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On Weak Approximation of Stochastic Differential Equations with Discontinuous Drift Coefficient[1]

Arturo Kohatsu-Higa
Department of Mathematical Sciences
Ritsumeikan University
1-1-1 Nojihigashi, Kusatsu, Shiga, 525-8577, Japan.
Antoine Lejay
Project-team TOSCA, Institut Élie Cartan Nancy
(Nancy-Université, CNRS, INRIA)
BP 239, F-54506 Vandoeuvre-les-Nancy, France.
Kazuhiro Yasuda
Faculty of Science and Engineering
Hosei University
3-7-2, Kajino-cho, Koganei-shi, Tokyo, 184-8584, Japan.

Abstract

In this paper, weak approximations of multi-dimensional stochastic differential equations with discontinuous drift coefficients are considered. Here as the approximated process, the Euler-Maruyama approximation of SDEs with approximated drift coefficients is used, and we provide a rate of weak convergence of them. Finally we present a rate of weak convergence of the Euler-Maruyama approximation of the original SDEs with constant diffusion coefficients.

1 Introduction

In mathematical finance, one describes asset price processes as the solution to the following stochastic differential equations (SDEs):

$$dX_t = b(t,X_t)\,\mathrm{d}t + \sigma(t,X_t)\,\mathrm{d}W_t. \tag{1.1}$$

where b and σ are certain functions and W_t is a Brownian motion. Then we consider a function f, which represents a payoff function in financial derivatives, and one write its associated option price as the expectation $E[f(X_T)]$, where T is a maturity of the option and X_T is the asset price at T. Note that we are using the interpretation of the expectation using a financial situation, but, of course, it is also important in many other fields and applications.

It is rare the occasion when one is able to calculate the previous expectation analytically. Therefore in order to obtain its value, one resorts to computer simulations and tries to obtain

[1]This paper is an abbreviated and preliminary version of A. Kohatsu-Higa, A. Lejay and K. Yasuda [5]. If ones are interested in this article, please send e-mail to k_yasuda@hosei.ac.jp.

an approximated value. In practice, two kinds of approximations are needed to simulate this expectation. One is an approximation of the SDEs (1.1) and the other is an approximation of the expectation. For the latter, one can typically use the Monte-Carlo method, which is based on law of large numbers in probability theory. On the other hand, for the former, the Euler-Maruyama approximation is often used. The Euler-Maruyama approximation can be described as follows: For simplicity, we split the interval $[0, T]$ equally in n subintervals and let the length of each time subinterval Δt be equal to $\frac{T}{n}$,

$$\bar{X}_0 = x, \quad \bar{X}_{i+1} = \bar{X}_i + b(i\Delta t, \bar{X}_i)\Delta t + \sigma(i\Delta t, \bar{X}_i)\sqrt{\Delta t}\xi_i,$$

where the random variables ξ_i, $i = 0, 1, \cdots, n-1$, are independent of each other and are distributed according to a $N(\mathbf{0}, I_d)$ law, where $\mathbf{0}$ is the d-dimensional zero vector and I_d is $d \times d$-unit matrix. When we approximate stochastic processes, one needs a criteria in order to determine the quality of the approximation. One mainly uses the following two criteria (strong error and weak error): the definition of an approximation with strong error of order $\gamma > 0$ is that there exists a positive constant C, which does not depend on Δt, such that

$$E\left[\left|X_T - \bar{X}_n\right|\right] \leq C\Delta t^{\gamma}.$$

Under enough regularity for coefficients b and σ, the strong error has the order $1/2$ for the above Euler-Maruyama approximation. For more details, readers can refer Exercise 9.6.3 in Kloeden and Platen [4]. The definition of weak error with order $\gamma > 0$ is that for all functions f in a certain class, there exists a positive constant C, which does not depend on Δt, such that

$$\left|E\left[f(X_T)\right] - E\left[f\left(\bar{X}_n\right)\right]\right| \leq C\Delta t^{\gamma}.$$

Here under enough regularity on the coefficients σ and b and on f, we have the weak error with order 1 for the Euler-Maruyama approximation.

The purpose of this paper is to treat an SDE with discontinuous drift coefficients and obtain an order of weak error for its approximation. Precisely speaking, we consider an SDE with an approximated drift coefficient b_ϵ, which is approximated using the Euler-Maruyama approximation. Then, one uses the approximated process as the approximation of the original SDEs. Then we estimate an order of the weak error between the original SDEs and the approximated process. In the latter part of this article, we deal with an SDE with constant diffusion coefficients and obtain an order of the weak error between the SDEs and their approximated process to which the Euler-Maruyama approximation is directly applied.

SDEs with discontinuous drift coefficients are of course used in various fields. For instant, in mathematical finance, if one wants to model a stock price process whose trend dramatically changes when a factor goes down a threshold value. In this case, the drift can be modeled as taking two values specified by some indicator function. This kind of SDE also appears in some control problems.

Weak error of SDEs with discontinuous coefficients (not only drift coefficients, but also diffusion coefficients) have been studied in Chan and Stramer [2] and Yan [12]. However in their papers, they only proved weak convergence of the Euler-Maruyama approximation, not mentioned an order of the weak convergence. And also strong error and the rate are studied in Przybylowicz [10] for SDEs with some type of discontinuous coefficients. Note that in this paper, the diffusion coefficients of our SDEs have enough regularity.

This paper is organized as follows: Some notations and assumptions are given in Section 2. We provide our main result on a rate of weak errors under SDEs with discontinuous drift and nonlinear diffusion coefficient in Section 3, and also give results under constant diffusion coefficients in Section 4. Finally we give some numerical results in Section 5. Proofs of theorems and so on below can be found in Kohatsu-Higa, Lejay and Yasuda [5].

2 Notations and Hypotheses

Let $d \in \mathbb{N}$. The space of continuous functions that are slowly increasing is denoted by $C_{Sl}(\mathbb{R}^d)$. A function f in $C_{Sl}(\mathbb{R}^d)$ is such that for every $k > 0$,

$$\lim_{|x|\to\infty} |f(x)|e^{-k|x|^2} = 0.$$

Fix $T > 0$. Let H be the set $[0,T) \times \mathbb{R}^d$ and $\overline{H} = [0,T] \times \mathbb{R}^d$.

Let σ be a measurable function on $[0,T] \times \mathbb{R}^d$ with values in the space of symmetric $d \times d$-matrices. We set $a = \sigma\sigma^*$ and assume that

$$\begin{aligned}&\text{there exist some positive constants } \Lambda \text{ and } \lambda \ (\Lambda \geq \lambda > 0)\\ &\text{such that } \lambda|\xi|^2 \leq \xi^* a(t,x)\xi \leq \Lambda|\xi|^2, \text{ for all } (t,x) \in \overline{H}, \text{ and all } \xi \in \mathbb{R}^d,\end{aligned} \tag{H1}$$

$$\sigma \text{ is uniformly continuous on } \overline{H}. \tag{H2}$$

Remark 2.1 *Note that* (H1) *gives a lower and upper bound on the eigenvalues of a, which are from the very construction equal to the eigenvalues of σ (we have chosen σ to be symmetric) for which* (H1) *holds with λ and Λ replaced by $\sqrt{\lambda}$ and $\sqrt{\Lambda}$.*

Let us also consider a measurable function b from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}^d$ such that

$$|b(t,x)| \leq \Lambda \text{ for all } (t,x) \in \overline{H}. \tag{H3}$$

From now on, we always assume (H1), (H2) and (H3) for b and σ.

Now, we give some notations. Fix $\alpha > 0$. Let $\mathrm{H}^\alpha(\mathbb{R}^d)$ be the space of continuous, bounded functions with continuous, bounded derivatives up to order $\lfloor\alpha\rfloor$ and such that $\partial_x^{\lfloor\alpha\rfloor} f$ is $(\alpha-\lfloor\alpha\rfloor)$-Hölder continuous. Let $\mathrm{H}^{\alpha/2,\alpha}(\overline{H})$ be the set of continuous functions with continuous derivatives $\partial_t^r\partial_x^s u$ for all $2r+s<\alpha$ and such that

$$\begin{aligned}\|u\|_{\mathrm{H}^{\alpha/2,\alpha}} = &\sum_{2r+s\leq\lfloor\alpha\rfloor} \sup_{(t,x)\in\overline{H}} |\partial_t^r\partial_x^s u(t,x)| + \sum_{2r+s=\lfloor\alpha\rfloor} \sup_{(t,x),(t,y)\in\overline{H}} \frac{|\partial_t^r\partial_x^s u(t,x) - \partial_t^r\partial_x^s u(t,y)|}{|x-y|^{\alpha-\lfloor\alpha\rfloor}}\\ &+ \sum_{0<\alpha-2r-s<2} \sup_{(t,x),(v,x)\in\overline{H}} \frac{|\partial_t^r\partial_x^s u(t,x) - \partial_t^r\partial_x^s u(v,x)|}{|t-v|^{(\alpha-2r-s)/2}}\end{aligned}$$

is finite.

3 Main Theorems

Let σ and b satisfy (H1)–(H3). These conditions are sufficient to ensure the existence of a unique weak solution $(X, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ to

$$X_t = x + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s + \int_0^t b(s, X_s)\,\mathrm{d}s \tag{3.1}$$

for a Brownian motion B.

Remark 3.1 *If $X_t = x + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s$ has a strong solution, then* (3.1) *also admits a strong solution (See Veretennikov* [11]*).*

Let b_ϵ be a family of measurable coefficients on $\overline{H}$ with $|b_\epsilon(t, x)| \leq \Lambda$ for $(t, x) \in \overline{H}$. Let us consider the unique weak solution $(X^\epsilon, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ to

$$X_t^\epsilon = x + \int_0^t \sigma(s, X_s^\epsilon)\,\mathrm{d}B_s + \int_0^t b_\epsilon(s, X_s^\epsilon)\,\mathrm{d}s. \tag{3.2}$$

Since b_ϵ and b are bounded, the distribution of X^ϵ may be deduced from the distribution of X through a Girsanov transform.

For $T > 0$, let $\overline{X}^\epsilon$ be the continuous solution of the Euler-Maruyama scheme of step size T/n. If $\phi(s) = \sup\{t \leq s \mid t = k/n \text{ for } k \in \mathbb{N}\}$, then

$$\overline{X}_t^\epsilon = x + \int_0^t \sigma\left(\phi(s), \overline{X}_{\phi(s)}^\epsilon\right)\mathrm{d}B_s + \int_0^t b_\epsilon\left(\phi(s), \overline{X}_{\phi(s)}^\epsilon\right)\mathrm{d}s. \tag{3.3}$$

When σ and b_ϵ belong to an appropiate class of functions $\mathfrak{M}$ (for example $\mathfrak{M} = \mathrm{H}^{\alpha/2,\alpha}(\overline{H})$ for some $\alpha > 0$ or $\mathfrak{M} = C_\mathrm{b}^{1,3}(\overline{H})$), and when f belongs to a proper class of functions $\mathfrak{F}$ (for example, $\mathfrak{F} = \mathrm{H}^{2+\alpha}(\mathbb{R}^d)$ or $\mathfrak{F} = C^3(\mathbb{R}^d) \cap C_{Sl}(\mathbb{R}^d)$), a rate of weak convergence of the Euler-Maruyama scheme $\overline{X}^\epsilon$ is known. This means that there exists some constant C_ϵ such that

$$\left|\mathbb{E}\left[f(X_T^\epsilon)\right] - \mathbb{E}\left[f(\overline{X}_T^\epsilon)\right]\right| \leq \frac{C_\epsilon}{n^\delta}.$$

Assume that $C_\epsilon = \mathrm{O}(\epsilon^{-\beta})$. This is in general the case when one chooses to use a regularization b_ϵ of b by using mollifiers.

On the other hand, as we will show below in Proposition 3.2 and Remarks 3.3 and 3.5, one has

$$\left|\mathbb{E}\left[f(X_T)\right] - \mathbb{E}\left[f(X_T^\epsilon)\right]\right| \leq C'\mathbb{E}\left[\left(\int_0^T |b(s, Y_s) - b_\epsilon(s, Y_s)|^p\,\mathrm{d}s\right)^{q/p}\right]^{1/q} \tag{3.4}$$

for some appropriate values of p and q and positive constant C'.

Assume that the quantity in the right-hand side of (3.4) decreases to 0 as $\mathrm{O}(\epsilon^\gamma)$. Optimizing over the choice of ϵ leads to the following theorem.

Assume that f belongs to some appropriate class of functions $\mathfrak{F}$, and an approximation b_ϵ of the drift b belongs to some class of functions $\mathfrak{M}$ in a way such that

$$\left|\mathbb{E}\left[f(X_T)\right] - \mathbb{E}\left[f(X_T^\epsilon)\right]\right| = \mathrm{O}(\epsilon^\gamma) \tag{3.5}$$

and

$$\left|\mathbb{E}\left[f(X_T^\epsilon)\right] - \mathbb{E}\left[f(\overline{X}_T^\epsilon)\right]\right| = \mathrm{O}\left(\frac{1}{\epsilon^\beta n^\delta}\right). \tag{3.6}$$

Then for $\epsilon = \mathrm{O}(n^{-\delta/(\gamma+\beta)})$,

$$\left|\mathbb{E}\left[f(X_T)\right] - \mathbb{E}\left[f(\overline{X}_T^\epsilon)\right]\right| \leq \mathrm{O}(n^{-\kappa}) \textit{where } \kappa = \frac{\delta\gamma}{\gamma+\beta}.$$

Under the assumptions (3.5) and (3.6), we have the order κ of the weak error among the SDEs (3.1) and the approximated process (3.3). Therefore, from now on, our interest is to find some conditions that the assumptions (3.5) and (3.6) hold.

3.1 A Perturbation Formula

Through Theorem 3.2 and the remarks below, we can find some situations where Assumption (3.5) holds.

Let X be the solution to (3.1) and X^ϵ be the solution to (3.2).

Theorem 3.2 *For $\alpha > 2$ and $p > 2$ such that $1/\alpha + 1/p < 1/2$ and $f \in C_{Sl}(\mathbb{R}^d)$,*

$$\left|\mathbb{E}\left[f(X_T)\right] - \mathbb{E}\left[f(X_T^\epsilon)\right]\right| \leq C_2(\alpha, p, T) A_T(\epsilon) \sqrt{\mathrm{Var}_{\mathbb{P}}(f(X_T))}$$

with

$$C_2(\alpha, p, T) = T^{1/2-1/p} \exp\left(T\Lambda^2\lambda^{-1}\left(\alpha - \frac{1}{2} + \left(1 - \frac{2}{\alpha}\right)\frac{\alpha\left(\frac{1}{2}+\frac{1}{p}\right)-1}{\alpha\left(\frac{1}{2}-\frac{1}{p}\right)-1}\right)\right),$$

$$A_T(\epsilon) = \mathbb{E}^0\left[\int_0^T |b_\epsilon(s, Y_s) - b(s, Y_s)|^p \,\mathrm{d}s\right]^{1/p},$$

where $(Y, \mathbb{P}^0)$ is the weak solution to $Y_t = x + \int_0^t \sigma(s, Y_s)\,\mathrm{d}W_s$ for some Brownian motion W.

Remark 3.3 *Let us assume that an upper Gaussian estimate holds for the transition density function $p(t, x, y)$ of Y defined by $Y_t = x + \int_0^t \sigma(s, Y_s)\,\mathrm{d}W_s$. This means that for some constants C_1 and C_2,*

$$p(t, x, y) \leq \frac{C_1}{t^{d/2}} \exp\left(\frac{-C_2|y - x|^2}{t}\right), \tag{3.7}$$

for all $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$. *Then for any* $1 < r, q \leq +\infty$ *satisfying* $d/2r + 1/q < 1$, *it follows that*

$$A_T(\epsilon) \leq C_3 \left(\int_0^T \left(\int_{\mathbb{R}^d} |b(s,y) - b_\epsilon(s,y)|^{pq}\, \mathrm{d}y \right)^{r/q} \mathrm{d}s \right)^{1/rp} = C_3 \|b - b_\epsilon\|_{\mathrm{L}^{rp,qp}(H)},$$

where C_3 *is a certain positive constant* C_3, *and for* $r < +\infty$ *set*

$$\|f\|_{\mathrm{L}^{r,q}(H)} = \left(\int_0^T \left(\int_0^T |f(s,x)|^q\, \mathrm{d}x \right)^{r/q} \mathrm{d}s \right)^{1/r},$$

and also set $\|f\|_{\mathrm{L}^{\infty,q}(H)} = \sup_{t \in [0,T]} \|f(t,\cdot)\|_{\mathrm{L}^q}$.

Remark 3.4 *Such estimate* (3.7) *holds for example if the diffusion coefficient a belongs to* $\mathrm{H}^{\alpha/2,\alpha}(H)$ *for some* $\alpha > 0$ *(See for example Ladyženskaja* [7, § IV.13, p. 377]*).*

Remark 3.5 *Even in absence of a Gaussian upper bounds, the Krylov estimate (Krylov* [6] *or Bass* [1, Theorem 7.6.2, p. 114]*) could also be used with Hypothesis* (H1) *in order to get an estimate on* $A_T(\epsilon)$. *In this case of a homogeneous coefficient b, from the Krylov estimate, we have*

$$|A_T(\epsilon)| \leq C(\lambda, \Lambda) e^T \|b - b_\epsilon\|_{\mathrm{L}^{dp}}.$$

In case of a time-inhomogeneous coefficient, a similar estimate could be obtained but on the bounded domain case and one should then estimate the exit time from such domains.

3.2 Rates of Convergence of the Euler-Maruyama Approximation with Regular Enough Coefficients

We now exhibit some situations where Assumption (3.6) holds, under the weakest possible assumptions on the regularity of the coefficients. Note that other results may hold (See Theorem 4.3 below).

3.2.1 Case of Hölder continuous coefficients

The weak rate of convergence of the Euler scheme when the coefficients of the PDE are Hölder continuous has been studied by R. Mikulevicius and E. Platen [9].

Theorem 3.6 (R. Mikulevicius and E. Platen [9]**)** *If for* $\alpha \in (0,1) \cup (1,2) \cup (2,3)$, *b and a belongs to* $\mathrm{H}^{\alpha/2,\alpha}(\overline{H})$ *and* $f \in \mathrm{H}^{2+\alpha}(\mathbb{R}^d)$, *then there exists a constant K such that*

$$\left| \mathbb{E}\left[f(X_T)\right] - \mathbb{E}\left[f\left(\overline{X}_T\right)\right] \right| \leq \frac{K}{n^{E(\alpha)}}$$

with

$$E(\alpha) = \begin{cases} \alpha/2 & \text{if } \alpha \in (0,1), \\ 1/(3-\alpha) & \text{if } \alpha \in (1,2), \\ 1 & \text{if } \alpha \in (2,3). \end{cases}$$

Besides, the constant K is linear in $\|b\|_{\mathrm{H}^{\alpha/2,\alpha}}$ *and* $\|a\|_{\mathrm{H}^{\alpha/2,\alpha}}$.

3.2.2 Case of smooth coefficients

Theorem 3.6 requires the coefficients to be Hölder continuous. Of course, the convergence rate is better for smooth coefficients. But in order to achieve a rate equal to 1, it requires a to be in $\mathrm{H}^{\alpha/2,\alpha}(\overline{H})$ with $\alpha > 2$ and a terminal condition in $\mathrm{H}^{2+\alpha}(\mathbb{R}^d)$ and then with a better regularity than C_p^4.

With a bit more regularity on a and b (if we use molifier for the approximation, b^ε has enough regularity), we see that we achieve a convergence rate equal to 1 provided that f in only in $C^3(\mathbb{R}^d) \cap C_{Sl}(\mathbb{R}^d)$ by using Malliavin calculus.

Theorem 3.7 *Assume that f in $C^3(\mathbb{R}^d) \cap C_{Sl}(\mathbb{R}^d)$, $b_\epsilon \in C_b^{1,3}(\overline{H})$ and $\sigma \in C_b^{1,3}(\overline{H})$. Then for a uniform step size T/n,*

$$\left|\mathbb{E}\left[f(X_T^\epsilon)\right] - \mathbb{E}\left[f(\overline{X}_T^\epsilon)\right]\right| \leq \frac{C}{n}\|b_\epsilon\|_{3,\infty},$$

where C is some positive constant and $\|b_\epsilon\|_{3,\infty}$ is defined as follows;

$$\|b_\epsilon\|_{3,\infty} = \sum_{j=0}^{3}\left\|\frac{\partial^j b_\epsilon}{\partial x^j}\right\|_\infty .$$

3.3 Example

Here we provide an example of order of ϵ in the case of the indicator function $b(t,x) = \mathbf{1}_{[\zeta_1,\zeta_2]}(x)$ for $x \in \mathbb{R}$ and $\zeta_1 < \zeta_2$. If we use the following b_ϵ for an approximation of b, b_ϵ has the Lipschitz continuity: for $\epsilon > 0$,

$$b_\epsilon(x) = \begin{cases} 0, & (-\infty, \zeta_1 - 2\epsilon) \cup (\zeta_2 + 2\epsilon, \infty), \\ \frac{1}{2\epsilon}x - \frac{\zeta_1 - 2\epsilon}{2\epsilon}, & [\zeta_1 - 2\epsilon, \zeta_1), \\ -\frac{1}{2\epsilon}x + \frac{\zeta_2 + 2\epsilon}{2\epsilon}, & (\zeta_2, \zeta_2 + 2\epsilon], \\ 1, & [\zeta_1, \zeta_2]. \end{cases}$$

Then we have the following orders: for $p > 2$,

$$\left(\int_{-\infty}^{\infty} |b_\epsilon(x) - b(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} = \left(\frac{4\epsilon}{p+1}\right)^{\frac{1}{p}} = O\left(\epsilon^{\frac{1}{p}}\right). \tag{3.8}$$

And the rate of the divergence of $\|b_\epsilon\|_{H^\alpha}$ is ϵ^{-1}. Now if we write the constant K in Theorem 3.6 as $K_1\|b\|_{\mathrm{H}^{\alpha/2,\alpha}} + K_2$ for some constants K_1 and K_2, which do not depend on ϵ and n, then an optimal size of ϵ is given as

$$\epsilon = \frac{1}{n^{p/2(1+p)}}\left\{\frac{pK_1}{C_2(\alpha,p,T)C_3\sqrt{\mathrm{Var}_{\mathbb{P}}(f(X_T))}(4/(p-1))^{1/p}}\right\}^{\frac{p}{1+p}},$$

where $C_2(\alpha, p, T)$ is the same as in Theorem 3.2 and C_3 is the same as in Remark 3.3.

If we use a mollifier with the Gaussian kernel as b_ϵ:

$$b_\epsilon(x) = \int_{-\infty}^{\infty} b\left(\frac{x-u}{\epsilon}\right) \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{u^2}{2\epsilon^2}\right) \mathrm{d}u,$$

then we have the same order of the convergence as the above (3.8) and this b_ϵ has enough regularity. And also the rate of the divergence of $\|b_\epsilon\|_{3,\infty}$ is ϵ^{-3}. Hence we obtain an optimal size of ϵ:

$$\epsilon = \frac{1}{n^{p/(1+3p)}} \left\{ \frac{3pC'}{C_2(\alpha, p, T)C_3C'' \sqrt{\mathrm{Var}_{\mathbb{P}}(f(X_T))}} \right\}^{\frac{p}{1+3p}},$$

where assume that we have the following estimations: $C\|b_\epsilon\|_{3,\infty} \leq C'/\epsilon^3$ for some positive constant C' in Theorem 3.7 and $\|b - b_\epsilon\|_{\mathrm{L}^p} \leq C''\epsilon^{1/p}$ for some positive constant C'' in the above estimation with the mollifier.

4 Constant Diffusion Case

We now consider a simple case of a time-homogeneous coefficient and a constant diffusion coefficient. To keep it simple, we assume that σ is the identity matrix and then that X is solution to

$$X_t = x + B_t + \int_0^t b(X_s)\,\mathrm{d}s \tag{4.1}$$

for a Brownian motion B with distribution $\mathbb{P}$. Let b_ϵ be a family of approximations of b satisfying (H3).

Let $\overline{X}$ and $\overline{X}^\epsilon$ be the continuous Euler-Maruyama schemes

$$\overline{X}_t = x + B_t + \int_0^t b\left(\overline{X}_{\phi(s)}\right) \mathrm{d}s \quad \text{and} \quad \overline{X}^\epsilon_t = x + B_t + \int_0^t b_\epsilon\left(\overline{X}^\epsilon_{\phi(s)}\right) \mathrm{d}s.$$

Lemma 4.1 *For $p > 2$, there exists a constant $C_3(p, \Lambda, T)$ such that*

$$\left|\mathbb{E}\left[f\left(\overline{X}_T\right)\right] - \mathbb{E}\left[f\left(\overline{X}^\epsilon_T\right)\right]\right| \leq C_3(p, \Lambda, T) \sqrt{\mathrm{Var}(f(x + B_T))}\|b - b_\epsilon\|_{\mathrm{L}^p}.$$

The next lemma is a direct consequence of Theorem 3.2 and the Hölder inequality of the Gaussian density.

Lemma 4.2 *For $p > d \vee 2$, there exists a constant $C_4(p, \Lambda, T)$ such that*

$$\left|\mathbb{E}\left[f(X_T)\right] - \mathbb{E}\left[f(X^\epsilon_T)\right]\right| \leq C_4(p, \Lambda, T) \sqrt{\mathrm{Var}(f(x + B_T))}\|b - b_\epsilon\|_{\mathrm{L}^p}.$$

The rate of weak convergence of the Euler-Maruyama scheme to the solution to (4.1) has been studied by V. Mackevičius in [8] for a drift coefficient which is Lipschitz continuous. The proof is given for the dimension $d = 1$, but it is remarked in the article that it is suitable whatever the dimension (See Remark below Theorem 1 in [8]).

Let us denote by $C^3_{\mathrm{p}}(\mathbb{R}^d)$ the space of functions on $\mathbb{R}^d$ that are three times continuously differentiable with all the derivatives up to order 3 of polynomial growth. Of course, $C^3_{\mathrm{p}}(\mathbb{R}^d) \subset C_{Sl}(\mathbb{R}^d)$.

Theorem 4.3 (R. Mackevičius, [8, Theorem 1]**)** *If b_ϵ is bounded Lipschitz continuous with constant* $\mathrm{Lip}(b_\epsilon)$ *and $f \in C^3_{\mathrm{p}}(\mathbb{R}^d)$, then there exists a constant $C_5(T, \Lambda, f)$ such that*

$$\left|\mathbb{E}\left[f(X^\epsilon_T)\right] - \mathbb{E}\left[f\left(\overline{X}^\epsilon_T\right)\right]\right| \leq \frac{C_5(T, \Lambda, f)}{n} \mathrm{Lip}(b_\epsilon).$$

Remark 4.4 *The statement of Theorem 1 in Mackevičius* [8] *is slightly different since b is not assumed to be bounded. Yet it is clear from the proof that the constant is linear in* $\mathrm{Lip}(b_\epsilon)$ *if b is also bounded.*

For a set G in $\mathbb{R}^d$, we define $G(\epsilon) = \{x \in \mathbb{R}^d | d(x, G) \leq \epsilon\}$, where $d(x, G) = \inf_{y \in G} |x - y|$ is the distance between x and G.

Theorem 4.5 *Let b be a bounded function on $\mathbb{R}^d$ which is Lipschitz except on a set G such that the Lebesgue* $\mathrm{meas}(G(\epsilon)) = \mathrm{O}(\epsilon^d)$*. Then for any $f \in C^3_{\mathrm{p}}(\mathbb{R})$ and $p > d \vee 2$,*

$$\left|\mathbb{E}\left[f(X_T)\right] - \mathbb{E}\left[f\left(\overline{X}_T\right)\right]\right| = \mathrm{O}\left(n^{-\frac{d}{p+d}}\right).$$

Remark 4.6 *We see that the rate of weak error converges to* $1/2$ *(resp.* $1/3$*) when $d > 2$ (resp. $d = 1$) when $p \to d$ (resp. $p \to 2$). However, the constants hidden in the* $\mathrm{O}(n^{-d/(p+d)})$ *explode to infinity as $p \to d \vee 2$. This means that with our estimates, a better rate of convergence is obtained at the cost of a bigger constant in front of the rate.*

Remark 4.7 *In the proof of Theorem 4.5, we choose an optimal size of ϵ as*

$$\epsilon = \frac{1}{n^{p/(p+d)}} \left\{ \frac{pC_5(T, \Lambda, f)C}{d(C_3(p, \Lambda, T) + C_4(p, \Lambda, T)) \sqrt{\mathrm{Var}(f(x + B_T))} C'} \right\},$$

where assume that we have the following estimations: $\mathrm{Lip}(b_\epsilon) = C/\epsilon$ *for some positive constant C in Theorem 4.3 and $\|b - b_\epsilon\|_{L^p} \leq C' \epsilon^{d/p}$ for some positive constant C'. Then we obtain the above rate of the weak error.*

5 Numerical Results

In this section, we give some preliminary numerical experiments in order to determine if the rates of weak convergence are optimal and to which extent the slower rate of convergence can be observed. Here we consider the following SDE:

$$X_t = x + \int_0^t b(X_s)\,\mathrm{d}s + W_t, \tag{5.1}$$

where

$$b(x) = \begin{cases} \theta_1, & x \le 0, \\ \theta_0, & x > 0. \end{cases}$$

This process is called a Brownian motion with two-valued, state-dependent drift, which is related to a stochastic control problem. Then from Karatzas and Shreve [3, Section 6.5], the transition density function is given as follows:

$$p_t(x,z) = \begin{cases} 2\displaystyle\int_0^{\infty}\int_0^t e^{2b\theta_1} h(t-s; y-z, -\theta_1) h(s; x+y, -\theta_0)\,\mathrm{d}s\,\mathrm{d}y, & x \ge 0,\ z \le 0, \\ 2\displaystyle\int_0^{\infty}\int_0^t e^{2(b\theta_1 + z\theta_0)} h(t-s; y, -\theta_1) h(s; x+y+z, -\theta_0)\,\mathrm{d}s\,\mathrm{d}y & \\ \quad + \dfrac{1}{\sqrt{2\pi t}}\left\{\exp\left(-\dfrac{(x-z+\theta_0 t)^2}{2}\right) - \exp\left(-\dfrac{(x+z-\theta_0 t)^2}{2} - 2\theta_0 x\right)\right\}, & x \ge 0,\ z > 0, \end{cases}$$

where set

$$h(t; x, \mu) = \frac{|x|}{\sqrt{2\pi t^3}} \exp\left(-\frac{(x-\mu t)^2}{2t}\right), \quad t > 0,\ x \ne 0,\ \mu \in \mathbb{R}.$$

Note that if $\theta_1 = -\theta_0 = \theta > 0$ and $x = 0$, the distribution of X_t is symmetric with respect to y-axis. So that when f is an odd function, we have $\mathbb{E}[f(X_t)] = 0$.

Two approximated processes are attempted: one is the Euler-Maruyama approximation of the original SDE (5.1), and the other is the Euler-Maruyama approximaton of SDE with the approximated drift coefficient

$$b_\epsilon(x) = \begin{cases} \theta_1, & x \le -\epsilon, \\ \dfrac{\theta_0 - \theta_1}{2\epsilon} x + \dfrac{\theta_0 + \theta_1}{2}, & -\epsilon < x \le \epsilon, \\ \theta_0, & x > \epsilon, \end{cases}$$

for $\epsilon > 0$. From Remark 4.7, set $\epsilon = n^{-\frac{2}{3}}$, where n is a number of time steps of the Euler-Maruyama approximation.

5.1 Case: $\theta_1 = -\theta_0 = 1$ and $f(x) = x$

In this section, we show a numerical result in the case of $\theta_1 = -\theta_0 = 1$, $f(x) = x$ and the initial value $X_0 = 0$. Then the true value of $\mathbb{E}[f(X_1)] = 0$ since $f(x) = x$ is an odd function.

Through Figure 1 to Figure 3, x-axis denotes the number of time steps n until time 1 from 10 to 150 with logarithmic scale. Weak errors of simulation results are reported at a logarithmic scale on the y-axis, that is $|\mathbb{E}[f(X_1)] - \mathbb{E}[f(\overline{X}_1)]|$ (thin line) and $|\mathbb{E}[f(X_1)] - \mathbb{E}[f(\overline{X}_1^{\varepsilon})]|$ (dotted line), where to obtain their expectation values, we use the Monte-Carlo method with 10^7 simulations for each n. If they are parallel to the thick straight line, the convergence rate has the order 1.

The numerical result in the case of $f(x) = x$ is the following:

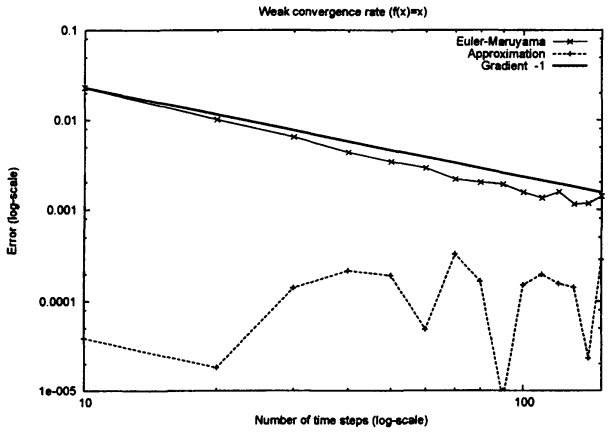

Figure 1: No. of time steps – weak error ($f(x) = x$).

From Figure 1, it is easy to find that the convergence rate of the Euler-Maruyama method has order 1, but for the Euler-Maruyama method with the approximated drift, the approximation converges much faster than the uncorrected one.

5.2 Case: $\theta_1 = -\theta_0 = 1$ and $f(x) = x^2$

Here we use the same values of parameters in the previous section and let $f(x) = x^2$. From Karatzas and Shreve [3, Exercise 6.5.3, pp.441], we have

$$\mathbb{E}\left[X_t^2\right] = \frac{1}{2} + \sqrt{\frac{t}{2\pi}}(|x| - t - 1)\exp\left(-\frac{(|x| - t)^2}{2t}\right) + \left\{(|x| - t)^2 + t - \frac{1}{2}\right\}\Phi\left(\frac{|x| - t}{\sqrt{t}}\right)$$
$$+ e^{2|x|}\left(|x| + t - \frac{1}{2}\right)\left[1 - \Phi\left(\frac{|x| + t}{\sqrt{t}}\right)\right],$$

where set

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{u^2}{2}}\,du.$$

And in the case of $x = 0$ and $t = 1$, we obtain $\mathbb{E}[f(X_1)] = 0.333369$.
The numerical result in the case of $f(x) = x^2$ is the following:

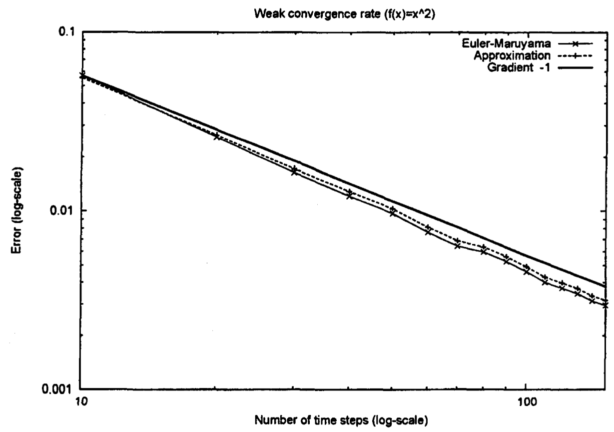

Figure 2: No. of time steps – weak error ($f(x) = x^2$).

From Figure 2, we easily find that the rate of convergence in the both methods is 1.

5.3 Case: $\theta_1 = -\theta_0 = 1$ and $f(x) = \mathbf{1}(x > 0) - \mathbf{1}(x \leq 0)$

In this section, we use $f(x) = \mathbf{1}(x > 0) - \mathbf{1}(x \leq 0)$ which does not have regurality and does not belong to our theorem. Note that the funciton f is symmetric with respect to the origin a.e. and X_t has the continuous and symmetric density function so that we have $\mathbb{E}[f(X_1)] = 0$.
The numerical result in the case of $f(x) = \mathbf{1}(x > 0) - \mathbf{1}(x \leq 0)$ is the following:

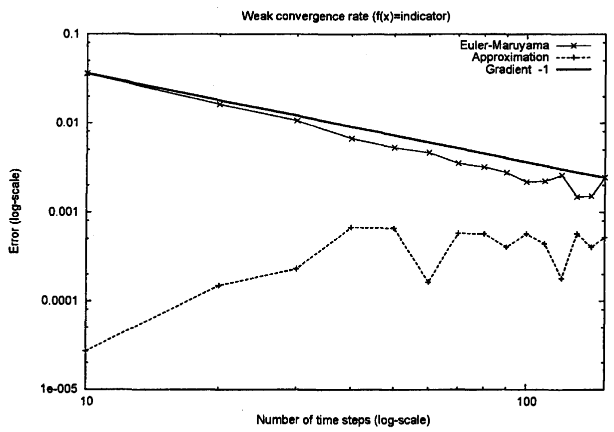

Figure 3: No. of time steps – weak error ($f(x) = \mathbf{1}(x > 0) - \mathbf{1}(x \leq 0)$).

From Figure 3, it is easy to find that the convergence rate of the Euler-Maruyama method has order 1, but as before the Euler-Maruyama method with the approximated drift, converges faster.

We have tested three cases above, the weak convergence rate of the Euler-Maruyama approximation in all of them is 1. And in the case of the Euler-Maruyama approximation with the approximated drift, we could not obtain the rate of convergence because the approximation converges too fast for $f(x) = x$ and $\mathbf{1}(x > 0) - \mathbf{1}(x \leq 0)$, but for $f(x) = x^2$, we find that the convergence rate is 1. This is probably due to how ϵ is chosen. In this case, we have chosen this example because we can obtain the weak limit in closed form. In order to have slower orders, we need to consider more complicated situations.

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