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Boundedness and Stability of Semi-linear Dynamic Equations on Time Scales

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Abstract: This paper explores the boundedness and asymptotic stability of a kind of semi-linear dynamic equations on time scales. We derive sufficient criteria for the uniform boundedness and the uniformly ultimate boundedness and establish necessary and sufficient criterion for the asymptotic stability. The approach is rather nontrivial and is based on the application of the contraction mapping principle.

Keywords: Time scales; Boundedness; Asymptotic stability; Contraction mapping principle.

1 Introduction

The theory of dynamic equations on time scale [3] has a tremendous potential for applications and has recently received much attention and seen many progresses in the past two decades due to the fact that a dynamic equation on time scales is related not only to the set of real numbers (continuous time scale, differential equations) and the set of integers (discrete time scale, difference equations) but also to more general time scales (an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$). Its history can be traced back to the calculus on time scales–the foundational work initiated by Stefan Hilger in his PhD thesis [9] in order to unify continuous and discrete analysis.

Stability plays an important role in the theory of dynamic equations on time scales. Since the pioneer work of Liapunov more than 100 years ago, Liapunov's direct method has been the primary tool to deal with stability problems in various type of dynamical systems such as differential equations[18], difference equations [1] and dynamic equations on time scales[3, 10]. However, the construction of appropriate Liapunov functions or functionals are technical, empirical and are not universally applicable. Criteria deduced from the direct method usually requires point-wise conditions, while many of the real-world dynamical models call for averages; calculations involved in the direct method are very technical and sophisticated. New methods and techniques are needed to address those difficulties.

Recently, the fixed point theory has been proven to be a powerful tool for dealing with the stability of differential equations. Burton [5] was among the first who study the stability using fixed point theory. When applicable, by the fixed point theory, one can try to avoid certain difficulties incurred in applying the Liapunov method but usually achieve conditions of average type [5]. For the comparison of these two approaches in dealing with the stability of differential equations, we refer to [4], [5] and [19, 20]. Though there have been studies of the stability of differential equations using the fixed point theory, it still remains open whether it can be applied to explore the stability of dynamic equations on time scales.

In this paper, the main approach is based on the contraction mapping principle (also Banach fixed point theorem). The tree of this paper is as following. In Section 2, we introduce some basic results of the calculus on general time scale. In Section 3, we first derive sufficient criteria for the uniform boundedness and the uniform ultimate boundedness of (2.2) and then establish necessary and sufficient criterion for the asymptotic stability of (2.2)

2 Preliminaries

Let $\mathbb{T}$ be a *time scale*, i.e., an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For more details of the time scale, one can see [3, 9]. To facilitate the discussion below, we introduce many notations $\varsigma = \min\{[0,\infty) \cap \mathbb{T}\}, \mathbb{T}^+ = [\varsigma, \infty) \cap \mathbb{T}, \mathbb{R}^+ = [0, \infty)$.

Now, we propose some definitions of boundedness and stability of dynamic equations on time scales. Consider the following nonlinear dynamic equations on time scales

$$x^{\Delta}(t) = F(t, x), \tag{2.1}$$

where $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, $F(\cdot, x)$ is rd-continuous on $\mathbb{T}$ for all $x \in \mathbb{R}$ and $F(t, \cdot)$ is continuous on $\mathbb{R}$ for all $t \in \mathbb{T}$. Moreover, for clarity we denote by $x(t, x_0, t_0)$ the solution of (2.1) with initial values $x(t_0) = x_0$.

Definition 2.1. The solutions of (2.1) are uniformly bounded, if for any $\alpha > 0$ and $t_0 \in \mathbb{T}^+$, there exists a $\beta_1(\alpha) > 0$ such that $|x_0| \leq \alpha, |x(t, x_0, t_0)| < \beta_1$ for all $t \geq t_0$.

Definition 2.2. The solutions of (2.1) are uniformly ultimately bounded for bound β_2, if there exists a $\beta_2 > 0$, for any $\alpha > 0$ and $t_0 \in \mathbb{T}^+$, there exists a $T(\alpha) > \varsigma$ such that $|x_0| \leq \alpha$, $|x(t, x_0, t_0)| < \beta_2$ for all $t \geq t_0 + T$.

Definition 2.3. [see [10]] The zero solution of (2.1) is said to be stable, if for any $\varepsilon > 0$ and $t_0 \in \mathbb{T}$, there exists a $\delta(t_0, \varepsilon) > 0$ such that $|x_0| \leq \delta$, $|x(t, x_0, t_0)| < \varepsilon$ for all $t \geq t_0$.

Definition 2.4. [see [10]] The zero solution of (2.1) is said to be asymptotically stable, if the zero solution is stable and if there exists a $\delta(t_0) > 0$ such that if $|x_0| \leq \delta$, $x(t, x_0, t_0) \to 0$ as $t \to \infty$.

In this paper, we will explore the boundedness and asymptotic stability of the following semi-linear equation on general time scale $\mathbb{T}$

$$x^{\Delta}(t) = -a(t)x(\sigma(t)) + f(t, x(t)). \tag{2.2}$$

where $a \in \mathcal{R}^+$, $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, $f(\cdot, x)$ is rd-continuous on $\mathbb{T}$ for all $x \in \mathbb{R}$ and $f(t, \cdot)$ is continuous on $\mathbb{R}$ for all $t \in \mathbb{T}$. When $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, (2.2) reduces to the semi-linear differential equations or difference Equations, whose boundedness and stability have been extensively studied. For example, see [1, 18].

Carrying out similar arguments as those in Theorem 2.74 in [3], we can easily obtain the following theorem.

Theorem 2.1. *Suppose $a \in \mathcal{R}$, $u \in C(\mathbb{T}, \mathbb{R})$. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$x^{\Delta}(t) = -a(t)x(\sigma(t)) + f(t, u(t)), \qquad x(t_0) = x_0 \tag{2.3}$$

is given by $x(t) = e_{\ominus a}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus a}(t, \tau) f(\tau, u(\tau))\Delta\tau$.

3 Boundedness and asymptotic stability

In this section, we explore the boundedness and asymptotic stability of the solutions of (2.2). The approach will base on the famous contraction mapping principle (also known as the Banach fixed point theorem or the contraction mapping theorem).

Lemma 3.1. *Let (X, d) be a complete metric space and $P : X \to X$ be a contraction mapping (that is, there exists a constant λ with $0 \leq \lambda < 1$ such that $d(P(x), P(y)) \leq \lambda d(x, y), x, y \in X$), then P has a unique fixed point in X.*

Theorem 3.1. *Assume that*

(i) *there is a function $b : \mathbb{T} \to \mathbb{R}^+$ such that $|f(t, x_1) - f(t, x_2)| \leq b(t)|x_1 - x_2|$ for any $x_1, x_2 \in \mathbb{R}, t \in \mathbb{T}$;*

(ii) $\lim_{t\to\infty}\int_\varsigma^t \xi_{\mu(\tau)}(a(\tau))\Delta\tau = \infty$ *and there are a* $0<\chi<1$ *and a* $M>0$ *such that*

$$\int_\varsigma^t e_{\ominus a}(t,\tau)|f(\tau,0)|\Delta\tau < M, \qquad \int_\varsigma^t e_{\ominus a}(t,\tau)b(\tau)\Delta\tau \le \chi, \quad t\ge\varsigma.$$

Then the solutions of (2.2) *are uniformly bounded.*

Proof. According to the condition (ii), for any fixed $t_0\ge\varsigma$, we have

$$\int_{t_0}^t \xi_{\mu(\tau)}(a(\tau))\Delta\tau = \int_\varsigma^t \xi_{\mu(\tau)}(a(\tau))\Delta\tau - \int_\varsigma^{t_0} \xi_{\mu(\tau)}(a(\tau))\Delta\tau \to\infty \quad \text{as} \quad t\to\infty$$

and $e_{\ominus a}(t,t_0)=1/e_a(t,t_0)\to 0$ as $t\to\infty$. Hence, we can find a positive number b_1 such that $e_{\ominus a}(t,t_0)\le b_1$ for all $t\ge t_0$. For any α_1, let $\beta_1=(\alpha_1 b_1+M)/(1-\chi)$ and define

$$S_1=\{u\in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})|u(t_0)=x_0, \text{ and } |u(t)|<\beta_1 \text{ for } t\ge t_0,\ |x_0|\le\alpha_1\},$$

then it is not difficult to show that the set S_1 is a complete metric space endowed with metric $d(u_1,u_2)=\|u_1-u_2\|=\sup_{t\in[t_0,\infty)}|u_1(t)-u_2(t)|$. In view of Theorem 2.1, for any $u\in S_1$, we let $Z_u(t)=e_{\ominus a}(t,t_0)u(t_0)+\int_{t_0}^t e_{\ominus a}(t,\tau)f(\tau,u(\tau))\Delta\tau$. Obviously, $Z_u(t_0)=u(t_0)=x_0$ and it is easy to show that $Z_u\in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. In addition, for any $t\ge t_0$, we have

$$\begin{aligned}|Z_u(t)| &\le e_{\ominus a}(t,t_0)|x_0| + \int_{t_0}^t e_{\ominus a}(t,\tau)b(\tau)|u(\tau)|\Delta\tau + \int_{t_0}^t e_{\ominus a}(t,\tau)|f(\tau,0)|\Delta\tau \\ &< b_1\alpha_1+\chi\beta_1+M=\beta_1.\end{aligned}$$

Therefore, we can define a mapping $P: S_1\to S_1$ by $(Pu)(t)=Z_u(t)$. By the condition (i), for any $u_1,u_2\in S_1$, we have, for any $t\ge t_0$,

$$\begin{aligned}|(Pu_1-Pu_2)(t)| &= \left|\int_{t_0}^t e_{\ominus a}(t,\tau)(f(\tau,u_1(\tau))-f(\tau,u_2(\tau)))\Delta\tau\right| \\ &\le \int_{t_0}^t e_{\ominus a}(t,\tau)b(\tau)\|u_1-u_2\|\Delta\tau \le \chi\|u_1-u_2\|.\end{aligned}$$

Therefore, P is a contraction mapping and has a unique fixed point in S_1, which is the unique solution of (2.2) in S_1. That is, for any $\alpha_1>0$ and $t_0\in\mathbb{T}^+$, there exists a $\beta_1(\alpha_1)>0$ such that, for any $|x_0|\le\alpha_1$, the solutions of (2.2) satisfies $|x(t,x_0,t_0)|<\beta_1, t\ge t_0$, that is, the solutions of (2.2) are uniformly bounded. □

Theorem 3.2. *Assume that the conditions* (i) *and* (ii) *in Theorem* 3.1 *hold. Then the solutions of* (2.2) *are uniformly ultimately bounded for bound* $\beta_2>M/(1-\chi)$.

Proof. By Theorem 3.1, for any $\alpha_2 > 0$ and $t_0 \geq \varsigma$, there exists a $\beta_1(\alpha_2) > 0$ such that, for any $|x_0| \leq \alpha_2$, $|x(t, x_0, t_0)| < \beta_1$ for all $t \geq t_0$. In order to prove the conclusion, we define

$$S_2 = \left\{ u \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}) \;\middle|\; \begin{array}{l} u(t_0) = x_0, |x_0| \leq \alpha_2, |u(t)| < \beta_1 \text{ for } t \geq t_0, \\ d(u(t), B(0, \beta_3)) \to 0 \text{ as } t \to \infty \end{array} \right\},$$

where $\beta_3 = M/(1-\chi)(< \beta_1)$ is a fixed positive constant number and $B(0, \beta_3)$ is a sphere with center 0 and radius β_3. Then S_2 is a complete metric space endowed with metric $d(u_1, u_2) = \|u_1 - u_2\| = \sup_{t \in [t_0, \infty)} |u_1(t) - u_2(t)|$.

For any $\varepsilon > 0$ and $u \in S_2$, there is a T_1 such that $|u(t)| < \beta_3 + \varepsilon/2$ for any $t \geq T_1$. It follows from the condition (ii) that, for sufficiently large $T_2 > T_1$, $\alpha_2 e_{\ominus a}(t, t_0) < \varepsilon/4$ and $\beta_1 \chi e_{\ominus a}(t, T_1) < \varepsilon/4$, $t \geq T_2$. For $u \in S_2$, we consider $Z_u(t) = e_{\ominus a}(t, t_0) u(t_0) + \int_{t_0}^{t} e_{\ominus a}(t, \tau) f(\tau, u(\tau)) \Delta\tau$. Obviously, $Z_u(t_0) = u(t_0) = x_0$ and $Z_u(t) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$. Moreover, by the proof of Theorem 3.1, we have $|Z_u(t)| < \beta_1$ for any $t \geq t_0$. If $t \geq T_2$, then we have

$$\begin{aligned}
|Z_u(t)| &\leq e_{\ominus a}(t, t_0)|u(t_0)| + \left| \int_{t_0}^{t} e_{\ominus a}(t, \tau) f(\tau, u(\tau)) \Delta\tau \right| \\
&= e_{\ominus a}(t, t_0)|u(t_0)| + \int_{t_0}^{T_1} e_{\ominus a}(t, \tau) b(\tau) |u(\tau)| \Delta\tau \\
&\quad + \int_{T_1}^{t} e_{\ominus a}(t, \tau) b(\tau) |u(\tau)| \Delta\tau + \int_{t_0}^{t} e_{\ominus a}(t, \tau) |f(\tau, 0)| \Delta\tau \\
&\leq \frac{\varepsilon}{4} + \beta_1 e_{\ominus a}(t, T_1) \int_{t_0}^{T_1} e_{\ominus a}(T_1, \tau) b(\tau) \Delta\tau \\
&\quad + (\beta_3 + \frac{\varepsilon}{2}) \int_{T_1}^{t} e_{\ominus a}(t, \tau) b(\tau) \Delta\tau + \beta_3(1 - \chi) \\
&< \frac{\varepsilon}{4} + \beta_1 \chi e_{\ominus a}(t, T_1) + (\beta_3 + \frac{\varepsilon}{2})\chi + \beta_3(1 - \chi) \\
&< \frac{\varepsilon}{2} + \beta_3 + \frac{\varepsilon}{2} = \varepsilon + \beta_3.
\end{aligned}$$

Since ε is arbitrary, we can conclude that $Z_u(t)$ approaches $B(0, \beta_3)$ as $t \to \infty$. Now we define a mapping $P : S_2 \to S_2$ by $(Pu)(t) = Z_u(t)$. It is not difficult to show that P is a contraction mapping by the same arguments as those in Theorem 3.1. Hence, P has a unique fixed point in S_2, which is a solution of (2.2). Therefore, for any fixed positive number c, we can choose $\beta_2 = \beta_3 + c$ as the bound of uniform ultimate boundedness. □

Theorem 3.3. *Assume that*

(i) $\liminf_{t \to \infty} \int_{\varsigma}^{t} \xi_{\mu(\tau)}(a(\tau)) \Delta\tau > -\infty$ *and* $f(t, 0) = 0$ *for all* $t \in \mathbb{T}$;

(ii) *there is a function* $b : \mathbb{T} \to \mathbb{R}^+$ *and an* $N > 0$ *such that* $|f(t, x_1) - f(t, x_2)| \leq b(t)|x_1 - x_2|$, $|x_1|, |x_2| \leq N$, $t \in \mathbb{T}$;

(iii) *there is a* $0 < \chi < 1$ *such that* $\int_{\varsigma}^{t} e_{\ominus a}(t, \tau)b(\tau)\Delta\tau \leq \chi$ *for* $t \geq \varsigma$.

Then the zero solution of (2.2) *is asymptotically stable if and only if*

$$\text{(iv)} \qquad \int_{\varsigma}^{t} \xi_{\mu(\tau)}(a(\tau))\Delta\tau \to \infty \text{ as } t \to \infty.$$

Proof. (Sufficiency) If the condition (iv) is satisfied, then there is a $b_2 > 0$ such that, for any $t_0 \geq \varsigma$, $|e_{\ominus a}(t, t_0)| \leq b_2$ for all $t \geq t_0$. Next, we choose a $\delta_1 > 0$ such that $\delta_1 b_2 + \chi N \leq N$. For any $|x_0| \leq \delta_1$, define

$$S_3 = \{u \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})|, u(t_0) = x_0, |u(t)| \leq N \text{ for } t \geq t_0, u(t) \to 0 \text{ as } t \to \infty\}.$$

It is easy to show that S_3 is a complete metric space endowed with metric $d(u_1, u_2) = \|u_1 - u_2\| = \sup_{t\in[t_0,\infty)} |u_1(t) - u_2(t)|$.

For any $\varepsilon > 0$ and $u \in S_3$, we can easily find a $T_3 > t_0$ such that $|u(t)| < \varepsilon/3$ for all $t \geq T_3$. It follows from the condition (iv) that there is a $T_4 > T_3$ such that $\delta_1 e_{\ominus a}(t, t_0) < \varepsilon/3$ and $N\chi e_{\ominus a}(t, T_3) < \varepsilon/3$ for $t \geq T_4$. Define $Z_u(t) = e_{\ominus a}(t, t_0)u(t_0) + \int_{t_0}^{t} e_{\ominus a}(t, \tau)f(\tau, u(\tau))\Delta\tau$, then $Z_u(t_0) = u(t_0) = x_0$ and $Z_u(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. If $t \geq T_4$, then

$$\begin{aligned}
|Z_u(t)| &\leq e_{\ominus a}(t, t_0)|u(t_0)| + \left|\int_{t_0}^{t} e_{\ominus a}(t, \tau)f(\tau, u(\tau))\Delta\tau\right| \\
&= e_{\ominus a}(t, t_0)|u(t_0)| + \int_{t_0}^{T_3} e_{\ominus a}(t, \tau)b(\tau)|u(\tau)|\Delta\tau + \int_{T_3}^{t} e_{\ominus a}(t, \tau)b(\tau)|u(\tau)|\Delta\tau \\
&\leq \frac{\varepsilon}{3} + Ne_{\ominus a}(t, T_3)\int_{t_0}^{T_3} e_{\ominus a}(T_3, \tau)b(\tau)\Delta\tau + \frac{\varepsilon}{3}\int_{T_3}^{t} e_{\ominus a}(t, \tau)b(\tau)\Delta\tau \\
&< \frac{\varepsilon}{3} + N\chi e_{\ominus a}(t, T_3) + \frac{\varepsilon}{3}\chi < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

We now define a mapping $P : S_3 \to S_3$ by $(Pu)(t) = Z_u(t)$. Then P is a contraction mapping. Thus, P has a unique fixed point in S_3, which is a solution of (2.2) and $x(t) = x(t, t_0, x_0) \to 0$ as $t \to \infty$.

Now we show that the zero solution of (2.2) is stable. For any $\varepsilon > 0(\varepsilon < b_2)$, choose a $\delta_2 > 0(\delta_2 < \varepsilon)$ such that $\delta_2 b_2 + \chi\varepsilon < \varepsilon$. To prove the conclusion, we will show that, for any $|x_0| < \delta_2$, $|x(t, x_0, t_0)| < \varepsilon$ for any $t \geq t_0$. Assume that there exists a $t^* > t_0$ such that $|x(t^*)| = \varepsilon$ and $|x(\tau)| < \varepsilon$ for $t_0 \leq \tau < t^*$. From Theorem 2.1, the solutions of (2.2) can be expressed as

$$x(t) = e_{\ominus a}(t, t_0)x(t_0) + \int_{t_0}^{t} e_{\ominus a}(t, \tau)f(\tau, x(\tau))\Delta\tau. \tag{3.1}$$

Hence, we have $|x(t^*)| \leq \delta_2 e_{\ominus a}(t^*, t_0) + \int_{t_0}^{t^*} e_{\ominus a}(t^*, \tau)b(\tau)|x(\tau)|\Delta\tau \leq \delta_2 b_2 + \chi\varepsilon < \varepsilon$, which contradicts to the definition of t^*, then the zero solution of (2.2) is stable. Therefore, if (iv) is satisfied, then the zero solution of (2.2) is asymptotically stable.

(Necessity) If (iv) fails, then there exist a sequence $\{t_n\}(t_n \to \infty$ as $n \to \infty)$ and some real number m_1 such that $\lim_{n\to\infty} \int_{\varsigma}^{t_n} \xi_{\mu(\tau)}(a(\tau))\Delta\tau = m_1$. It is easy to show that there is a positive constant L such that $|\int_{\varsigma}^{t_n} \xi_{\mu(\tau)}(a(\tau))\Delta\tau| \leq L$ and $e_a(t_n, \varsigma) \leq e^L$ for all $n = 1, 2, \cdots$. Therefore, it follows from the condition (iii) that

$$\int_{\varsigma}^{t_n} e_a(\tau, \varsigma)b(\tau)\Delta\tau = \int_{\varsigma}^{t_n} e_a(t_n, \varsigma)e_{\ominus a}(t_n, \tau)b(\tau)\Delta\tau \leq \chi e_a(t_n, \varsigma) \leq e^L.$$

This implies that there exists a convergent subsequence. Without loss of generality, we still assume that it is $\{t_n\}$ such that $\lim_{n\to\infty} \int_{\varsigma}^{t_n} e_a(\tau, \varsigma)b(\tau)\Delta\tau = m_2$ for some positive constant m_2. Hence, we can find sufficiently large k^* such that $\int_{t_{k^*}}^{t_n} e_a(\tau, \varsigma)b(\tau)\Delta\tau < (1-\chi)/(2Q^2)$ for $n \geq k^*$, where $Q = \sup_{t\geq\varsigma} e_{\ominus a}(t, \varsigma)$. Since the zero solution of (2.2) is asymptotically stable, for given a real number $B > 0$, there exits a $\delta_0 > 0(\delta_0 < B)$ such that $|x(t, x(t_{k^*}), t_{k^*})| < B$ for $t \geq t_{k^*}$ with the initial value $|x(t_{k^*})| = \delta_0$. For all $t \geq t_{k^*}$, we have $|x(t)| \leq x(t_{k^*})e_{\ominus a}(t_n, t_{k^*}) + \int_{t_{k^*}}^{t} e_{\ominus a}(t, \tau)|f(\tau, x(\tau))|\Delta\tau \leq \delta_0 Q + \chi \sup_{t\geq t_{k^*}} |x(t)|$. This shows that $|x(t)| \leq \delta_0 Q/(1-\chi)$ for all $t \geq t_{k^*}$. Meanwhile, it is easy to show that

$$\begin{aligned}
|x(t_n)| &\geq \delta_0 e_{\ominus a}(t_n, t_{k^*}) - \int_{t_{k^*}}^{t_n} e_{\ominus a}(t_n, \tau)b(\tau)|x(\tau)|\Delta\tau \\
&\geq \delta_0 e_{\ominus a}(t_n, t_{k^*}) - \frac{\delta_0 Q}{1-\chi} e_{\ominus a}(t_n, \varsigma) \int_{t_{k^*}}^{t_n} e_a(\tau, \varsigma)b(\tau)\Delta\tau \\
&\geq e_{\ominus a}(t_n, t_{k^*}) \left[\delta_0 - \frac{\delta_0 Q}{1-\chi} Q \int_{t_{k^*}}^{t_n} e_a(\tau, \varsigma)b(\tau)\Delta\tau\right] \\
&\geq \frac{1}{2}\delta_0 e_{\ominus a}(t_n, t_{k^*}) \geq \frac{1}{2}\delta_0 e^{-2L}
\end{aligned}$$

for $n \geq k^*$. This implies that $x(t) \nrightarrow 0$ as $t \to \infty$, which is contradiction. That is, (iv) is necessary for asymptotically stable of the zero solution of (2.2). This completes the proof. $\square$

Next, we consider the existence and stability of periodic solutions of (2.2) by the famous contraction mapping principle. Therefore, the time scale $\mathbb{T}$ is supposed to be

ω*-periodic*, i.e., $t \in \mathbb{T}$ implies $t \pm \omega \in \mathbb{T}$. This implies that the graininess μ is also ω-periodic, that is, $\mu(t \pm \omega) = \mu(t)$. Some examples of such time scales are $\mathbb{R}, \mathbb{Z}, \bigcup_{k\in\mathbb{Z}}[2k, 2k+1], \bigcup_{k\in\mathbb{Z}} \bigcup_{n\in\mathbb{N}} \left\{k + \frac{1}{n}\right\}$. Moreover, in (2.2), we assume that $a \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, and $a(t), f(t, x)$ are both ω-periodic in t.

Lemma 3.2. *$x(t)$ is an ω-periodic solution of* (2.2) *if and only if $x(t)$ is a solution of*

$$x(t) = \int_t^{t+\omega} \frac{e_a(\tau, t)}{e_a(\varsigma + \omega, \varsigma) - 1} f(\tau, x(\tau))\Delta\tau. \tag{3.2}$$

Proof. Let $x(t)$ be an ω-periodic solution of (2.2), then one has $x^{\Delta}(t) + a(t)x(\sigma(t)) = f(t, x(t))$. Multiplying both sides of the above equation by $e_a(t, \varsigma)$ leads to $(e_a(t, \varsigma)x(t))^{\Delta} = e_a(t, \varsigma)f(t, x(t))$. Integrating from t to $t + \omega$, we have $x(t + \omega)e_a(t + \omega, \varsigma) - x(t)e_a(t, \varsigma) = \int_t^{t+\omega} e_a(\tau, \varsigma)f(\tau, x(\tau))\Delta\tau$. Since $x(t)$ is ω-periodic, one can easily reach (3.2). Therefore, the necessity of the claim is valid. The proof of the sufficiency is trivial. □

Theorem 3.4. *Assume that the conditions* (ii), (iii) and (iv) *of Theorem* 3.3 *are satisfied. If $A \int_\varsigma^{\varsigma+\omega} b(\tau)\Delta\tau < 1$ holds, where $A = e_a(\varsigma + \omega, \varsigma)/(e_a(\varsigma + \omega, \varsigma) - 1)$, then* (2.2) *has a unique asymptotically stable periodic solution.*

Proof. Define

$$S_4 = \{u \in C(\mathbb{T}, \mathbb{R}) |\ u(t + \omega) = u(t) \text{ for all } t \in \mathbb{T}\}, \quad \|u\| = \max_{t\in I_\omega} |u(t)| \text{ for } u \in S_4.$$

It is not difficult to show that $(S_4, \|\cdot\|)$ is a Banach space. Define a mapping T as follows:

$$Tu(t) = \int_t^{t+\omega} \frac{e_a(\tau, t)}{e_a(\varsigma + \omega, \varsigma) - 1} f(\tau, u(\tau))\Delta\tau.$$

Obviously, $T : S_4 \to S_4$. Meanwhile, for any $u_1, u_2 \in S_4$, we have

$$\begin{aligned} |Tu_1(\tau) - Tu_2(\tau)| &= \int_t^{t+\omega} \frac{e_a(\tau, t)}{e_a(\varsigma + \omega, \varsigma) - 1} |f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))|\Delta\tau \\ &\leq A \int_\varsigma^{\varsigma+\omega} b(\tau)\Delta\tau \|u_1 - u_2\|. \end{aligned}$$

Then, T is a contraction mapping with a unique fixed point in S_4, which is a unique periodic solution of (2.2) by Lemma 3.2. Proceeding the same as those in the proof of Theorem 3.3, we can easily show that the periodic solution is asymptotically stable. Therefore, (2.2) has a unique asymptotically stable periodic solution. □

4 Application

Now, we turn to some concrete dynamic equations on time scales, which incorporate many mathematical models in real-world applications when the time scale reduces to $\mathbb{R}$ or $\mathbb{Z}$.

Example 4.1. Consider the following autonomous dynamic equation

$$x^{\Delta}(t) = -rx(\sigma(t)) + \eta e^{-\gamma x(t)} \tag{4.1}$$

where r, η, γ are all positive constants and the initial values of (4.1) are positive.

Theorem 4.1. *If $\eta\gamma < r$, then the solutions of* (4.1) *are uniformly bounded and uniformly ultimately bounded.*

Proof. By Theorem 2.1, it is not difficult to show that the solutions of (4.1) are always positive for all $t \geq \varsigma$ if the initial values are positive. Obviously, $\lim_{t\to\infty} \int_{\varsigma}^{t} \xi_{\mu(\tau)}(r)\Delta\tau = \infty$ and for any $x_1, x_2 \in \mathbb{R}^+$, we have $|\eta e^{-\gamma x_1} - \eta e^{-\gamma x_2}| \leq \eta\gamma|x_1 - x_2|$. Moreover,

$$\eta\gamma \int_{\varsigma}^{t} e_{\ominus r}(t,\tau)\Delta\tau = \frac{\eta\gamma}{r}(1 - e_{\ominus r}(t,\varsigma)) < \frac{\eta\gamma}{r} < 1, \quad \eta \int_{\varsigma}^{t} e_{\ominus r}(t,\tau)\Delta\tau \leq \frac{\eta}{r}, \quad t \geq \varsigma.$$

Theorem 3.1 and Theorem 3.2 imply the claims. □

Remark 1. Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$. Then (4.1) can been reformulated as continuous or discrete Lasota-Wazewska model without delay. This kind of models with delay have been extensively discussed in differential equations [6, 16] and difference equations [11].

Example 4.2. Consider the following nonautonomous dynamic equation

$$x^{\Delta}(t) = -a(t)x(\sigma(t)) + b(t), \tag{4.2}$$

where $a, b \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ and b is bounded on $\mathbb{T}$.

In fact, (4.2) is general to incorporate many single species models as special cases. For example, if we let $\mathbb{T} = \mathbb{R}$ and $x(t) = 1/N(t)$, then (4.2) reduces to the famous Verhulst logistic equation $\dot{N}(t) = N(t)(a(t) - b(t)N(t))$. If $\mathbb{T} = \mathbb{Z}$ and $x(t) = 1/N(t)$, then (4.2) reduces to the famous Beverton-Holt equation [2, 13], $N(t+1) = (1 + a(t))N(t)/[1 + b(t)N(t)]$. If $b(t) = a(t)\ln(c(t))$ and $x(t) = \ln(N(t))$, then (4.2) reduces to the continuous Gompertz single species model [7, 14], $\dot{N}(t) = a(t)N(t)\ln(c(t)/N(t))$. When $\mathbb{T} = \mathbb{R}$ or discrete Gompertz single species model [15], $N(t+1) = N(t)^{\frac{1}{1+a(t)}}c(t)^{\frac{a(t)}{1+a(t)}}$ when $\mathbb{T} = \mathbb{Z}$.

Applying those theorems in Section 3 to (4.2), one can easily reach the following claims.

Theorem 4.2. *If* $\bar{a} = \inf_{t\in\mathbb{T}}(a(t)) > 0$, *then the solutions of* (4.2) *are uniformly bounded and uniformly ultimately bounded. Moreover, if* a, b *are* ω*-periodic, then* (4.2) *has a unique periodic solution* $x(t) = \int_{-\infty}^{t} e_{\ominus a}(t,\tau)b(\tau)\Delta\tau$, *which is asymptotically stable.*

Example 4.3. Consider

$$x^{\Delta}(t) = -a(t)x(\sigma(t)) + \frac{b(t)}{1+x^2(t)}, \tag{4.3}$$

where $a, b \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ and b is bounded on $\mathbb{T}$. When $\mathbb{T} = \mathbb{R}$, (4.3) is a particular case of physiological control systems [12, 17].

Theorem 4.3. *Let* $\bar{a} = \inf_{t\in\mathbb{T}}(a(t)) > 0$ *and* $\|b\| = \sup_{t\in\mathbb{T}}(b(t))$. *If* $\|b\| < \bar{a}$, *then the solutions of* (4.3) *are uniformly bounded and uniformly ultimately bounded.*

Proof. Obviously, $\lim_{t\to\infty}\int_{\varsigma}^{t} \xi_{\mu(\tau)}(a(\tau))\Delta\tau = \infty$. For $x_1, x_2 \in \mathbb{R}$, one can reach $|1/(1+x_1^2) - 1/(1+x_2^2)| \leq |x_1 - x_2|$. In addition, for any $t \geq \varsigma$, we have $\int_{\varsigma}^{t} e_{\ominus a}(t,\tau)b(\tau)\Delta\tau = \|b\|/\bar{a}(1 - e_{\ominus a}(t,\varsigma)) < \|b\|/\bar{a} < 1$. It follows from Theorem 3.1 and Theorem 3.2 that the solutions of (4.3) are uniformly bounded and uniformly ultimately bounded. $\square$

By Theorem 3.4, we can easily get

Theorem 4.4. *Assume that* a, b *are both* ω*-periodic and the conditions of Theorem* 4.3 *hold. If* $\frac{e_a(\varsigma+\omega,\varsigma)}{e_a(\varsigma+\omega,\varsigma)-1}\int_{\varsigma}^{\varsigma+\omega} b(\tau)\Delta\tau < 1$, *then* (4.3) *has a unique asymptotically stable periodic solution.*

Example 4.4. Consider the dynamic equation

$$x^{\Delta}(t) = -a(t)x(\sigma(t)) + b(t)\tanh(x(t)) + \gamma(t) \tag{4.4}$$

where $a, b, \gamma \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, b, r are both bounded on $\mathbb{T}$.

When $\mathbb{T} = \mathbb{R}$, (4.4) reduces to a single artificial effective neuron with dissipation [7, 8]. It is clear that $|\tanh(x_1) - \tanh(x_2)| \leq |x_1 - x_2|$ for $x_1, x_2 \in \mathbb{R}$. Applying Theorem 3.1, Theorem 3.2 and Theorem 3.4 to (4.4), one can easily find that Theorem 4.3 and Theorem 4.4 hold for (4.4).

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