数理解析研究所講究録
第 1786 巻 2012 年 116-121

Reconstruction of restoring forces from periods and amplitudes

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Abstract

We consider an inverse problem to reconstruct a restoring force of an autonomous differential equation. The restoring force is obtained globally by a Lipschitz continuous function of each amplitude to corresponding amplitude of each periodic solution to the differential equation and uniquely determined by using a map, a C^{1}-diffeomorphism of each positive amplitude the corresponding negative amplitude of the solution.

1 Main Theorem

This article is a concise introduction to a recent work [4] by the authors. The subject is an inverse problem about nonlinear oscillations on the differential equation

$$\ddot{u}+g(u)=0, \quad \dot{}=\frac{d}{dt}. \tag{1.1}$$

Here a nonlinearity g is a continuous function on $\mathbb{R}$ and satisfies the so-called signal condition

$$ug(u)>0, \quad u\neq 0. \tag{1.2}$$

The condition (1.2) means that the nonlinearity g works as a restoring force. If the solution u of (1.1) has a periodic motion then we can denote the positive and negative half-amplitudes of the solution by a and b, respectively; a full amplitude A is defined by $A=\frac{a-b}{2}$ (see Figure 1). Then the period T of the solution is a function of A.

We consider an inverse problem to reconstruct g of (1.1) from $T(A)$. Namely, $T(A)$ is given and g is unknown function to be determined from $T=T(A)$, and, this problem involves the following three questions:

Problem 1.1

(i) *Given a function $T=T(A)$, does there exist g realizing T?*

(ii) *If such g exist, how many such g do there exist?*

(iii) *What an additional condition determines g uniquely?*

This problem to determine g from a relation T between periods and amplitudes has been studied by many authors. We pick out some results.

- Urabe [6, 7] established that, given a positive function T with a Lipschitz continuous derivative, there exist infinitely many functions g realizing T locally, namely, for sufficiently small amplitudes.
- Alfawicka [1] proved that the local existence of g under a weaker assumption that T itself is Lipschitz continuous.

- Cima, Mañosas and Villadelprat [2] showed that, to determine g uniquely, a map assigning each negative half-amplitude to each positive half-amplitude is useful.
- The global existence of g realizing the half-period function (a relation between half-periods and half-amplitudes) was established by Kamimura [3].

As is mentioned above, the nonlinearity g is not determined uniquely only by the period function T. Hence, as an additional data, we employ a function φ which assigns the negative half-amplitude b to the positive half-amplitude a. Note that, φ is a decreasing function. We call φ a pairing function.

We now prepare three notations. Let $C[\alpha, \beta]$ denote the set of continuous functions on the interval $[\alpha, \beta]$, let $Lip_+(I)$ denote the set of Lipschitz continuous, positive functions on I, and let $Diff^1(I, J)$ denote the set of C^1–diffeomorphisms of I onto J. We suppose that a period function $T \in Lip_+[0, A_{\max}]$ and a pairing function $\varphi \in Diff^1([0, a_{\max}], [b_{\min}, 0])$, where $A_{\max} = \frac{a_{\max}-b_{\min}}{2}$. We obtain the following result, when they are given:

Theorem 1.2 *A nonlinearity $g \in C[b_{\min}, a_{\max}]$ is uniquely reconstructed by T and φ.*

Strictly, Theorem 1.2 implies that: *let $b_{\min} < 0 < a_{\max}$, $A_{\max} = \frac{a_{\max}-b_{\min}}{2}$, then, given a positive, Lipschitz continuous function T on $[0, A_{\max}]$ and a C^1-diffeomorphism φ of $[0, a_{\max}]$ onto $[b_{\min}, 0]$ with $\varphi(0) = 0$, there exist a unique $g \in C[b_{\min}, a_{\max}]$ with* (1.2) *such that* (i) *for each $A \in [0, A_{\max}]$, the period of the solution to* (1.1) *with each amplitude $A \in [0, A_{\max}]$ coincides with T;* (ii) *the correspondence assigning each negative half-amplitude to each positive half-amplitude is given by φ.*

We here present a rough, physical interpretation of our problem and result based upon our own experience. When the earthquake shook the east-northern area of Japan at the 11th of March, 2011, the authors observed that tall buildings around our university located near the Shinagawa station in Tokyo were swinging very hard. The top of the building was swinging from the right edge a to the left edge b and then, from the left to the right (see Figure 1, which is drawn somewhat exaggeratedly). We could get a time (period) which the building took for this one process corresponding to an amplitude A. At the same time we get a correspondence which assigns b to a. When the swing became bigger we get another observed data, a set of a period for another amplitude and another correspondence from a to b. If the data are obtained continuously for $0 < A \le A_{\max}$ (of course, it is possible only theoretically) then one can estimate completely how the restoring force of the building worked.

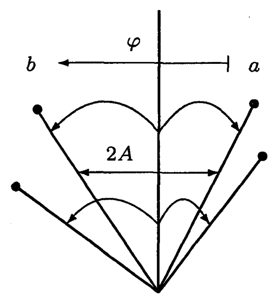

Figure 1: Physical interpretation

The outline of the proof of Theorem 1.2 will be given in Section 2. In Section 3, we shall explain how we obtain g by giving an example. In Section 4, we shall establish a characterization of g, and as a special case, we shall study a problem of isochronicity. In Section 5, we shall pick out some open problems related with Theorem1.2.

2 Outline of the proof

The first important equation is the conservation law

$$\frac{1}{2}\dot{u}(t)^2 + G(u(t)) = E, \tag{2.1}$$

where E is a positive constant expressing the total energy of the system (1.1), and, throughout the paper, we use the notation: $G(u) := \int_0^u g(\xi)d\xi$. From this equation (2.1), we get

$$\dot{u}^2 = 2(E - G(u)).$$

Clearly, this leads to

$$\frac{dt}{du} = \pm\frac{1}{\sqrt{2(E-G(u))}}.$$

Hence, by an elementary calculation, we obtain

$$T = \sqrt{2}\left(\int_0^a \frac{du}{\sqrt{E-G(u)}} - \int_0^b \frac{du}{\sqrt{E-G(u)}}\right),$$

$$G(a) = G(b) = E,$$

where $a = \max u(t)$, $b = \min u(t)$. We call a and b a positive and a negative half-amplitude of the solution u, respectively. If the velocity $\dot{u}$ vanishes, then (2.1) implies the second condition. From this relation, by the substitution $s = G(u)$, which is written as $u = x_+(s)$ $(u > 0)$ and $u = x_-(s)$ $(u < 0)$ by using a new function $x_\pm(s)$, we obtain

$$T = \sqrt{2}\left(\int_0^E \frac{x'_+(s)}{\sqrt{E-s}}ds - \int_0^E \frac{x'_-(s)}{\sqrt{E-s}}ds\right).$$

Therefore, when we define a function $x(E)$ by

$$x(E) := \frac{x_+(E) - x_-(E)}{2},$$

we arrive at

$$T(A) = 2\sqrt{2}\int_0^E \frac{x'(s)}{\sqrt{E-s}}ds \quad \text{with} \quad A = x(E).$$

This function $x(E)$ is a key function in this article. Since $x(E)$ gives the amplitude A, this equality is written as

$$T(x(E)) = 2\sqrt{2}\int_0^E \frac{x'(s)}{\sqrt{E-s}}ds. \tag{2.2}$$

We have to solve (2.2). But, instead of (2.2), it is convenient to use $\frac{1}{2}$-integration of (2.2), where, $\frac{1}{2}$-integration means to apply the integral operator $I^{\frac{1}{2}}$, which is defined by the so-called Riemann-Liouville integral operator

$$I^{\frac{1}{2}}\phi(t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{\phi(s)}{\sqrt{t-s}}ds,$$

of the order $\frac{1}{2}$. Hence, we get

$$\frac{1}{2\sqrt{2}\,\pi}\int_0^E \frac{T(x(s))}{\sqrt{E-s}}ds = x(E), \tag{2.3}$$

which is a key equation of our task. Note that, firstly, $x(E)$ is an unknown function of (2.3) and that, secondly, the numerator is the composition of T and x, which is unknown, and therefore (2.3) is a nonlinear integral equation. Fortunately (see [3]), (2.3) can be solved by a method of successive approximations, $x(E)$ is a C^1–function with $x'(E) > 0$ for $E > 0$, and $x(E)$ attains $A_{\max}$ at some point which is denoted by q.

Now it is possible to show how we construct g in an algorithm (see Figure 2). Firstly, by a given function T, we solve the key equation (2.3), and for $A_{\max} > 0$, determine q such that $x(q) = A_{\max}$. The number q means the maximum value of E. Secondly, $x_{\pm}(E)$ is defined by

$$x(E) = \frac{x_+(E) - x_-(E)}{2}, \quad x_-(E) = \varphi(x_+(E)). \tag{2.4}$$

Then $x_+(q) = a_{\max}$, $x_-(q) = b_{\min}$. At last, by setting $u = x_{\pm}(E)$, we obtain the nonlinearity g on $[b_{\min}, a_{\max}]$ by the definition

$$g(u) = \begin{cases} \dfrac{1}{x_+'(x_+^{-1}(u))}, & 0 \le u \le a_{\max}, \\[2ex] \dfrac{1}{x_-'(x_-^{-1}(u))}, & b_{\min} \le u \le 0. \end{cases} \tag{2.5}$$

When we determine g in this way, one can show that the period of u to (1.1) coincides with the given function T, and such g is unique. This is the outline of the proof of our main theorem.

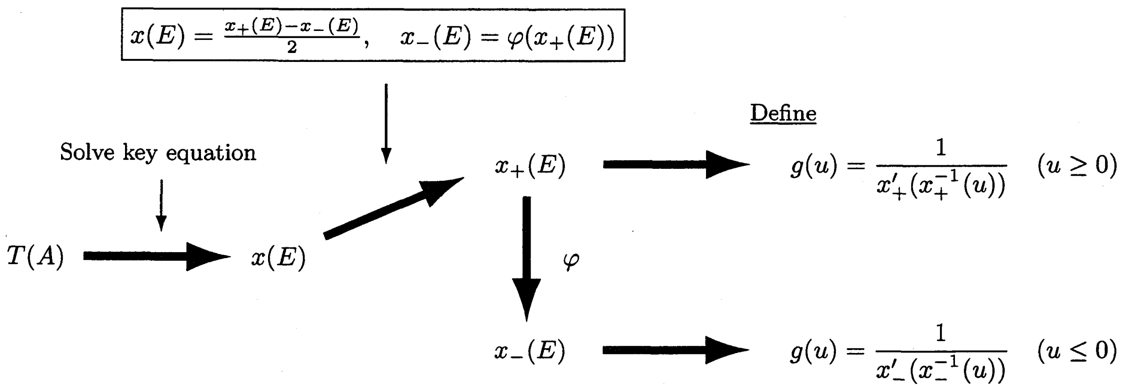

Figure 2: Algorithm

3 Example

In order to see how we get g, let us now explore an example. We give $T(A) = 2\sqrt{2}\,\pi\cosh\frac{A}{2}$ for $A \ge 0$ as a period function and also give $\varphi(a) = 2\log\left(2 - e^{\frac{a}{2}}\right)$ for

$0 < a < 2\log 2$ as a pairing function. By the first step of the algorithm that we gave in the previous section, we obtain the solution

$$x(E) = \log\frac{1+\sqrt{E}}{1-\sqrt{E}}, \quad 0 \le E < 1.$$

In fact, this function satisfies the key equation,

$$\int_0^E \frac{\cosh\left(\frac{1}{2}\log\frac{1+\sqrt{s}}{1-\sqrt{s}}\right)}{\sqrt{E-s}}ds = \int_0^E \frac{ds}{\sqrt{E-s}\sqrt{1-s}} = \log\frac{1+\sqrt{E}}{1-\sqrt{E}}$$

for $0 \le E < 1$. By the second step, namely solving

$$x(E) = \frac{x_+(E) - 2\log\left(2 - e^{\frac{x_+(E)}{2}}\right)}{2},$$

we have

$$x_\pm(E) = 2\log(1 \pm \sqrt{E}\,), \quad 0 \le E < 1.$$

Therefore, by applying (2.5) to these functions $x_\pm(E)$, we finally obtain

$$g(u) = e^u - e^{\frac{u}{2}}, \quad b_{\min} \le u \le a_{\max},$$

for any $b_{\min} < 0 < a_{\max} < 2\log 2$.

4 Characterization

Now we characterize a function G. By the key function and the pairing function defined by (2.4), it leads to

$$E = x^{-1}\left(\frac{x_+(E) - x_-(E)}{2}\right) = x^{-1}\left(\frac{x_+(E) - \varphi(x_+(E))}{2}\right).$$

Hence, by the substitution $E = G(u)$, which is written as $u = x_+(E)$, we have

$$G(u) = x^{-1}\left(\frac{u - \varphi(u)}{2}\right), \quad 0 \le u \le a_{\max}.$$

Also, in a similar way, we get

$$G(u) = x^{-1}\left(\frac{\varphi^{-1}(u) - u}{2}\right), \quad b_{\min} \le u \le 0.$$

Thus, we obtain the following:

Theorem 4.1

$$G(u) = x^{-1}\left(\left|\frac{u - \sigma(u)}{2}\right|\right), \quad b_{\min} \le u \le a_{\max}, \tag{4.1}$$

where the function $\sigma(u)$ is defined by

$$\sigma(u) = \begin{cases} \varphi(u), & 0 \le u \le a_{\max}, \\ \varphi^{-1}(u), & b_{\min} \le u \le 0. \end{cases}$$

The function $\sigma(u)$ has the following properties: (i) $\sigma \circ \sigma = \mathrm{Id}$ (the identity map), $\sigma \neq \mathrm{Id}$; (ii) $\sigma(0) = 0$; (iii) $\sigma \in C^1[0, a_{\max}] \cap C^1[b_{\min}, 0]$. By (iii), the function σ can obtain a cusp at the origin. By differentiating this $G(u)$, we get a nonlinearity $g(u)$. Since x^{-1} and σ are determined from T and φ , respectively, the nonlinearity $g(u)$ can be indexed by the period function T and the pairing function φ through (4.1).

An important, special case of this characterization appears in a problem of isochronicity (see [2, 5]). If given a period T is a constant ω, then we have the function $x(E) = \frac{\omega}{\sqrt{2}\pi}\sqrt{E}$. Therefore, we get the following:

Corollary 4.2 *The period of all the periodic motions of* (1.1) *is a constant* ω *if and only if* $G(u) = \frac{\pi^2}{2\omega^2}(u - \sigma(u))^2$, $u \in \mathbb{R}$, *where* $\sigma(u)$ *is a function satisfying the properties* (i), (ii), *and* (iii)' $\sigma \in C^1[0, \infty) \cap C^1(\infty, 0]$.

5 Future works

Without a doubt there are many open problems concerning this research that should be settled. One of important problems is to make a stability result. This asks whether the correspondence $(T, \varphi) \mapsto g$ is continuous in some appropriate topology (metric). Since, in practical problems, the data set observed is usually discrete, prescribed pair of function T and φ becomes an approximation necessarily. This is why a good stability result is desired. Another is to establish global existence theorems corresponding to Theorem1.2 for other, general equations, for example, $\ddot{u} + g(u, \dot{u}) = 0$ (g depends on the velocity), $\frac{d}{dt}\left(\left|\frac{du}{dt}\right|^{p-2}\frac{du}{dt}\right) + g(u) = 0$ (p–Laplacian), $\Delta u + g(u) = 0$ (multi–dimensional case), and so on. We point out that, for the equation $\ddot{u} + g(x) sgn\, \dot{u} + u = 0$, the readers may refer to [5]. Also, recently, the global existence result for the half-period function concerning $\frac{d}{dt}((\frac{du}{dt})^{2m-1}) + g(u) = 0$ has been established by Usami and Yoshimi (see [8]).

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