Semipositivity of relative canonical bundles via Kähler-Ricci flows

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Abstract
In this paper, we shall discuss the fact that the fiberwise Kähler-Ricci flow preserves the semipositivity on a smooth projective family. The full accounts will be given in [B-T].

1 Introduction
In [Ka1], Y. Kawamata proved a semipositivity of the direct image of a relative pluricanonical systems. The second author extended the result to the case of logpluricanonical systems in terms of the generalized Kähler-Einstein metric by using the method in [T4] ([T7]).

In February in 2010, the second author attended the talk given by R. Berman in Luminy about [B].

Inspired by this talk the authors began to work on the stability of the semipositivity of the fiberwise Kähler-Ricci flows on a smooth projective family. This enables us to provide the homotopy version of the semipositivity of relative canonical bundles (cf. Theorem 7). This provides us a new tool to explore the projective (or possibly) Kähler families. For example, as a consequence we may give an alternative proof of the quasiprojectivity of the moduli space of polarized varieties with semiample canonical sheaves.

This is a research announcement and the full accounts will be given in [B-T].

1.1 Kähler-Einstein metrics
Let $X$ be a compact Kähler manifold. It is important to construct a canonical Kähler metric on $X$.

Let $(X, \omega)$ be a compact Kähler manifold. $(X, \omega)$ is said to be Kähler-Einstein, if there exists a constant $c$ such that

$$\text{Ric}(\omega) = c \cdot \omega$$

holds, where the Ricci tensor: $\text{Ric}(\omega)$ is defined by

$$\text{Ric}(\omega) = -\sqrt{-1} \partial \overline{\partial} \log \det \omega.$$

This means that $X$ admits a Kähler-Einstein metrics, then $c_1(X)$ is positive or negative or 0.

**Theorem 1** ([A, Y1]) Let $X$ be a compact Kähler manifold.
(1) If $c_1(X) < 0$, then there exists a Kähler-Einstein metric $\omega$ such that

$$-\text{Ric}(\omega) = \omega.$$ 

(2) If $c_1(X) = 0$, for every Kähler class $c$, there exists a Ricci flat Kähler metric $\omega$ such that $[\omega] = c$ and

$$\text{Ric}(\omega) = 0.$$ 

1.2 Twisted Kähler-Einstein metrics

Let $X$ be a smooth projective variety defined over $\mathbb{C}$ and let $(L, h_L)$: a (singular) hermitian $\mathbb{Q}$-line bundle on $X$ with $\sqrt{-1} \Theta_{h_L} \geq 0$.

$\omega$ is said to be a twisted Kähler-Einstein metrics associated with $(L, h_L)$, if

$$-\text{Ric}(\omega) + \sqrt{-1} \Theta_{h_L} = \omega$$

holds in the sense of current.

Theorem 2 ([T7]) If $h_L$ is $C^\infty$ on a nonempty Zariski open subset and $I(h_L) \cong \mathcal{O}_X$. Then there exists a closed positive current $\omega$ on $X$ such that

(1) There exists a nonempty Zariski open subset $U$ of $X$ such that $\omega|U$ is $C^\infty$,

(2) $-\text{Ric}(\omega) + \sqrt{-1} \Theta_{h_L} = \omega$ holds on $U$,

(3) $(\omega^n)^{-1} \cdot h_L$ is an AZD of $K_X + L$. 

1.3 Bergman metrics

Let $X$ be a smooth projective variety and let $(L, h_L)$ be a singular hermitian line bundle on $X$. We set

$$K(X, K_X + L, h_L) := \sum |\sigma_i|^2,$$

where $\{\sigma_i\}$ is an orthonormal basis of $H^0(X, \mathcal{O}_X(K_X + L) \otimes I(h_L))$ with respect to the inner product:

$$(\sigma, \tau) := \int_X \sigma \cdot \bar{\tau} \cdot h_L.$$ 

We call $K(X, K_X + L, h_L)$ the Bergman kernel of $X$ with respect to $(L, h_L)$. If $|H^0(X, \mathcal{O}_X(K_X + L) \otimes I(h_L))|$ is very ample, then the pull back of the Fubini-Study metric

$$\omega := \sqrt{-1} \partial \bar{\partial} \log K(X, K_X + L, h_L)$$

is a Kähler form on $X$. We call it the Bergman metric on $X$ with respect to $(L, h_L)$. 

$\square$
1.4 Dynamical construction of K-E-metrics

Let $X$ be a smooth projective $n$-fold with ample $K_X$ and $(A, h_A)$ be a sufficiently ample line bundle with $C^\infty$-metric $h_A$. We set $K_1 = K(X, K_X + A, h_A), h_1 = K_1^{-1}$. And inductively we define

$$K_m = K(X, mK_X + A, h_{m-1}), h_m = K_m^{-1}$$

for $m \geq 2$. Then we have the following rather unexpected result.

**Theorem 3** ([T]) $dV_E = \lim_{m \to \infty} \sqrt[n]{(m!)^{-1}K_m}$ is the K-E volume form on $X$, i.e., $\omega_E = -\text{Ric}dV_E$ is K-E-form.

1.5 Kähler-Ricci flow

Let $X$ be a compact Kähler manifold and let $\omega_0$: $C^\infty$-Kähler form on $X$.

We consider the initial value problem:

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}(\omega(t)) - \omega(t)$$

(1)

on $X \times [0, T)$,

$$\omega(0) = \omega_0,$$

where $\text{Ric}(\omega(t)) = -\sqrt{-1}\partial\bar{\partial} \log \det \omega(t)$ and $T$ is the maximal existence time for the $C^\infty$-solution. This type of Kähler-Ricci flow was first considered by the second author in [T1]. Then by taking the exterior derivative of the both sides of (1),

$$[\omega(t)] = (1 - e^{-t})2\pi c_1(K_X) + e^{-t}[\omega_0] \in H^{1,1}(X, \mathbb{R})$$

Let $\mathcal{K}(X)$ denote the Kähler cone of $X$. Then the following holds:

**Proposition 1** ([T1])

$$T = \sup\{t|[\omega(t)] \in \mathcal{K}(X)\}$$

holds.

The next question is what happens on $\omega(t)$ after exiting the Kähler cone. Let $PE(X)$ denote the pseudoeffective cone $\subseteq H^{1,1}(X, \mathbb{R})$.

**Definition 1** Let $T$ be a closed positive $(1, 1)$ current on $X$. $T$ is said to be of minimal singularities, if for every closed positive $(1, 1)$-current $T'$ with $[T'] = [T]$, there exists a $L^1$-function $\varphi$ such that

$$T' = T + \sqrt{-1}\partial\bar{\partial}\varphi$$

and is bounded from above.

The following proposition is an easy consequence of [Le, p.26, Theorem 5].

**Proposition 2** Let $\eta \in PE(X)$ be a pseudoeffective class. Then there exists a closed positive $(1, 1)$-current $T_{\min}$ with minimal singularities which represents $\eta$. □
A closed semipositive current $T$ with $[T] \in PE(X)$ is said to be of almost minimal singularities if we write $T$ as $T = T_{\text{min}} + \sqrt{-1} \partial \overline{\partial} \varphi$ for some $\varphi \in L^1(X)$, $e^{-\varphi} \in L^p(X)$ holds for every $p \geq 1$.

For a pseudoeffective $\mathbb{R}$-line bundle $F$ on a smooth projective manifold $M$, we say that the decomposition:

$$F = P + N(P, N \in \text{Div}(M) \otimes \mathbb{R})$$

is said to be a Zariski composition, if there exists a closed semipositive $(1,1)$ current $T$ on $M$ such that

1. $T$ is a closed semipositive current of almost minimal singularities in $2\pi c_1(F)$,
2. $T_{\text{sing}} = 2\pi N$ in the sense of currents, where $T = T_{\text{abc}} + T_{\text{sing}}$ is the Lebesgue decomposition.

Let $X$ be a smooth projective variety with pseudoeffective $K_X$. Then we have the following lemma by [B-C-H-M].

**Lemma 1** There exists a sequence: $T = T_0 < T_1 < \cdots < T_j < \cdots$ such that for each $j$, there exists a modification $\pi_j : X_j \rightarrow X$ such that $\pi_j^*(e^{-t}L + (1 - e^{-t})K_X)$ admits a Zariski decomposition:

$$\pi_j^*(e^{-t}L + (1 - e^{-t})K_X) = P_t + N_t$$

such that $N_t$ is independent of $t \in [T_j, T_{j+1})$. \[ \square \]

Then we have the following theorem.

**Theorem 4** Let $X$ be a smooth projective variety with pseudoeffective canonical class. Let $(L, h_L)$ be a $C^\infty$-hermitian line bundle such that $\omega_0 := \sqrt{-1} \Theta_{h_L}$ is a Kähler form on $X$. Then the initial value problem:

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}(\omega(t)) - \omega(t) \quad \text{on} \quad X \times [0, \infty), \quad (2)$$

$\omega(0) = \omega_0$ has the unique long time solutiton $\omega(t)$ such that

1. For $t \in [T_j, T_{j+1})$, $\omega(t)$ is $C^\infty$ on a nonempty Zariski open subset $U(T_j)$ depending on $T_j \in [0, \infty)$ defined as in Lemma 1.
2. For $t \in [T_j, T_{j+1})$, $\omega(t)$ satisfies the equation (2) on $U(T_j)$.
3. $\omega(t)$ is a closed semipositive current with almost minimal singularity in $(1 - e^{-t})2\pi c_1(K_X) + e^{-t}c_1(L)$. \[ \square \]

2 Proof of Theorem 4

Let $X$ be a smooth projective variety with pseudoeffective canonical class and let $(L, h_L)$ be a $C^\infty$-hermitian line bundle on $X$ such that $\omega_0 = \sqrt{-1} \Theta_{h_L}$ is a Kähler form.
2.1 Discretization of Kähler-Ricci flows

Let $a$ be a positive integer. We consider the following successive equations:

$$a(\omega_{m,a} - \omega_{m-1,a}) = -\text{Ric}_{\omega_{m,a}} - \omega_{m,a}$$  \hspace{1cm} (3)

for $m \geq 1$ under the initial condition $\omega_{0,a} = \omega_0$. We see that the cohomology class $[\omega_{m,a}]$ satisfies the equations:

$$a([\omega_{m,a}] - [\omega_{m-1,a}]) = 2\pi c_1(K_X) - [\omega_{m,a}]$$  \hspace{1cm} (4)

Hence we see that

$$[\omega_{m,a}] = \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right)2\pi c_1(K_X) + \left(1 + \frac{1}{a}\right)^{-m}[\omega_0]$$  \hspace{1cm} (5)

We define the singular hermitian metric

$$h_{m,a} := n!(\omega_{m,a}^n)^{-\frac{1}{a+1}} \cdot h_{m-1}^{\frac{a}{m-a+1}}$$  \hspace{1cm} (6)

on

$$(1 - t_{m,a})L + t_{m,a}K_X,$$  \hspace{1cm} (7)

where

$$t_{m,a} = 1 - \left(1 + \frac{1}{a}\right)^{-m}.$$  \hspace{1cm} (8)

$$\omega(m,a) := t_{m,a}(-\text{Ric} \Omega) + (1 - t_{m,a})\omega_0$$  \hspace{1cm} (9)

Then the $\{u_{m,a}\}_{m=0}^{\infty}$ satisfies the successive differential equations:

$$a(u_{m,a} - u_{m-1,a}) = \log\frac{(\omega(m,a) + \sqrt{-1}\partial\overline{\partial}u_{m,a})^n}{\Omega} - u_{m,a}.$$  \hspace{1cm} (10)

Now we introduce the following notation:

$$\delta_a u_{m,a} := a(u_{m,a} - u_{m-1,a}),$$  \hspace{1cm} (11)

i.e., $\delta_a u_{m,a}$ denotes the (backward) difference at $u_{m,a}$.

Then (10) is denoted as:

$$\delta_a u_{m,a} = \log\frac{(\omega(m,a) + \sqrt{-1}\partial\overline{\partial}u_{m,a})^n}{\Omega} - u_{m,a}.$$  \hspace{1cm} (12)

Later we shall see that this equation corresponds to the parabolic Monge-Ampère equation:

$$\frac{\partial u}{\partial t} = \log\frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}u)^n}{\Omega} - u,$$  \hspace{1cm} (13)

where

$$\omega_t := (1 - e^{-t})(-\text{Ric} \Omega) + e^{-t}\omega_0$$  \hspace{1cm} (14)
with the initial condition: \( u = 0 \) on \( X \times \{0\} \).
And there are correspondences:
\[
\frac{m}{a} \leftrightarrow t, \ u_{m,a} \leftrightarrow u(t, a), \ \omega(m, a) \leftrightarrow \omega_t
\]
and
\[
\delta_a u_{m,a} \leftrightarrow \frac{\partial u}{\partial t}.
\]
We set
\[
T := \sup \{ t \in \mathbb{R} \mid 2\pi(1 - e^{-t})c_1(K_X) + e^{-t}[\omega_0] \in \mathcal{K} \}.
\] (15)
Since the Kähler-Ricci flow corresponds to the minimal model with scalings in [B-C-H-M] in an obvious manner, we have the following lemma.

**Lemma 2** ([B-C-H-M]) The followings holds:
(1) \( e^{-T} \in \mathbb{Q} \),
(2) \((1 - e^{-T})K_X + e^{-T}L\) is semample. \(\square\)

By Lemma 2, there exists a \( C^\infty \)-function \( \phi \) such that
\[
\omega_{T, \phi} := (1 - e^{-T})(\text{Ric} \Omega + \sqrt{-1}\partial\overline{\partial} \phi) + e^{-T}\omega_0
\] (16)
is a \( C^\infty \)-semipositive form on \( X \) and is strictly positive on a nonempty Zariski open subset of \( X \). We set
\[
\omega(m, a) := \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right)(\text{Ric} \Omega + \sqrt{-1}\partial\overline{\partial} \phi) + \left(1 + \frac{1}{a}\right)^{-m}\omega_0
\] (17)
\[
= \omega(m, a) + \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right)\sqrt{-1}\partial\overline{\partial} \phi
\]
We set
\[
m(a) := \sup \left\{ m \left| \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right)c_1(K_X) + \left(1 + \frac{1}{a}\right)^{-m}[\omega_0] \in \mathcal{K} \right. \right\}.
\] (18)
Then since
\[
\omega(m, a) = \frac{1 - (1 + \frac{1}{a})^{-m}}{1 - e^{-T}} \omega_{T, \phi} + \frac{(1 + \frac{1}{a})^{-m} - e^{-T}}{1 - e^{-T}}\omega_0.
\] (19)
for every \( m < m(a) \), \( \omega(m, a) \) is a \( C^\infty \)-Kähler form on \( X \) and for \( m = m(a) \), \( \omega(m, a) = \omega_{T, \phi} \) holds.

**Theorem 5** (3) has a smooth solution \( \omega_{m,a} \) as long as \( [\omega(m, a)] \in \mathcal{K} \). And (10) has \( C^\infty \)-solution as \( [\omega(m, a)] \in \mathcal{K} \). \(\square\)

**Lemma 3** Suppose that \( T \) is finite, then we see that
\[
\omega(T) := \lim_{t \uparrow T} \omega(t)
\]
exists in \( C^\infty \)-topology on \( X \setminus E \) and is a well defined as a limit of closed positive current on \( X \). \(\square\)
2.2 Beyond the Kähler cone

After exiting the Kähler cone, the singular solution of the Kähler-Ricci flow can be constructed as follows.

**Theorem 6** There exists a sequence of closed semipositive currents \( \{\omega_{m,a}\}_{m=0}^{\infty} \) such that

1. For every \( m \geq 0 \), \( \omega_{m,a} \) is a closed semipositive current on \( X \),
2. There exists a nonempty Zariski open subset \( U_m \) of \( X \) such that \( h_{m,a}|U_m \) is \( C^\infty \),
3. \( h_{m,a} \) is an AZD of the \( \mathbb{Q} \)-line bundle \( (1-t_{m,a})L+t_{m,a}K_X \),
4. \( \omega_{m,a} = \sqrt{-1} \Theta_{h_{m,a}} \) is a well defined closed semipositive current on \( X \),
5. \( \{\omega_{m,a}\}_{m=0}^{\infty} \) satisfies the equations (3) on \( U_m \). \( \square \)

The following lemma is a slight refinement of Lemma 1.

**Lemma 4** There exists a sequence of positive number \( T = T_0 < T_1 < \cdots < T_j < \cdots \) such that for every \( t \in [T_j, T_{j+1}) \)

1. There exists a modification \( \pi_j : X_j \to X \) such that \( \pi_j^*(e^{-t}L+(1-e^{-t})K_X) \) admits a Zariski decomposition:
   \[
   \pi_j^*(e^{-t}L+(1-e^{-t})K_X) = P_t + N_t (P_t, N_t \in \text{Div}(X_j) \otimes \mathbb{R}),
   \]
   where \( P_t \) is nef and \( N_t \) is effective and
   \[
   H^0(X_j, \mathcal{O}_{X_j}(\lfloor mP_j \rfloor)) \simeq H^0(X_j, \mathcal{O}_{X_j}(m\pi_j^*(e^{-t}L+(1-e^{-t})K_X)))
   \]
   holds for every \( m \) such that \( me^{-t} \in \mathbb{Z} \).
2. \( N_t \) is independent of \( t \in [T_j, T_{j+1}) \),
3. If \( e^{-t} \in \mathbb{Q} \), then \( P_t \) is semiample. \( \square \)

We set \( N_j := N_t(t \in [T_j, T_{j+1})) \) and \( \tau_j \) be the multivalued holomorphic section of \( N_j \) with divisor \( N_j \). Then there exists a \( C^\infty \)-hermitian metric \( \| \cdot \| \) such that \( \omega_{\tau_j} + \sqrt{-1} \partial \overline{\partial} \log \| \tau_j \| \) is a closed semipositive current. We set
\[
\phi_j := \log \| \tau_j \|^2.
\]
Suppose that we have already defined \( u_{0,a}(\phi_j) \) such that for every \( \varepsilon > 0 \), there exists a constant \( C(\varepsilon) \)
\[
\omega_{0,a}(\phi_j) \geq \varepsilon \phi_j + C(\varepsilon)
\]
holds. We set
\[
\omega_j(m,a) := \left( 1 - e^{-T_j} \left( 1 + \frac{1}{a} \right)^{-m} \right) (-\text{Ric} \Omega) + e^{-T_j} \left( 1 + \frac{1}{a} \right)^{-m} \omega_0.
\]
We consider the Ricci iteration:
\[
\delta_a u_{m,a}(\phi_j) = \log \frac{(\omega(m,a)_{\phi_j} + \sqrt{-1} \partial \overline{\partial} u_{m,a}(\phi_j))^n}{\Omega \cdot e^{-\phi_j}} - u_{m,a}(\phi_j).
\]
The rest of the proof is similar to the case \( t \in [0, T) \).
3 Semipositivity of a Kähler-Ricci flow

In this section we shall sketch the proof of the fact that the relative Kähler-Ricci flows preserve the semipositivity in the horizontal direction on projective families.

3.1 Main results

Let $f : X \to S$ be a smooth projective family and let $\omega$ be a relative Kähler form on $X$. We set $n := \dim X - \dim S$ and $k := \dim S$. We define the relative Ricci form $\text{Ric}_{X/S,\omega}$ of $\omega$ by

$$\text{Ric}_{X/S,\omega} = -\sqrt{-1}\partial\bar{\partial} \log (\omega^n \wedge f^*|ds_1 \wedge \cdots \wedge ds_k|^2),$$

(24)

where $(s_1, \cdots, s_k)$ is a local coordinate on $S$. Then it is easy to see that $\text{Ric}_{X/S,\omega}$ is independent of the choice of the local coordinate $(s_1, \cdots, s_k)$. The Kähler-Ricci flow preserves the semipositivity in the following sense.

**Theorem 7** Let $f : X \to S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let $L$ be an ample line bundle on $X$ and let $h_L$ be a $C^\infty$-hermitian metric on $L$ with strictly positive curvature. Suppose that there exists a $C^\infty$-relative volume form $\Omega$ on $f : X \to S$ such that $\text{Ric} \Omega + \sqrt{-1}\Theta_{h_L}$ is also a Kähler form on $X$. We set $\omega_0 := \sqrt{-1}\Theta_{h_L}$. We consider the normalized Kähler-Ricci flow:

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}_{X/S,\omega(t)} - \omega(t)$$

on $X$ with the initial condition $\omega(0) = \omega_0$, where $\text{Ric}_{X/S,\omega(t)}$ denotes the relative Ricci form of $\omega(t)$ on $X$.

Then $\omega(t)$ is a closed semipositive current on $X$ for every $t \in [0, \infty)$. □

In Theorem 7, the semipositivity of $\omega(t)$ corresponds to the pseudoeffectivity of $(1 - e^{-t})K_{X/S} + e^{-t}L$. And as $t$ goes to infinity, we observe that the relative canonical bundle $K_{X/S}$ is pseudoeffective.

Similarly we have the following theorem.

**Theorem 8** Let $f : X \to S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let $L$ be an ample line bundle on $X$ and let $h_L$ be a $C^\infty$-hermitian metric on $L$ with strictly positive curvature. Let $K$ be a closed semipositive current on $X$ such that $K$ is $C^\infty$ on a nonempty Zariski open subset of $X$ and $[K] \in 2\pi c_1(K_{X/S})$. We set $\omega_0 := \sqrt{-1}\Theta_{h_L}$. We consider the Kähler-Ricci flow:

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}_{X/S,\omega(t)} - K$$

on $X$ with the initial condition $\omega(0) = \omega_0$, where $\text{Ric}_{X/S,\omega(t)}$ denotes the relative Ricci form of $\omega(t)$ on $X$.

Then $\omega(t)$ is a closed semipositive current on $X$ for every $t \in [0, \infty)$. Moreover as $t$ goes to infinity, $\omega(t)$ converges to a current solution of $-\text{Ric}_{X/S,\omega(t)} = K$. □
3.2 Some conjecture for the Kähler case

We expect that the similar statement holds even in the case that $f : X \to S$ is a smooth Kähler fibration.

**Conjecture 1** Let $X$ be a compact Kähler manifold with pseudoeffective canonical bundle. And let $\omega_0$ be a $C^\infty$-Kähler form on $X$. Suppose that there exists a $C^\infty$-volume form $\Omega$ such that

$$\text{Ric} \Omega + \omega_0$$

is also a Kähler form on $X$. Then there exists a family of closed semipositive current $\omega(t)$ on $X$ such that

1. $\omega(0) = \omega_0$,
2. For every $T > 0$, there exists a nonempty Zariski open subset $U(T)$ depending on $T$ such that $\omega(t)$ is Kähler form on $U(T) \times [0, T)$,
3. $|\omega(t)| = 2\pi(e^{-t}|\omega_0| + (1 - e^{-t})c_1(K_X))$ holds for every $t \in [0, \infty)$,
4. On $U(t) \times [0, T) \omega(t)$ satisfies the differential equation:

$$\frac{\partial \omega(t)}{\partial t} = -\text{Ric}_{\omega(t)} - \omega(t).$$

This conjecture will lead us to the invariance of plurigenera in the Kähler case.

**Conjecture 2** Let $f : X \to S$ be a smooth Kähler family with pseudoeffective canonical bundles. Let $\omega_0$ be a $C^\infty$-Kähler form on $X$. Suppose that there exists a $C^\infty$-relative volume form $\Omega$ on $f : X \to S$ such that $\text{Ric} \Omega + \omega_0$ is also a Kähler form on $X$. We consider the normalized Kähler-Ricci flow:

$$\frac{\partial \omega(t)}{\partial t} = -\text{Ric}_{X/S, \omega(t)} - \omega(t)$$

on $X$ with the initial condition $\omega(0) = \omega_0$, where $\text{Ric}_{X/S, \omega(t)}$ denotes the relative Ricci form of $\omega(t)$ on $X$.

Then $\omega(t)$ is a closed semipositive current on $X$ for every $t \in [0, \infty)$. □

This conjecture will lead us to the invariance of plurigenera in the Kähler case.

4 Proof of Theorem 7

The essential technical difficulty here is the fact that we cannot apply the direct calculation of the variation, since the Kähler-Ricci flow in Theorem 4 has singularities. We overcome this difficulty by using the dynamical construction of the solution of the Ricci iterations as in [LC]
4.1 The relative Ricci iterations to the relative Kähler-Ricci flow

Let $f : X \to S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let $L$ be an ample line bundle on $X$ and let $h_L$ be a $C^\infty$-hermitian metric on $L$ with strictly positive curvature. Suppose that there exists a $C^\infty$-relative volume form $\Omega$ on $f : X \to S$ such that $\text{Ric} \Omega + \sqrt{-1} \Theta_{h_L}$ is also a Kähler form on $X$. We set $\omega_0 := \sqrt{-1} \Theta_{h_L}$. We consider the normalized Kähler-Ricci flow:

$$\frac{\partial \omega(t)}{\partial t} = -\text{Ric}_{X/S, \omega(t)} - \omega(t)$$  \hspace{1cm} (25)

on $X$ with the initial condition $\omega(0) = \omega_0$, where $\text{Ric}_{\omega(t)}$ denotes the relative Ricci form on $X$.

For every $s \in S$, we consider Lemma 1. Then by the invariance of the twisted plurigenra, we see that for every $C > 0$ the sequence

$$T = T_0 < T_1 < \cdots < T_j < \cdots < C$$  \hspace{1cm} (26)

in Lemma 1 are constant on a nonempty Zariski open subset $S(C)$ of $S$.

Suppose that we have already proven the (logarithmic) plurisubharmonic variation property of the solution $\omega(t)$ of (25) for every $t < C$ on $f^{-1}(S(C))$. Then the removable singularity theorem for plurisubharmonic function implies the logarithmic plurisubharmonic variation property of the solution $\omega(t)$ over the whole $X$.

Hence we may and do assume that the sequence $T_0 < \cdots < T_j < \cdots$ are constant over the whole $S$ without loss of generality. Moreover since the assertion of Theorem 7 is local in $S$, we may and do assume that $S$ is the unit open polydisk $\Delta^k$ in $\mathbb{C}^k$.

The plurisubharmonic variation property of the Ricci iteration is proven by the parallel argument as follows.

We set

$$m(a) := \sup \left\{ m \left| \left( 1 + \frac{1}{a} \right)^{-m} > e^{-T_0} \right. \right\}. \hspace{1cm} (27)$$

First we shall consider the relative Ricci iteration:

$$\delta_a \omega_{m,a} = -\text{Ric}_{\omega_{m,a}/S} - \omega_{m,a}, \omega_{0,a} = \omega_0$$  \hspace{1cm} (28)

on $X$ for $0 \leq m < m(a)$. This is equivalent to the fiberwise Ricci iteration:

$$\delta_a \omega_{m,a,z} = -\text{Ric}_{\omega_{m,a}/S,s} - \omega_{m,a,s}, \omega_{0,a} = \omega_0|X_s,$$  \hspace{1cm} (29)

on $X_s$ for $0 \leq m < m(a)$. Then by the proof of Theorem 4, letting $a$ tends to infinity, we may construct the solution of the relative Kähler-Ricci flow:

$$\frac{\partial \omega(t)}{\partial t} = -\text{Ric}_{X/S, \omega(t)} - \omega(t)$$  \hspace{1cm} (30)

on $X \times [0, T_0)$.

Then as in the previous section, we may continue this process beyond the critical time $T_0$ and we obtain the long time existence of the current solution of the relative Kähler-Ricci flow on $X$. 
4.2 Auxiliary Ricci iterations

We prove Theorem 7 by decomposing the Ricci iterations by a dynamical system of Bergman kernels and apply the plurisubharmonic variation properties of Bergman kernels due to Berndtsson. The main difficulty is to deal with \( \mathbb{Q} \)-line bundles. We deal with \( \mathbb{Q} \)-line bundles in terms of the auxiliary Ricci iterations.

**Lemma 5** For every \( 0 \leq m \leq m(a) \), \( \omega_{m,a} \) is semipositive on \( X \). \( \square \)

We prove Lemma 5 by induction on \( m \).

For \( m = 0 \) \( \omega_{0,a} = \omega_{0} \) is a Kähler form on \( X \) by the assumption. Hence Lemma 5 holds for \( m = 0 \). Suppose that \( \omega_{m,a} \) is semipositive on \( X \). We shall prove that \( \omega_{m+1,a} \) is also semipositive on \( X \).

To prove this assertion, we consider the auxiliary Ricci iteration which connects \( \omega_{m,a} \) and \( \omega_{m+1,a} \).

First we define the \( \mathbb{Q} \)-line bundle \( L_{m} \) by

\[
L_{m} := \left( 1 - \left( 1 + \frac{1}{a} \right)^{-m} \right) K_{X/S} + \left( 1 + \frac{1}{a} \right)^{-m} L.
\]  

(31)

Let \( q = q(m+1) \) be a postive integer such that \( qL_{m+1} \) is a genuine line bundle on \( X \). Since

\[
L_{m+1} = \left( 1 - \left( 1 + \frac{1}{a} \right)^{-(m+1)} \right) K_{X/S} + \left( 1 + \frac{1}{a} \right)^{-(m+1)} L
\]  

is of the form \( \beta(K_{X/S} + \alpha L) \) for some positive rational numbers \( \alpha \) and \( \beta \). By B-C-H-M, we have that the relative logcanonical ring:

\[
R(X, K_{X/S} + \alpha L) = \oplus_{\nu=0}^{\infty} f_{*} \mathcal{O}_{X}(\lfloor \nu(K_{X/S} + \alpha L) \rfloor)
\]  

is a finitely generated algebra over \( \mathcal{O}_{S} \). By the invariance of twisted plurigeera, we see that each \( f_{*} \mathcal{O}_{X}(\lfloor \nu(K_{X/S} + \alpha L) \rfloor) \) is a vector bundle over \( S \) which is biholomorphic to the unit open polydisk \( \Delta^{k} \). We take a sufficiently large positive integer \( \nu_{0} \) and take a set of generators \( \{\sigma_{i}\} \) of \( f_{*} \mathcal{O}_{X}(\nu_{0}!(K_{X/S} + \alpha L)) \) (In this case \( K_{X/S} + \alpha L \) is relatively ample. But later we also consider the case \( K_{X/S} + \alpha L \) is big, but not relatively ample). Then we set

\[
h_{m,a,0} := \left( \sum_{i} |\sigma_{i}|^{2} \right)^{-\frac{\beta}{\nu_{0}}} \left( \sum_{i} |\sigma_{i}|^{2} \right)^{-\frac{\alpha}{\nu_{0}}}
\]  

(32)

and

\[
\omega_{m,a,0} := \sqrt{-1} \Theta_{h_{m,a,0}}.
\]  

(33)

Then \( h_{m,a,0} \) is a hermitian metric of \( L_{m+1} = \beta(K_{X/S} + \alpha L) \) with semipositive curvature on \( X \). Now we shall consider the following Ricci iteration:

\[
-Ric_{\omega_{m,a,\ell}} + (q - a - 1)\omega_{m,a,\ell-1} + a\omega_{m,a} = q\omega_{m,a,\ell}
\]  

(34)

for \( \ell \geq 1 \). The following lemma follows entirely the same way as the dynamical construction of Kähler-Einstein metrics.
Lemma 6 \( \lim_{\ell \to \infty} \omega_{m,a,\ell} \) exists in \( C^\infty \)-topology on \( X \). And
\[
\lim_{\ell \to \infty} \omega_{m,a,\ell} = \omega_{m+1,a}
\] (35)
holds. \( \Box \)

We use this auxiliary Ricci iteration to connect \( \omega_{m,a} \) and \( \omega_{m+1,a} \) by a dynamical system of Bergman kernels. This method is exactly the same one in [T7].

4.3 Dynamical systems of Bergman kernels

To prove the semipositivity of \( \omega(t) \) on \( X \) for \( t \in [0, T_0] \), it is enough to prove the following lemma.

Lemma 7 \( h_{m,a} \) has semipositive curvature on \( X \). \( \Box \)

We now use the strategy as in [T7]. We shall prove Lemma 7 by induction on \( m \). Since \( h_L \) has positive curvature, \( h_{0,a} = h_L \) has semipositive curvature.

Suppose that we have already proven that \( h_{m-1,a} \) has semipositive curvature.

Let \( A \) be a sufficiently ample line bundle on \( X \) and let \( h_A \) be a \( C^\infty \)-hermitian metric on \( X \) with strictly positive curvature.

Now we shall define the metric on \( L_{m+1} \) by
\[
h_{m,a,\ell}|X_s = h_{m,a,\ell,s}(s \in S).
\]
(36)

By induction on \( \ell \), we shall prove the following lemma.

Lemma 8 \( h_{m,a,\ell} \) has semipositive curvature on \( X \) for every \( \ell \geq 0 \). \( \Box \)

Proof of Lemma 8. By the construction (cf. (32)), \( h_{m,a,0} \) has semipositive curvature.

Suppose that we have already proven that \( h_{m,a,\ell-1} \) is a hermitian metric with semipositive curvature on \( X \). For every \( s \in S \), we shall consider the dynamical system of Bergman kernels as follows. We set
\[
K_{1,s} := K \left( X_s, A + K X_s + (q - a - 1)L_{m+1} + aL_m|X_s), h_A \cdot h_{m,\ell-1}^{q-a-1} \cdot h_{m,a}|X_s \right)
\]
and
\[
h_{1,s} := K_{1,s}^{-1}
\]
(37)
(38)

Suppose that we have already constructed \( K_{p-1,s} \) and \( h_{p-1,s} \) for some \( p \geq 2 \). Then we define \( K_{p,s} \) and \( h_{p,s} \) by
\[
K_{p,s} := K \left( X_s, A + p(K X_s + (q - a - 1)L_{m+1} + aL_m|X_s), h_{m,\ell-1}^{q-a-1} \cdot h_{m,a} \cdot h_{p-1}|X_s \right)
\]
(39)

and
\[
h_{p,s} := \frac{1}{K_{p,s}}
\]
(40)

Similarly as in [T4, T7] we have the following lemma.
Lemma 9

\[ K_{\infty,s} := \lim_{p \to \infty} (p!)^{-n} h_A \cdot K_{p,s} \]

exists in \( L^1 \)-topology and

\[ h_{m,a,\ell,s} := K_{\infty,s}^{-1} \]

is a \( C^\infty \)-hermitian metric on \( L_{m+1}|X_s \). And the curvature

\[ \omega_{m,a,\ell,s} := \sqrt{-1} \Theta_{h_{m,a,\ell,s}} \]

satisfies the differential equation:

\[ -\text{Ric}_{\omega_{m,a,\ell,s}} + (q-a-1)\omega_{m,a,\ell-1,s} + a\omega_{m,a,s} = q\omega_{m,a,\ell,s} \]

on \( X \).

We define the relative Bergman kernel \( K_p \) on \( X \) by

\[ K_p|X_s = K_{p,s} \]

Then \( h_p = K_p^{-1} \) is a hermitian metric with semipositive curvature on \( A + p(K_{X/S} + (q-a-1)L_{m+1} + aL_m) \) by induction on \( p \) by the following theorem mainly due to B. Berndtsson.

**Theorem 9** ([B1, B2, B3, B-P]) Let \( f : X \to S \) be a projective family of projective varieties over a complex manifold \( S \). Let \( S^0 \) be the maximal nonempty Zariski open subset such that \( f \) is smooth over \( S^0 \).

Let \((L, h_L)\) be a pseudo-effective singular hermitian line bundle on \( X \).

Let \( K_s := K(X_s, K_X + L|X_s, h|X_s) \) be the Bergman kernel of \( K_{X,s} + (L|X_s) \) with respect to \( h|X_s \) for \( s \in S^0 \). Then the singular hermitian metric \( h \) of \( K_{X/S} + L|f^{-1}(S^0) \) defined by

\[ h|X_s := K_s^{-1}(s \in S^0) \]

has semipositive curvature on \( f^{-1}(S^0) \) and extends to \( X \) as a singular hermitian metric on \( K_{X/S} + L \) with semipositive curvature in the sense current.

Now we prove the semipositivity of \( \sqrt{-1} \Theta_{h_p} \) by induction on \( p \). First the semipositivity of \( \sqrt{-1} \Theta_{h_1} \) follows from Theorem 9 by the assumption that \( \sqrt{-1} \Theta_{h_{m,a,\ell-1}} \) and \( \sqrt{-1} \Theta_{h_{m-1,a}} \) are semipositive. Suppose that we have already proven the semipositivity of \( h_{p-1} \) for some \( p \geq 2 \). We note that \( h_{p-1}, h_{m,a,\ell-1} \) and \( h_{m,a} \) has semipositive curvature on \( X \) by the induction assumption. Then by the inductive definition of \( h_p \) (cf. (39) and (40)) and Theorem 9, we see that \( \sqrt{-1} \Theta_{h_p} \) is also semipositive.

Hence by induction, we see that \( \{h_{p}\}^\infty_{p=1} \) has semipositive curvature on \( X \). Then by Lemma 9, we see that \( h_{m,a,\ell} \) has semipositive curvature. This completes the proof of Lemma 8.

By Lemmas 6 and 8, we see that \( h_{m+1} \) is a metric on \( L_{m+1} \) with semipositive curvature. Hence by induction on \( m \), we complete the proof of Lemma 7.
Now by Lemma 7 and the proof of Theorem 1, we see that $\omega(t)$ is semipositive on $X$ for $t \in [0, T_0]$.

Now we complete the proof of Theorem 7 by repeating the similar argument inductively for $t \in [T_j, T_{j+1}](j \geq 0)$. This completes the proof of Theorem 7.

References


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