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Stabilization of unstable periodic orbits with complex characteristic multipliers via delayed feedback control

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1 Introduction

The delayed feedback control (DFC) is proposed by Pyragas [2] as a method of chaos controls. To stabilize unstable periodic orbits of a differential equation

(1) $$x'(t)=f(x) \quad (x\in \mathbb{R}^{d}),$$

DFC uses the feedback control input given by the difference between the delayed state and the current state:

(2) $$y'(t)=f(y)+K(y(t-\omega)-y(t)).$$

Here, K is a $d\times d$ matrix and we call it the **gain matrix**, and ω is a time delay. If this time delay coincides with the period of one of the unstable periodic orbits of Eq.(1), then the solution of Eq.(1) is also a solution of Eq.(2). To achieve the stabilization of the desired unstable periodic orbit, the time of delay ω and the gain matrix K should be adjusted. In many works ω and K are adjusted in numerical experiment, by using the amplitude of the feedback control input as a criterion: If the amplitude tends to zero as t increases, then DFC succeeds. But there are few analytical results how to choose ω and K to achieve the stabilization.

Recently, in [1], we give an analytical result on this subject for a special case where the gain matrix is given by $K=kE$ ($k\in \mathbb{R}$, E is $d\times d$ identity matrix). To judge the stability of a periodic orbit we can use characteristic multipliers of the first variational equation around the orbit. But there are few information about characteristic multipliers or periodic operators for delay differential equations. In the paper, we have introduced newly a mapping named "C-map". C-map gives a relationship between characteristic multipliers of the first variational equations around the unstable periodic orbit of Eqs.(1) and (2) and enables to judge the stability of periodic orbits of Eq.(2). This is a very useful and strong tool. In fact, we also give the sufficient conditions of for the DFC to succeed or not; If Eq. (1) has at least one characteristic multiplier lager than 1, then the DFC with the gain $K = kE$ does not work; if all the unstable characteristic multipliers of Eq. (1) are contained in a given region the DFC with the gain $K = kE$ succeeds.

The aim of the present article is to give a condition for the DFC to succeed in the case where Eq. (1) has an unstable complex characteristic multiplier.

2 C-map theorem

We will summarize the results in [1]. Let $\phi(t)$ be an unstable periodic orbit of Eq.(1) and ω the period of $\phi(t)$. Then $\phi(t)$ is also a solution of Eq.(2). Analytically, we could consider that the stabilization of $\phi(t)$ is successful when $\phi(t)$ is orbitally stable as a solution of Eq.(2). Consider the first variational equations around $\phi(t)$ such that

$$\text{(L)} \qquad x'(t) = A(t)x(t)$$

and

$$\text{(DFL)} \qquad y'(t) = A(t)y(t) + K(y(t-\omega) - y(t)),$$

where $A(t) = Df(\phi(t))$ is a Jacobian matrix of f. Obviously, $A(t)$ is an ω-periodic matrix. The stability of $\phi(t)$ as a solution of Eqs.(1) and (2) are governed by the characteristic multipliers (which will be defined in the next paragraph) of Eqs.(L) and (DFL), respectively.

Let $T(t,s)$ be the solution operator of Eq.(L) defined on $\mathbb{C}^d$. The eigenvalue μ of the periodic operator $T(\omega,0)$, i.e., $\mu \in \sigma(T(\omega,0))$, is called a **characteristic multiplier of Eq.**(L). Let $U(t,s)$ be the solution operator of Eq.(DFL) defined on $C([-\omega,0],\mathbb{C}^d)$. Note that the periodic operator $U(\omega,0)$ is a compact operator. The point spectrum ν of the operator, i.e., $\nu \in P_\sigma(U(\omega,0)) = \sigma(U(\omega,0)) \setminus \{0\}$, is called a **characteristic multiplier of Eq.**(DFL). Since $f(\phi(t))$ is a periodic solution of both Eqs.(L) and (DFL), $1 \in \sigma(T(\omega,0))$ and $1 \in P_\sigma(U(\omega,0))$. If any other points $\nu \in P_\sigma(U(\omega,0)) \setminus \{1\}$ have modulus less than one, i.e. $|\nu| < 1$, the periodic orbit $\phi(t)$ of the nonlinear Eq.(2) is orbitally stable; in this case we say that **the stabilization of $x = \phi(t)$ by DFC with feedback gain K succeeds.**

In [1] we obtained a relationship between the characteristic multipliers of Eq. (L) and those of Eq. (DFL), in the case when $K = kE$. Here E is an $n \times n$ identity matrix.

Theorem A ([1, Cor. 5.3]). *Let $K = kE$. Then $\nu \in P_\sigma(U(0))$ if and only if $g_k(\nu) \in \sigma(T(0))$. Here*

$$g_\kappa(z) := ze^{\omega(1-z^{-1})\kappa}, \qquad z \in \mathbb{C} \setminus \{0\}.$$

We call the map g_κ **characteristic map, in short, C-map**. C-map theorem enables us to estimate the characteristic multipliers of Eq. (DFL) from those of Eq. (L). In [1], by using this theorem, we presented a design method of the feedback gain K such that the DFC succeeds.

Let us introduce the design method. First, we classify $\sigma(T(0))$ as follows.

$$\sigma_U := \{\mu \in \sigma(T(0)) : |\mu| > 1\}$$
$$\sigma_N := \{\mu \in \sigma(T(0)) : |\mu| = 1\}$$
$$\sigma_S := \{\mu \in \sigma(T(0)) : |\mu| < 1\}$$

We note that σ_U is not empty, because $T(0)$ is the periodic operator of the first variational equation (L) around the UPO $\phi(t)$. We also note that the elements of σ_U indicate the type of the instability of $\phi(t)$.

Next, we will define a new function which is used in the design method. Consider a mapping from $s \in (0, \pi)$ to $\alpha \in (0, 2)$

$$\alpha = \frac{s(1+\cos s)}{\sin s}, \qquad 0 < s < \pi.$$

This is one-to-one and onto mapping, because

$$\frac{d\alpha}{ds} = \frac{(1+\cos s)(\sin s - s)}{\sin^2 s} < 0, \qquad \text{for} \quad 0 < s < \pi,$$

and

$$\lim_{s\to 0} \alpha = 2, \quad \lim_{s \to \pi} \alpha = 0.$$

So, there exists an inverse mapping and we write it as $s(\alpha)$. Using this mapping we define the new function $\beta(\alpha)$ as follows.

$$\beta(\alpha) = \frac{2s(\alpha)}{\sin s(\alpha)}, \qquad \text{for} \quad 0 < \alpha < 2.$$

Our design method of the feedback gain K such that the DFC succeeds is as follows.

Theorem B ([1, Th. 7.6]). Assume $k \in \mathbb{R}$ and $K = kE$.

(i) If there exists $\mu \in \sigma_U$ such that $\mu > 1$, then there exists $\nu \in P_\sigma(U(0))$ such that $\nu > 1$.

(ii) Let $\sigma_U \subset (-e^2, -1)$ and $\alpha_0 = \max\{\log|\mu| : \mu \in \sigma_U\}$. For any k, if

$$\frac{\alpha_0}{2\omega} < k < \frac{\beta(\alpha_0)}{2\omega}$$

holds, then $|\nu| < 1$ or $\nu = 1$ for any $\nu \in P_\sigma(U(0))$.

The statement (i) gives the limitation of DFC for unstable periodic orbits whose characteristic multiplier more than 1. The statement (ii) gives a design method of the feedback gain K for DFC to succeed. But $\sigma_U \subset (-e^2, -1)$ means that all the unstable characteristic multipliers are real. Therefore Theorem B is not applicable if there exists a complex unstable characteristic multiplier.

3 Complex characteristic multipliers

Before stating our main statement, we have to make some preparations.

Consider a mapping from $\hat{y} \in [0, \pi]$ to $s \in [0, \pi]$

$$s = \hat{y} + \sin \hat{y}.$$

This is one-to-one and onto mapping, because

$$\frac{ds}{d\hat{y}} = 1 + \cos \hat{y} > 0, \quad \text{for } \hat{y} \in [0, \pi),$$

and

$$s = \begin{cases} 0, & \text{if } \hat{y} = 0 \\ \pi, & \text{if } \hat{y} = \pi. \end{cases}$$

So, there exists an inverse mapping and we write it as $\hat{y}(s)$.

Let $b \in (0, \pi)$ be fixed. Consider a mapping from $s \in (\hat{y}(b), b)$ to $\alpha \in (0, 1 - \cos \hat{y}(b))$

$$\alpha = \frac{(b-s)(1-\cos s)}{\sin s}, \qquad \hat{y}(b) < s < b.$$

This is one-to-one and onto mapping, because

$$\frac{d\alpha}{ds} = \frac{b - s - \sin s}{1 + \cos s} < 0, \qquad \text{for} \quad \hat{y}(b) < s < b,$$

and

$$\lim_{s \to \hat{y}(b)} \alpha = 1 - \cos \hat{y}(b), \quad \lim_{s \to b} \alpha = 0.$$

So, there exists an inverse mapping and we write it as $s(\alpha)$. Using this mapping we define the new function $\hat{\beta}(\alpha)$ as follows.

$$\hat{\beta}(\alpha) = \frac{b - s(\alpha)}{\sin s(\alpha)}, \qquad \text{for} \quad 0 < \alpha < 1 - \cos \hat{y}(b).$$

The following statement might be true.

Proposition 1. Assume $k \in \mathbb{R}$ and $K = kE$, and

$$\sigma_U \subset \{z = Re^{is} \in \mathbb{C} \colon s \in [-\pi, \pi] \text{ and } 1 < R < e^{1-\cos \hat{y}(|s|)}\}.$$

For any $\mu \in \sigma_U$ and $k \in \mathbb{R}$, if either

$$b \neq \pi \quad \frac{a}{\omega(1 - \cos \hat{y}(b))} < k < \frac{\hat{\beta}(a)}{\omega};$$

or

$$b = \pi, \quad \frac{a}{2\omega} < k < \frac{\beta(a)}{2\omega}$$

holds, then $|\nu| < 1$ or $\nu = 1$ for any $\nu \in P_\sigma(U(0))$. Here $a = \log |\mu|$ and $b = |\arg \mu| \in [0, \pi]$.

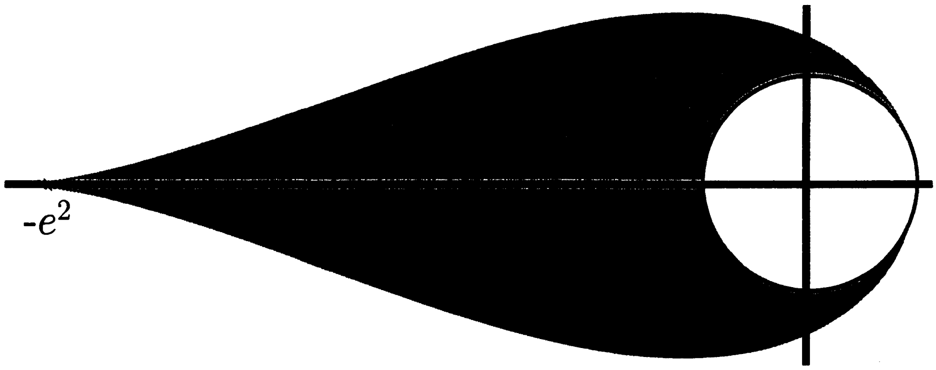

Figure 1. The domain of the characteristic multipliers in the complex plane which could be stabilized by DFC with the feedback gain $K = kE$.

References

[1] R. Miyazaki, T. Naito, and J. S. Shin, Delayed feedback control by commutative gain matrices, *SIAM J. Math. Anal.*, **43**, (2011), 1122–1144.

[2] Pyragas, K., Continuous control of chaos by self-controlling feedback, *Phys. Lett. A*, (1992), **170**, 421–428.