathcal{R}} \twoheadrightarrow\!\!\!\!\!\!\!/\;\; (\tilde{R})^*_{\mathcal{M}}$ whose example is also easy, simple, and handy.

THEOREM 5.9. *If R is a hyperbolic Riemann surface on which there is no bounded minimal harmonic function, then $(\tilde{R})^*_{\mathcal{R}} \twoheadrightarrow\!\!\!\!\!\!\!/\;\; (\tilde{R})^*_{\mathcal{M}}$ for any Tôki covering surface $\tilde{R}$ of R.*

PROOF: Contrariwise suppose there is a bounded minimal harmonic function $\tilde{u}$ on arbitrarily chosen Tôki covering surface $\tilde{R}$ of R. Let $u \in HB(R)$ satisfies $\tilde{u} = u \circ \pi$. Since $\tilde{u} > 0$ on $\tilde{R}$, we must have $u > 0$ on R. Let $u \geqq v > 0$ on R for some $v \in H(R)^+$, and actually $v \in HB(R)^+$. Then $\tilde{u} \geqq \tilde{v} := v \circ \pi > 0$ on $\tilde{R}$ yields the existence of a constant $c > 0$ such that $v \circ \pi = \tilde{v} = c\tilde{u} = cu \circ \pi$ on $\tilde{R}$ so that $v = cu$ on R. This proves the existence of a bounded minimal harmonic function u on R, a contradiction. Thus $\tilde{R}$ does not carry any bounded minimal harmonic function and , by Theorem 5.8,

$$\mathrm{hm}_{(\tilde{R})^*_{\mathcal{M}}}(\{\xi\}) = 0 \tag{5.10}$$

for every $\xi \in \gamma_{\mathcal{M}} := (\tilde{R})^*_{\mathcal{M}} \setminus \tilde{R}$. Recall that $\gamma_{\mathcal{R}} := (\tilde{R})^*_{\mathcal{R}} \setminus \tilde{R}$ contains a point ζ_0 with $\mathrm{hm}_{(\tilde{R})^*_{\mathcal{R}}}(\{\zeta_0\}) = 1$ (cf. (5.6)). If there exists a continuous extention $I^* : (\tilde{R})^*_{\mathcal{R}} \to (\tilde{R})^*_{\mathcal{M}}$ of the identity $I : \tilde{R} \to \tilde{R}$, then, on setting $I^*(\zeta_0) =: \xi_0 \in \gamma_{\mathcal{M}}$, (2.5) implies that $\mathrm{hm}_{(\tilde{R})^*_{\mathcal{M}}}(\{\xi_0\}) = \mathrm{hm}_{(\tilde{R})^*_{\mathcal{M}}}(I^*(\{\zeta_0\})) > 0$, contradicting (5.10) and thus the relation $(\tilde{R})^*_{\mathcal{R}} \twoheadrightarrow\!\!\!\!\!\!\!/\;\; (\tilde{R})^*_{\mathcal{M}}$ is established. □

Although we have finished our main object of this paper to prove (4.3) or equivalently (4.6), to get better understanding for the compactification theory, we add here one more item on Kuramochi compactifications $(\tilde{R})^*_{\mathcal{K}}$ of Tôki covering surfaces $\tilde{R}$. We will see that $(\tilde{R})^*_{\mathcal{K}}$ is considerably small if R is not too complicated in some sense. From now on we consider only, what we call *special*, Tôki covering surfaces. In addition to the defining conditions (a) and (b) in this section for Tôki covering surfaces we add one more condition:

(c) the projection $\mathcal{B} = \pi(\tilde{\mathcal{B}})$ in R of the set $\tilde{\mathcal{B}}$ of branch points in $\tilde{R}$ is isolated in R.

If a covering surface $(\tilde{R}, R, \pi)$ satisfies three conditions (a), (b), and (c), then we say $\tilde{R}$, or more precisely, $(\tilde{R}, R, \pi)$ is a *special Tôki covering surface* of R. Either in [13] or in [9] the proof for Theorem 5.2 is valid not only for plain Tõki covering surfaces but also, in reality, for special ones. Actually Tôki covering surfaces constructed in [13] and [9] were special ones. Thus we have the existence result: there always exists at least one special Tôki covering surface $\tilde{R}$ of any open Riemann surface R. Hereafter we always assume that Tôki covering surfaces $\tilde{R}$ are special. A Riemann surface R is said to be *regular* if $R \notin \mathcal{O}_G$, i.e., R carries the Green kernel $G(\cdot,\cdot)$, and

$$\lim_{z\to\infty_R} G(z,a) = 0, \tag{5.11}$$

where ∞_R is the Alexandroff point of R and a is an arbitrary reference point in R. The condition (5.11) is equivalent to the following: the set $\{z \in R : G(z,a) > \lambda\}$ is relatively copmpact in R for every $\lambda > 0$.

THEOREM 5.12. *Any special Tôki covering surface $\tilde{R}$ of any regular Riemann surface R is again regular.*

PROOF: Let $\tilde{G}(\tilde{z},\tilde{w})$ (resp. $G(z,w)$) be the Green kernel on $\tilde{R}$ (resp. R) and take an $\tilde{a} \in \tilde{R}$ arbitrarily fixed with $a = \pi(\tilde{a}) \in R$. We are to prove that $\lim_{z\to\infty_{\tilde{R}}} \tilde{G}(\tilde{z},\tilde{a}) = 0$, which we do by contradiction. Thus assume contrarily the existence of a sequence $(\tilde{b}_m)_{m\in\mathbb{N}}$ in $\tilde{R}$ with $\tilde{b}_m \to \infty_{\tilde{R}}$ $(m\to\infty)$ and

$$\lim_{m\to\infty} \tilde{G}(\tilde{b}_m,\tilde{a}) =: \varepsilon > 0. \tag{5.13}$$

Since $\tilde{G}(\tilde{z},\tilde{w}) \leqq G(\pi(\tilde{z}),\pi(\tilde{w}))$ on $\tilde{R}\times\tilde{R}$, we have $\tilde{G}(\tilde{b}_m,\tilde{a}) \leqq G(\pi(\tilde{b}_m),a)$. Let $b_m := \pi(\tilde{b}_m)$ $(m \in \mathbb{N})$. By (5.13), $\liminf_{m\to\infty} G(b_m,a) \geqq \tilde{G}(\tilde{z}_m,\tilde{a}) = \varepsilon > 0$. By the regularity of R, by choosing a subsequence if necessary, we can assume that $\lim_{m\to\infty} b_m = c \in R$. We can assume that $c \neq a$ by moving a a little if necessary. By virtue of the condition (c), we can find a closed neighborhood $V \subset R\setminus\{a\}$ of c as follows: $(V\setminus\{c\})\cap\mathcal{B} = \emptyset$. Let $\pi^{-1}(V) = \cup_{n\in\mathbb{N}}\tilde{V}_n$ and we can further take V small enough so as to make each connected component $\tilde{V}_n$ of $\pi^{-1}(V)$ compact for each $n\in\mathbb{N}$. Let $\pi^{-1}(c)\cap\tilde{V}_n = \{\tilde{c}_n\}$ so that $\pi^{-1}(c) = \{\tilde{c}_n : n \in \mathbb{N}\}$. Then $(\tilde{V}_n\setminus\{c_n\})\cap\tilde{\mathcal{B}} = \emptyset$ for every $n\in\mathbb{N}$. Let $k_n = k_n(\tilde{V}_n;\tilde{R}\setminus\{\tilde{a}\})$ be the Harnack constant for the compact set $\tilde{V}_n$ relative to the region $\tilde{R}\setminus\{\tilde{a}\}$. We can assume that $k := \sup\{k_n : n\in\mathbb{N}\} \in (1,,+\infty)$. Once more, by choosing a subsequence of $(b_m)_{m\in\mathbb{N}}$ if necessary, we can assume that $(b_m)_{m\in\mathbb{N}} \subset V$ and let $\tilde{b}_m \in \tilde{V}_{n(m)}$ for every $m\in\mathbb{N}$. Comparing two functions $\tilde{w} \mapsto \sum_{n=1}^N \tilde{G}(\tilde{c}_n,\tilde{w})$ for any fixed $N\in\mathbb{N}$ and $\tilde{w} \mapsto G(c,\pi(\tilde{w}))$, we have

$$\sum_{n-1}^{\infty} \tilde{G}(\tilde{c}_n,\tilde{w}) \leqq G(c,\pi(\tilde{w})) \qquad (\tilde{w}\in\tilde{R}).$$

Then we have

$$\sum_{n-1}^{\infty} \tilde{G}(\tilde{c}_n,\tilde{a}) \leqq G(c,a) < +\infty$$

so that we infer $\tilde{G}(\tilde{c}_n, \tilde{a}) \to 0$ as $n \to \infty$ and in particular

$$\lim_{m\to\infty} \tilde{G}(\tilde{c}_{n(m)}, \tilde{a}) = 0. \tag{5.14}$$

By virtue of

$$\tilde{G}(\tilde{b}_n, \tilde{a}) \leqq k_m \tilde{G}(\tilde{c}_{n(m)}, \tilde{a}) \leqq k\tilde{G}(\tilde{c}_{n(m)}, \tilde{a}),$$

we arrive, by (5.13) and (5.14), at a contradiction:

$$0 < \varepsilon = \lim_{m\to} \tilde{G}(\tilde{b}_m, \tilde{a}) \leqq k \cdot \lim m \to \infty \tilde{G}(\tilde{c}_{n(m)}, \tilde{a}) = 0,$$

and we are done. □

THEOREM 5.15. *The Kuramochi compactification $(\tilde{R})^*_K$ of any special Tôki covering surface $\tilde{R}$ of a regular Riemann surface R is a one point compactification of $\tilde{R}$ so that*

$$(\tilde{R})^*_K = (\tilde{R})^*_S = (\tilde{R})^*_A. \tag{5.16}$$

PROOF: We consider the subregion W of $\tilde{R}$ obtained from $\tilde{R}$ by removing a closed parametric disc K. We denote by β the relative boundary of W, a circle, and by $\gamma := (\tilde{R})^*_K \setminus \tilde{R}$ so that W is bounded by β and γ. We only have to show that γ consists of a single point. As the table (4.2) shows, $\overline{W} = W \cup (\beta \cup \gamma)$, the closure of W in $(\tilde{R})^*_K$, lies above $(\tilde{R})^*_A \setminus K = (\tilde{R} \setminus K) \cup \{\infty\}$, where $\infty = \infty_{\tilde{R}}$ is the point at infinity of $\tilde{R}$. Thus γ lies over ∞. We denote by $N(\cdot, w)$ the relative Neumann kernel on W and $G(\cdot, w)$ the relative Green kernel on W so that both of $N(\cdot, w)$ and $G(\cdot, w)$ have vanishing boundary values on β. An N fundamental sequence $(w_n)_{n\in\mathbf{N}}$ in W (or rather in $\tilde{R}$) is a sequence of points $w_n \in W$ such that $w_n \to \infty$ and $N(\cdot, w_n)$ converges locally uniformly on W (so that on $W \cup \beta$) to a harmonic function $N(\cdot, (w_n)_{n\in\mathbf{N}})$ on W having boundary values zero on β. Two N fundamental sequences $(w_n)_{n\in\mathbf{N}}$ and $(w'_n)_{n\in\mathbf{N}}$ are said to be equivalent if $N(\cdot, (w_n)_{n\in\mathbf{N}}) \equiv N(\cdot, (w'_n)_{n\in\mathbf{N}})$ on W. If we denote by ζ an equivalence class of N fundamental sequences, then the totality of such ζ is γ and $N(\cdot, \zeta) = N(\cdot, (w_n)_{n\in\mathbf{N}})$ if $(w_n)_{n\in\mathbf{N}} \in \gamma$. This is the procedure of constructing $(\tilde{R})^*_K$ by the completion of $\tilde{R}$ with respect to the metric determined by the Neumann kernel. Thus we only have to show that $(N(\cdot, w_n))_{n\in\mathbf{N}}$ converges to a fixed harmonic function on W for any sequence $(w_n)_{n\in\mathbf{N}}$ in W converging to $\infty = \infty_{\tilde{R}}$ in $(\tilde{R})^*_A$. Let $HD(W; \partial W) := \{u \in HD(W) \cap C(W \cup \beta) : u|\beta = 0\}$. Then $HD(W; \partial W)$ is a Hilbert space under the inner product $D(u, v; W) := \int_W du \wedge *dv$ for u and v in $HD(W; \partial W)$. The Hilbert space $HD(W; \partial W)$ has a reproducing kernel $B(z, w)$, usually called the (harmonic) Bergman kernel on W, and it is seen that

$$B(z, w) = N(z, w) - G(z, w). \tag{5.17}$$

Since $\tilde{R}$ is regular, we can ascertain that W is also regular. We see that $HD(W;\partial W)$ is linearly isomorphic to $HD(\tilde{R})$ by the correspondence $u \leftrightarrow v$ with $u = v$ on the Royden harmonic boundary of δ of $\tilde{R}$. By Theorem 5.4, $HD(\tilde{R}) = \mathbb{R}$ or $\dim HD(\tilde{R}) = 1$. Then $\dim HD(W;\partial W) = 1$ so that $HD(W,\partial W) = \{cw | c \in \mathbb{R}\}$, where $w \in HD(W;\partial W)$ with $w = 1$ on the Royden harmonic boundary δ of $\tilde{R}$. Suppose $(w_n)_{n\in\mathbb{N}}$ is any N fundamental sequence in W. Observe that $G(\cdot, w_n) \to 0$ $(n \to \infty)$ in view of the regularity of $\tilde{R}$. Thus (5.17) assures that

$$\lim_{n\to\infty} B(\cdot, w_n) = \lim_{n\to\infty} N(\cdot, w_n) \tag{5.18}$$

locally uniformly on $W \cup \beta$. Since $B(\cdot, w_n) \in HD(W;\partial W) = \mathbb{R}w$, we see that $B(\cdot, w_n) = c_n w$ for some $c_n \in \mathbb{N}$. From (5.18), it follows the existence of a $c \in \mathbb{R}$ such that $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty} B(\cdot, w_n) = cw$. Hence (5.18) shows that $N(\cdot, w_n) \to cw$ uniformly in the vicinity of β. Thus

$$c\int_\beta *dw = \int_\beta *d(cw) = \lim_{n\to\infty}\int_\beta *dB(\cdot, w_n) = \lim_{n\to\infty}\int_\beta *dN(\cdot, w_n).$$

In view of the Stokes formula

$$0 = \int_{\beta+\alpha_n+\delta} *dN(\cdot, w_n) = \int_\beta *dN(\cdot, w_n) + \int_{\alpha_n} *dN(\cdot, w_n) + \int_\delta *dN(\cdot, w_n),$$

where α_n is a small circle centered at w_n. Since $*dN(\cdot, w_n) = 0$ on δ, and $\int_{\alpha_n} *dN(\cdot, w_n) = 1$, we deduce $\int_\beta *dN(\cdot, w_n) = -1$. Hence $c = -1/\int_\beta *dw$ so that

$$\lim_{n\to\infty} N(\cdot, w_n) = -\left(\int_\beta *dw\right)^{-1} w$$

for any N fundamental sequence $(w_n)_{n\in\mathbb{N}}$ so that γ is a singleton. □

As we have already remarked, an example of R for which $R^*_{\mathcal{R}} \not\twoheadrightarrow R^*_{\mathcal{M}}$ also works as an example for $R^*_{\mathcal{K}} \not\twoheadrightarrow R^*_{\mathcal{M}}$. However, (5.16) can be used to construct an example showing $R^*_{\mathcal{K}} \not\twoheadrightarrow R^*_{\mathcal{M}}$ *directly.* Let R be an open Riemann surface satisfying the following two conditions: R is regular; $R \notin \mathcal{O}_{HB}$. Concrete examples of such R as above are plenty; the simplest one is the unit disc $\mathbb{D}$. Then the special Tôki covering surface $\tilde{R}$ of R is the required example: $(\tilde{R})^*_{\mathcal{K}} \not\twoheadrightarrow (\tilde{R})^*_{\mathcal{M}}$. In fact, Theorem 5.15 shows that $(\tilde{R})^*_{\mathcal{K}} = \tilde{R} \cup \{\infty\}$. Since $\tilde{R} \notin \mathcal{O}_{HB}$ along with $R \notin \mathcal{O}_{HB}$, the Martin representation theorem assures that the Martin boundary $\gamma_{\mathcal{M}}$ of $\tilde{R}$ cannot be a singleton. Thus $\tilde{R} \cup \{\infty\}$ cannot lie over $\tilde{R} \cup \gamma_{\mathcal{M}}$ since there is no surjective mapping of $\{\infty\}$, a one point set, onto $\gamma_{\mathcal{M}}$, a multiple points set. Now every relation in the table (4.2) has at least one each publicized source (including our present paper) assuring its validity as we aimed for. Nevertheless we still strongly feel it would be extremely desirable to have another proof for the relation

$$(\tilde{R})^*_{\mathcal{M}} \not\twoheadrightarrow (\tilde{R})^*_{\mathcal{K}}$$

other than ingenious but formidably difficult one given by Kuramoci himself.

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