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Precise asymptotic behavior of positive solutions of generalized Thomas – Fermi differential equations

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1 Introduction

1.1. This paper is concerned with positive solutions of generalized Thomas-Fermi differential equations of the form

$$(|x'|^{\alpha}\operatorname{sgn} x')' = q(t)|x|^{\beta}\operatorname{sgn} x, \tag{A}$$

where α and β are positive constants and $q : [a, \infty) \to (0, \infty)$ is a continuous function. Equation (A) is said to be *half-linear, super-half-linear* or *sub-half-linear* according as $\alpha = \beta$, $\alpha < \beta$ or $\alpha > \beta$.

Our analysis will be performed in the framework of *regular variation* (in the sense of Karamata).

For the readers benefit we recall here the definition and some basic properties of regularly varying functions.

A measurable function $f : (0, \infty) \to (0, \infty)$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for } \forall\lambda > 0,$$

or equivalently, if it is expressed in the form

$$f(t) = c(t)\exp\left\{\int_{t_0}^{t} \frac{\delta(s)}{s} ds\right\}, \quad t \geqq t_0,$$

for some $t_0 > 0$ and for some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty), \qquad \lim_{t\to\infty} \delta(t) = \rho.$$

We denote by RV(ρ) the set of all regularly varying functions of index ρ. If in particular $\rho = 0$, then we use the symbol SV for RV(0) and refer to members of SV as *slowly varying functions.*

By definition a function $f(t) \in$ RV(ρ) can be expressed as

$$f(t) = t^{\rho} g(t) \quad \text{with} \quad g(t) \in \mathrm{SV} \tag{1.1}$$

and so the class of slowly varying functions is of fundamental importance in the theory of regularly varying functions. If $c(t) \equiv c_0$, then $f(t)$ is called a *normalized* regularly varying function of index ρ. Furthermore, a function $f(t) \in$ RV(ρ) satisfying

$$\lim_{t\to\infty} \frac{f(t)}{t^{\rho}} = \text{const} > 0 \tag{1.2}$$

is termed a *trivial* regularly varying function of index ρ, and a *nontrivial* regularly varying function of index ρ otherwise. The set of all trivial (resp. nontrivial) regularly varying functions of index ρ is denoted by tr-RV(ρ) (resp. ntr-RV(ρ)).

We quote the following result - Karamata integration theorem, which is frequently used throughout the paper and is of the highest importance in the application of regularly varying functions.

Proposition 1.1. *Let $L(t)$ be a slowly varying function. Then we have as $t \to \infty$*

(i) $$\int_{t_0}^{t} s^{\gamma} L(s)ds \sim \frac{t^{\gamma+1}}{\gamma+1} L(t) \quad \textit{if} \quad \gamma > -1;$$

(ii) $$\int_{t}^{\infty} s^{\gamma} L(s)ds \sim -\frac{t^{\gamma+1}}{\gamma+1} L(t) \quad \textit{if} \quad \gamma < -1;$$

(iii) *If $\gamma = -1$ the occurring integrals are new slowly varying functions.*

A comprehensive treatment of the theory and application of regular variation is presented by Bingham, Goldie and Teugels in [1]. Also, properties often needed for the analysis of regularly varying solutions of differential equations can be found in [4].

1.2. The study of the half-linear equation (A) in the framework of regular variation (in the sense of Karamata) was first attempted by Jaroš, Kusano and Tanigawa [2] who obtained the following result.

Proposition 1.2. *Equation* (A) *with $\alpha = \beta$ possesses a regularly varying solution of index ρ if and only if*

$$\rho \in (-\infty, 0] \cup [1, \infty) \tag{1.3}$$

and

$$\lim_{t\to\infty} t^{\alpha} \int_t^{\infty} q(s)ds = |\rho|^{\alpha-1}\rho(\rho-1).$$

The non-half-linear case of equation (A) has been analyzed from the viewpoint of regular variation in a recent paper [3] by the present authors, in which the existence of slowly and regularly varying solutions of index 1 and their asymptotics are studied for both super-half-linear and sub-half-linear cases of (A).

Here we consider the sub-half-linear case of equation (A) and assume that the coefficient $q(t)$ is regularly varying of index $\sigma \in \mathbb{R}$ i.e.

$$\alpha > \beta, \quad q(t) \in \mathrm{RV}(\sigma) \quad \text{i.e.} \quad q(t) = t^{\sigma} L(t), \ L(t) \in \mathrm{SV}. \tag{1.4}$$

We establish some simple necessary and sufficient conditions for the existence of nontrivial regularly varying solutions $x(t)$ of index ρ in the range (1.3) using Schauder-Tychonoff fixed point theorem, and determine the precise asymptotic behavior for $t \to \infty$ of such solutions $x(t)$.

These results (in Section 3) are preceded by a short section showing that information about the surprisingly simple and clear structure of regularly varying solutions of (A) can be obtained on the basis of Proposition 1.2.

Let us emphasize that in the fundamental paper on the subject by Mizukami, Naito and Usami [5] cases (2.1) a) and (2.2) a) (i.e. the trivial SV and RV(1) ones) are completely resolved, both for the super- and sub- half-linear equation (A) with the continuity of $q(t)$ as the basic hypothesis on $q(t)$ (i.e. without the regular variation).

2 Structure of regularly varying solutions

Let $x(t)$ be a positive solution of equation (A) on $[t_0, \infty)$. Then, we see from (A) that $|x'(t)|^{\alpha-1}x'(t)$ is increasing for $t \geqq t_0$, which means that $x'(t)$ is either positve or negative for all large t. If $x'(t)$ is positive, then it is increasing and tends to a positive constant or grows to infinity as $t \to \infty$, which implies that either

$$\text{a)} \ \lim_{t\to\infty} \frac{x(t)}{t} = \text{const} > 0 \quad \text{or} \quad \text{b)} \ \lim_{t\to\infty} \frac{x(t)}{t} = \infty, \tag{2.1}$$

while if $x'(t)$ is negative, then $x'(t) \to 0$ as $t \to \infty$, and $x(t)$ is decreasing and satisfiess either

$$\text{a)} \ \lim_{t\to\infty} x(t) = \text{const} > 0 \quad \text{or} \quad \text{b)} \ \lim_{t\to\infty} x(t) = 0. \tag{2.2}$$

Note that if $x(t) \in \mathrm{RV}(\rho)$ is an increasing (resp. a decreasing) solution of (A), then (2.1) (resp. (2.2)) implies that $\rho \geqq 1$ (resp. $\rho \leqq 0$).

This shows that the restriction (1.3) on ρ holds also for sub- (and super) half-linear cases.

Let $\mathcal{R}_+$ (resp. $\mathcal{R}_-$) denote the totality of increasing (resp. decreasing) regularly varying solutions of (A). The symbol $\mathcal{R}(\rho)$ is used to mean the set of all regularly varying solutions of index ρ of (A). Then, from what is remarked above we have the following schematic representation for $\mathcal{R}_+$ and $\mathcal{R}_-$:

$$\mathcal{R}_+ = \bigcup_{\rho \geqq 1} \mathcal{R}(\rho), \qquad \mathcal{R}_- = \bigcup_{\rho \leqq 0} \mathcal{R}(\rho). \tag{2.3}$$

It turns out, however, that use of Proposition 1.2 for half-linear equations enables us to make a deeper analysis of (2.3), depicting a surprisingly simple picture of the structure of increasing and decreasing regularly varying solutions of the sub-half-linear equation (A).

Theorem 2.1. *Let $\alpha > \beta$ and suppose that $q(t)$ is a regularly varying function. Then, the structure of regularly varying solutions of* (A) *is as follows:*

$$\mathcal{R}_+ = \mathcal{R}(\rho_+) \quad \textit{for some single} \quad \rho_+ \in [1, \infty), \tag{2.4}$$

$$\mathcal{R}_- = \mathcal{R}(0) \cup \mathcal{R}(\rho_-) \quad \textit{for some single} \quad \rho_- \in (-\infty, 0). \tag{2.5}$$

Remark 2.2. The class $\mathcal{R}_+$ is always non-empty, and the index ρ_+ in (2.4) is uniquely determined by $q(t)$ and its regularity index. It may happen that $\mathcal{R}_-$ is empty, but if $\mathcal{R}_- \neq \emptyset$, then the subclass $\mathcal{R}(0)$ in (2.5) is always non-empty, while $\mathcal{R}(\rho_-)$ may or may not be empty. In case $\mathcal{R}(\rho_-) \neq \emptyset$ the index ρ_- is uniquely determined by $q(t)$ and its regularity index.

3 The case $\rho \neq 0,\ 1$ and the case $\rho = 0,\ 1$

We prove

Theorem 3.1. *Suppose that* (1.4) *holds; then equation* (A) *possesses regularly varying solutions $x(t)$*

a) *of index* $\rho < 0$ *if and only if* $\sigma < -\alpha - 1$,

b) *of index* $\rho > 1$ *if and only if* $\sigma > -\beta - 1$.

In both cases ρ *is given by*

$$\rho = \frac{\sigma + \alpha + 1}{\alpha - \beta}. \tag{3.1}$$

Furthermore, all of such solutions are governed by the unique asymptotic formula

$$x(t) \sim \left[\frac{t^{\alpha+1}q(t)}{\alpha|\rho|^{\alpha-1}\rho(\rho-1)}\right]^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{3.2}$$

Here and throughout the symbol $\sim$ *is used to mean the asymptotic equivalence*

$$f(t) \sim g(t) \quad as \ \ t \to \infty \iff \frac{f(t)}{g(t)} \to 1 \quad as \ \ t \to \infty.$$

PROOF. The "only if"part.

a) Suppose that $x(t)$ is an RV(ρ)-solution of index $\rho < 0$. Then as is explained at the beginning of Section 2, it is decreasing and $x(t)$ and $x'(t)$ both tend to 0 as $t \to \infty$.

By integrating on both sides of (A) from t to ∞ one obtains

$$x(t) = \int_t^\infty \left(\int_s^\infty q(r)x(r)^\beta dr\right)^{\frac{1}{\alpha}} ds, \tag{3.3}$$

for all large t. We need to analyze the right-hand side of (3.3). Using the expression (1.1) for $q(t)$ and $x(t)$, i.e. by writing $q(t) = t^\sigma l(t)$, $x(t) = t^\rho \xi(t)$, $l(t), \xi(t) \in$ SV, we have

$$\int_t^\infty q(s)x(s)^\beta ds = \int_t^\infty s^{\sigma+\rho\beta} l(s)\xi(s)^\beta ds. \tag{3.4}$$

The convergence of the last integral means that $\sigma + \rho\beta \leqq -1$, but the case $\sigma + \rho\beta = -1$ is impossible. Therefore, we have $\sigma + \rho\beta < -1$. Karamata integration theorem i.e. Proposition 1.1 (ii) applied to (3.4) gives

$$\left(\int_t^\infty q(s)x(s)^\beta ds\right)^{\frac{1}{\alpha}} \sim \frac{t^{\frac{\sigma+\rho\beta+1}{\alpha}} l(t)^{\frac{1}{\alpha}} \xi(t)^{\frac{\beta}{\alpha}}}{[-(\sigma+\rho\beta+1)]^{\frac{1}{\alpha}}}, \quad t \to \infty. \tag{3.5}$$

Since the right-hand side function is by (3.5) integrable in a neighborhood of ∞, one has

$$\frac{\sigma + \rho\beta + 1}{\alpha} \leqq -1. \tag{3.6}$$

But the case of equality is excluded. For, because of (3.3), (3.5) and Proposition 1.1 (iii), this would lead to

$$x(t) \sim \frac{1}{\alpha^{\frac{1}{\alpha}}} \int_t^\infty s^{-1} l(s)^{\frac{1}{\alpha}} \xi(s)^{\frac{\beta}{\alpha}} ds \in \mathrm{SV}, \quad t \to \infty, \tag{3.7}$$

which implies that $\rho = 0$, contradicting the hypothesis $\rho < 0$.

We are left with the inequality case of (3.6) which permits another application of Proposition 1.1 (ii) giving

$$\int_t^\infty \left(\int_s^\infty q(r) x(r)^\beta dr \right)^{\frac{1}{\alpha}} ds \sim \lambda t^{\frac{\sigma+\rho\beta+1}{\alpha}+1} l(t)^{\frac{1}{\alpha}} \xi(t)^{\frac{\beta}{\alpha}}, \tag{3.8}$$

as $t \to \infty$, where

$$\lambda = [-(\sigma + \rho\beta + 1)]^{-\frac{1}{\alpha}} \left[- \left(\frac{\sigma + \rho\beta + 1}{\alpha} + 1 \right) \right]^{-1}. \tag{3.9}$$

Combining (3.3) with (3.8) and $x(t) = t^\rho \xi(t)$ gives for $t \to \infty$

$$x(t) \sim \lambda^{\frac{\alpha}{\alpha-\beta}} [t^{\sigma+\alpha+1} l(t)]^{\frac{1}{\alpha-\beta}} = \lambda^{\frac{\alpha}{\alpha-\beta}} [t^{\alpha+1} q(t)]^{\frac{1}{\alpha-\beta}}. \tag{3.10}$$

This means that ρ is given by (3.1), and hence the negativity of ρ implies $\sigma < -\alpha - 1$. Since λ defined by (3.9) can be expressed as

$$\lambda = (\alpha(-\rho)^\alpha (1-\rho))^{-\frac{1}{\alpha}} = (\alpha |\rho|^{\alpha-1} \rho(\rho - 1))^{-\frac{1}{\alpha}},$$

formula (3.10) also gives the desired asymptotic formula (3.2).

b) Suppose that $x(t)$ is an RV(ρ)-solution of index $\rho > 1$, then it is increasing and by [5, Th. 3.8], the integral $\int_{t_0}^{\infty} q(r) x(r)^\beta dr$ diverges. Hence by integrating on both sides of (A) twice from t_0 to t, we obtain the asymptotic relation

$$x(t) \sim \int_{t_0}^t \left(\int_{t_0}^s q(r) x(r)^\beta dr \right)^{\frac{1}{\alpha}} ds, \quad t \to \infty. \tag{3.11}$$

The divergence of the inner integral (3.11) implies

$$\sigma + \rho\beta \geqq -1. \tag{3.12}$$

But the equality, via Proposition 1.1 (i) applied to (3.11), would give for $t \to \infty$

$$x(t) \sim t \left(\int_{t_0}^t s^{-1} l(s) \xi(s)^\beta ds \right)^{\frac{1}{\alpha}}, \quad t \to \infty. \tag{3.13}$$

This shows that $x(t) \in \mathrm{RV}(1)$ contradicting $\rho > 1$.

We have yet to treat the inequality case in (3.12): A repeated application of Proposition 1.1 (i) to the integral in (3.11) and the use of the expression (1.1) for $x(t)$ i.e. $x(t) = t^{\rho}\xi(t)$ gives for $t \to \infty$

$$x(t) \sim \mu^{\frac{\alpha}{\alpha-\beta}}[t^{\sigma+\alpha+1}l(t)]^{\frac{1}{\alpha-\beta}} = \mu^{\frac{\alpha}{\alpha-\beta}}[t^{\alpha+1}q(t)]^{\frac{1}{\alpha-\beta}}, \tag{3.14}$$

where μ is given by

$$\mu = (\sigma + \rho\beta + 1)^{-\frac{1}{\alpha}}\Big(\frac{\sigma + \rho\beta + 1}{\alpha} + 1\Big)^{-1}.$$

This shows that the regularity index ρ of $x(t)$ is given by (3.1). In addition, since by hypothesis $\rho > 1$, this implies $\sigma > -\beta - 1$, and since $\mu = (\alpha\rho^{\alpha}(\rho-1))^{-1/\alpha}$, the asymptotic formula (3.14) is identical to (3.2).

The "if" part.

a) Suppose that $\sigma < -\alpha - 1$. Define the constant ρ by (3.1) and the function $X_1(t)$ by

$$X_1(t) = \left[\frac{t^{\alpha+1}q(t)}{\alpha|\rho|^{\alpha-1}\rho(\rho-1)}\right]^{\frac{1}{\alpha-\beta}}, \quad t \geqq a. \tag{3.15}$$

It is a matter of straightforward computation to verify that the integrals in (3.16) converge and via Proposition 1.1 (ii), that $X_1(t)$ satisfies the following asymptotic relation

$$\int_t^{\infty}\Big(\int_s^{\infty} q(r)X_1(r)^{\beta}dr\Big)^{\frac{1}{\alpha}}ds \sim X_1(t), \quad t \to \infty. \tag{3.16}$$

Therefore, there exists $T > a$ such that

$$\frac{1}{2}X_1(t) \leqq \int_t^{\infty}\Big(\int_s^{\infty} q(r)X_1(r)^{\beta}dr\Big)^{\frac{1}{\alpha}}ds \leqq 2X_1(t), \quad t \geqq T. \tag{3.17}$$

Let $\mathcal{X}_1$ denote the set consisting of all continuous functions $x(t)$ on $[T, \infty)$ satisfying

$$kX_1(t) \leqq x(t) \leqq KX_1(t), \quad t \geqq T, \quad \text{and} \quad x(t) \sim X_1(t), \quad t \to \infty, \tag{3.18}$$

where $0 < k < 1$ and $K > 1$ are constants such that

$$k^{1-\frac{\beta}{\alpha}} \leqq \frac{1}{2}, \quad \text{and} \quad K^{1-\frac{\beta}{\alpha}} \geqq 2. \tag{3.19}$$

It is clear that $\mathcal{X}_1$ is a closed convex subset of the locally convex space $C[T, \infty)$ with the topology of uniform convergence on compact subintervals of $[T, \infty)$. We now consider the integral operator $\mathcal{F}$ defined by

$$\mathcal{F}x(t) = \int_t^{\infty}\Big(\int_s^{\infty} q(r)x(r)^{\beta}dr\Big)^{\frac{1}{\alpha}}ds, \quad t \geqq T. \tag{3.20}$$

It will be shown that $\mathcal{F}$ is a self-map on $\mathcal{X}_1$, $\mathcal{F}$ is a continuous map and the set $\mathcal{F}(\mathcal{X}_1)$ is relatively compact in $C[T,\infty)$. Consequently, by the Schauder-Tychonoff fixed point theorem there exists a fixed point $x(t) \in \mathcal{X}_1$ of $\mathcal{F}$, which is a solution of the integral equation (3.3) and hence of the differential equation (A) on $[T,\infty)$. Since $x(t) \sim X_1(t)$ as $t \to \infty$, $x(t)$ provides a desired regularly varying solution of negative index $\rho < 0$ given by (3.1) and with the asymptotics given by (3.2).

b) Suppose that $\sigma > -\beta - 1$ and put

$$X_1(t) = \left[\frac{t^{\alpha+1}q(t)}{\alpha\rho^\alpha(\rho-1)}\right]^{\frac{1}{\alpha-\beta}}, \quad t \geqq a, \tag{3.21}$$

where $\rho > 1$ is defined by (3.1). As before it is verified without difficulty, $X_1(t)$ has the asymptotic property

$$\int_a^t \left(\int_a^s q(r)X_1(r)^\beta dr\right)^{\frac{1}{\alpha}} ds \sim X_1(t), \quad t \to \infty, \tag{3.22}$$

and one can choose $T > a$ large enough so that $X_1(t) \geqq 1$ and

$$\int_T^t \left(\int_T^s q(r)X_1(r)^\beta dr\right)^{\frac{1}{\alpha}} ds \leqq 2X_1(t) \quad \text{for} \quad t \geqq T. \tag{3.23}$$

Let $\mathcal{G}$ denote the integral operator

$$\mathcal{G}x(t) = 1 + \int_T^t \left(\int_T^s q(r)x(r)^\beta dr\right)^{\frac{1}{\alpha}} ds, \quad t \geqq T, \tag{3.24}$$

and define $\mathcal{X}_1$ to be the set of continuous functions $x(t)$ on $[T,\infty)$ satisfying

$$1 \leqq x(t) \leqq 2KX_1(t), \quad t \geqq T, \quad \text{and} \quad x(t) \sim X_1(t), \quad t \to \infty, \tag{3.25}$$

where $K > 1$ is a constant such that

$$K^{1-\frac{\beta}{\alpha}} \geqq 2^{1+\frac{\beta}{\alpha}}. \tag{3.26}$$

If $x(t) \in \mathcal{X}_1$, then using (3.22), (3.23), (3.25) and (3.26), one obtains

$$\begin{aligned} 1 \leqq \mathcal{G}x(t) &\leqq 1 + \int_T^t \left(\int_T^s q(r)(2KX_1(r))^\beta dr\right)^{\frac{1}{\alpha}} ds \\ &\leqq 1 + 2(2K)^{\frac{\beta}{\alpha}}X_1(t) \leqq 1 + KX_1(t) \leqq 2KX_1(t), \quad t \geqq T, \end{aligned}$$

and

$$\mathcal{G}x(t) \sim \mathcal{G}X_1(t) \sim X_1(t) \quad \text{as} \quad t \to \infty.$$

This implies that $\mathcal{G}$ maps $\mathcal{X}_1$ into itself. Furthermore one can prove in a routine manner the continuity of $\mathcal{G}$ and the relative compactness of $\mathcal{G}(\mathcal{X}_1)$. Therefore $\mathcal{G}$ has a fixed point $x(t) \in \mathcal{X}_1$, which clearly gives an RV(ρ)-solution of equation (A) of index $\rho > 1$ given by (3.1) with the asymptotics (3.2).

This completes the proof of Theorem 3.1.

Using a similar argument as in the proof of Theorem 3.1 we obtain analogous results for the cases $\rho = 0$, i.e. when $x(t) \in$ SV, and $\rho = 1$, i.e. when $x(t) \in$ RV(1). This is encompassed in the following two theorems.

Theorem 3.2. *Suppose that* (1.4) *holds; then equation* (A) *possesses nontrivial decreasing slowly varying solutions* $x(t)$ *if and only if*

$$\text{(3.27)} \qquad \text{(i)} \quad \sigma = -\alpha - 1, \qquad \text{(ii)} \quad \int_a^\infty (tq(t))^{\frac{1}{\alpha}} dt < \infty.$$

Furthermore, all such solutions are governed by the same asymptotic formula for $t \to \infty$

$$\text{(3.28)} \qquad x(t) \sim \left[\frac{\alpha - \beta}{\alpha^{1+\frac{1}{\alpha}}} \int_t^\infty (sq(s))^{\frac{1}{\alpha}} ds \right]^{\frac{\alpha}{\alpha-\beta}}.$$

Remark 3.3. Asymptotic formula (3.28) is identical to the formula (4.15) in [3].

Theorem 3.4. *Suppose that* (1.4) *holds; then equation* (A) *possesses nontrivial* (*increasing*) *regularly varying solutions of index* $\rho = 1$ *if and only if*

$$\text{(3.29)} \qquad \text{(i)} \quad \sigma = -\beta - 1, \qquad \text{(ii)} \quad \int_a^\infty s^\beta q(s) ds = \infty.$$

Furthermore, all such solutions are governed by the same asymptotic formula for $t \to \infty$

$$\text{(3.30)} \qquad x(t) \sim t \left[\frac{\alpha - \beta}{\alpha} \int_a^t s^\beta q(s) ds \right]^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Remark 3.5. Asymptotic formula (3.30) is identical to the last one in [3].

Remark 3.6. Theorem 3.1 reveals how the asymptotic behavior of regularly varying solutions of the sub-half-linear differential equation (A) is determined by its coefficient $q(t)$ which is regularly varying, but also conversely.

Suppose that the equation $((x')^5)' = q(t)x^3$ with regularly varying $q(t)$ has a solution $x(t) \in$ RV(-2) such that

$$x(t) \sim t^{-2}(2 + \sin \log \log t), \quad t \to \infty,$$

then by Theorem 3.1 a) $q(t)$ must satisfy

$$q(t) \sim 480t^{-6}\Big(t^{-2}(2+\sin\log\log t)\Big)^2 = 480t^{-10}(2+\sin\log\log t)^2, \quad t\to\infty.$$

If it is known that the equation $((x')^7)' = q(t)x^5$, has a solution $x(t) \in \mathrm{RV}(2)$ such that

$$x(t) \sim t^2 \exp\Big(\sqrt{\log t}\Big), \quad t\to\infty,$$

then, by Theorem 3.1 b) $q(t)$ must enjoy the asymptotic behavior

$$q(t) \sim 896t^{-4}\exp\Big(2\sqrt{\log t}\Big), \quad t\to\infty.$$

Example 3.7. Consider the equation

$$\text{(3.31)}\quad (|x'|^{\alpha-1}x')' = q(t)|x|^{\beta-1}x, \quad q(t) = \frac{\alpha\varphi(t)}{t^{\alpha+1}(\log t)^{\alpha}(\log\log t)^{2\alpha-\beta}}, \quad t>e,$$

where $\alpha > \beta > 0$ and $\varphi(t)$ is a continuous function such that $\lim_{t\to\infty}\varphi(t) = k > 0$. It is easy to see that (3.27) (ii) holds and

$$\int_t^\infty (sq(s))^{\frac{1}{\alpha}}ds \sim \frac{k^{\frac{1}{\alpha}}\alpha^{1+\frac{1}{\alpha}}}{(\alpha-\beta)(\log\log t)^{1-\frac{\beta}{\alpha}}}, \quad t\to\infty.$$

Hence Theorem 3.2 ensures the existence of a nontrivial slowly varying solution $x_1(t)$ of (3.31) such that

$$x_1(t) \sim \frac{k^{\frac{1}{\alpha-\beta}}}{\log\log t}, \quad t\to\infty.$$

If in particular

$$\varphi(t) = 1 + \frac{1}{\log t} + \frac{2}{\log t\cdot\log\log t},$$

then (3.31) has an exact SV-solution $1/\log\log t$. Note that (3.31) also has a trivial SV solution $x_2(t)$ decreasing to a positive constant as $t\to\infty$.

Example 3.8. Consider the differential equation

$$\text{(3.32)}\quad \Big(|x'|^{\alpha-1}x'\Big)' = q(t)|x|^{\beta-1}x, \quad q(t) = \alpha t^{-(\beta+1)}(\log t)^{\alpha-\beta-1}\varphi(t), \quad t \geqq e,$$

where $\alpha > \beta > 0$ and $\varphi(t)$ is a continuous function such that $\lim_{t\to\infty}\varphi(t) = k > 0$. Since $\sigma = -\beta - 1$ and

$$\int_e^t s^\beta q(s)ds \sim \frac{\alpha k}{\alpha-\beta}(\log t)^{\alpha-\beta}, \quad t\to\infty,$$

Theorem 3.8 ensures the existence of nontrivial RV(1)-solutions of (3.32), all of which satisfy

$$x(t) \sim k^{\frac{1}{\alpha-\beta}} t \log t, \quad t \to \infty.$$

If in particular

$$\varphi(t) = \left(1 + \frac{1}{\log t}\right)^{\alpha-1},$$

then (3.32) has an exact solution $t \log t$.

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