ALMOST DISJOINT AND INDEPENDENT FAMILIES

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ABSTRACT. I collect a number of proofs of the existence of large almost disjoint and independent families on the natural numbers. This is mostly the outcome of a discussion on mathoverflow.

1. INTRODUCTION

A family \( \mathcal{F} \subseteq \mathcal{P}(\omega) \) is an independent family (over \( \omega \)) if for every pair \( \mathcal{A}, \mathcal{B} \) of disjoint finite subsets of \( \mathcal{F} \) the set

\[ \cap \mathcal{A} \cap (\omega \setminus \cup \mathcal{B}) \]

is infinite. Fichtenholz and Kantorovich showed that there is an independent family on \( \omega \) of size continuum [3] (also see [6] or [8]). I collect several proofs of this fundamental fact. A typical application of the existence of a large independent family is the result that there are \( 2^{2^{\aleph_0}} \) ultrafilters on \( \omega \) due to Pospíšil [11]:

Given an independent family \( (A_\alpha)_{\alpha < 2^{\aleph_0}} \), for every function \( f : 2^{\aleph_0} \to 2 \) there is an ultrafilter \( p_f \) on \( \omega \) such that for all \( \alpha < 2^{\aleph_0} \) we have \( A_\alpha \in p_f \) iff \( f(\alpha) = 1 \). Now \( (p_f)_{f : 2^{\aleph_0} \to 2} \) is a family of size \( 2^{2^{\aleph_0}} \) of pairwise distinct ultrafilters.

Independent families in some sense behave similarly to almost disjoint families. Subsets \( A \) and \( B \) of \( \omega \) are almost disjoint if \( A \cap B \) is finite. A family \( \mathcal{F} \) of infinite subsets of \( \mathcal{P}(\omega) \) is almost disjoint any two distinct elements \( A, B \) of \( \mathcal{F} \) are almost disjoint.

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An easy diagonalisation shows that every countably infinite, almost disjoint family can be extended.

**Lemma 2.1.** Let \( (A_n)_{n \in \omega} \) be a sequence of pairwise almost disjoint, infinite subsets of \( \omega \). Then there is an infinite set \( A \subseteq \omega \) that is almost disjoint from all \( A_n, n \in \omega \).

**Proof.** First observe that since the \( A_n \) are pairwise almost disjoint, for all \( n \in \omega \) the set

\[ \omega \setminus \bigcup_{k < n} A_k \]

is infinite. Hence we can choose a strictly increasing sequence \( (a_n)_{n \in \omega} \) of natural numbers such that for all \( n \in \omega, a_n \in \omega \setminus \bigcup_{k < n} A_k \). Clearly, if \( k < n \), then \( a_n \not\in A_k \).

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It follows that for every $k \in \omega$ the infinite set $A = \{a_n : n \in \omega\}$ is almost disjoint from $A_k$. \hfill \square

A straightforward application of Zorn's Lemma gives the following:

**Lemma 2.2.** Every almost disjoint family of subsets of $\omega$ is contained in a maximal almost disjoint family of subsets of $\omega$.

**Corollary 2.3.** Every infinite, maximal almost disjoint family is uncountable. In particular, there is an uncountable almost disjoint family of subsets of $\omega$.

**Proof.** The uncountability of an infinite, maximal almost disjoint family follows from Lemma 2.1. To show the existence of such a family, choose a partition $(A_n)_{n \in \omega}$ of $\omega$ into pairwise disjoint, infinite sets. By Lemma 2.2, the almost disjoint family $\{A_n : n \in \omega\}$ extends to a maximal almost disjoint family, which has to be uncountable by our previous observation. \hfill \square

Unfortunately, this corollary only guarantees the existence of an almost disjoint family of size $\aleph_1$, not necessarily of size $2^{\aleph_0}$.

**Theorem 2.4.** There is an almost disjoint family of subsets of $\omega$ of size $2^{\aleph_0}$.

All the following proofs of Theorem 2.4 have in common that instead of on $\omega$, the almost disjoint family is constructed as a family of subsets of some other countable set that has a more suitable structure.

**First proof.** We define the almost disjoint family as a family of subsets of the complete binary tree $2^{<\omega}$ of height $\omega$ rather than $\omega$ itself. For each $x \in 2^{\omega}$ let $A_x = \{x \upharpoonright n : n \in \omega\}$.

If $x, y \in 2^{\omega}$ are different and $x(n) \neq y(n)$, then $A_x \cap A_y$ contains no sequence of length $> n$. It follows that $\{A_x : x \in 2^{\omega}\}$ is an almost disjoint family of size continuum. \hfill \square

Similarly, one can consider for each $x \in [0,1]$ the set $B_x$ of finite initial segments of the decimal expansion of $x$. $\{B_x : x \in [0,1]\}$ is an almost disjoint family of size $2^{\aleph_0}$ of subsets of a fixed countable set.

**Second proof.** We again identify $\omega$ with another countable set, in this case the set $\mathbb{Q}$ of rational numbers. For each $r \in \mathbb{R}$ choose a sequence $(q_n^r)_{n \in \omega}$ of rational numbers that is not eventually constant and converges to $r$. Now let $A_r = \{q_n^r : n \in \omega\}$.

For $s, r \in \mathbb{R}$ with $s \neq r$ choose $\varepsilon > 0$ so that

$$(s - \varepsilon, s + \varepsilon) \cap (r - \varepsilon, r + \varepsilon) = \emptyset.$$  

Now $A_s \cap (s - \varepsilon, s + \varepsilon)$ and $A_r \cap (r - \varepsilon, r + \varepsilon)$ are both cofinite and hence $A_s \cap A_r$ is finite. It follows that $\{A_r : r \in \mathbb{R}\}$ is an almost disjoint family of size $2^{\aleph_0}$. \hfill \square

**Third proof.** We construct an almost disjoint family on the countable set $\mathbb{Z} \times \mathbb{Z}$. For each angle $\alpha \in [0, 2\pi)$ let $A_\alpha$ be the set of all elements of $\mathbb{Z} \times \mathbb{Z}$ that have distance $\leq 1$ to the line $L_\alpha = \{(x, y) \in \mathbb{R}^2 : y = \tan(\alpha) \cdot x\}$. 

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For two distinct angles $\alpha$ and $\beta$ the set of points in $\mathbb{R}^2$ of distance $\leq 1$ to both $L_\alpha$ and $L_\beta$ is compact. It follows that $A_\alpha \cap A_\beta$ is finite. Hence $\{A_\alpha : \alpha \in [0, 2\pi)\}$ is an almost disjoint family of size continuum.

Fourth proof. We define a map $e : [0, 1] \to \omega^\omega$ as follows: for each $x \in [0, 1]$ and $n \in \omega$ let $e(x)(n)$ be the integer part of $n \cdot x$.

For every $x \in [0, 1]$ let $A_x = \{(n, e(x)(n)) : n \in \omega\}$. If $x < y$, then for all sufficiently large $n \in \omega$, $e(x)(n) < e(y)(n)$. It follows that $\{A_x : x \in [0, 1]\}$ is an almost disjoint family of subsets of $\omega \times \omega$.

Observe that $e$ is an embedding of $([0, 1], \leq)$ into $(\omega^\omega, \leq^*)$, where $f \leq^* g$ if for almost all $n \in \omega, f(n) \leq g(n)$.

3. INDEPENDENT FAMILIES

Independent families behave similarly to almost disjoint families. The following results are analogs of the corresponding facts for almost disjoint families.

Lemma 3.1. Let $m$ be an ordinal $\leq \omega$ and let $(A_n)_{n<m}$ be a sequence of infinite subsets of $\omega$ such that for all pairs $S, T$ of finite disjoint subsets of $m$ the set

$$\bigcap_{n \in S} A_n \setminus \left( \bigcup_{n \in T} A_n \right)$$

is infinite. Then there is an infinite set $A \subseteq \omega$ that is independent over the family $\{A_n : n < m\}$ in the sense that for all pairs $S, T$ of finite disjoint subsets of $m$ both

$$\left( A \cap \bigcap_{n \in S} A_n \right) \setminus \left( \bigcup_{n \in T} A_n \right)$$

and

$$\bigcap_{n \in S} A_n \setminus \left( A \cup \bigcup_{n \in T} A_n \right)$$

are infinite.

Proof. Let $(S_n, T_n)_{n \in \omega}$ be an enumeration of all pairs of disjoint finite subsets of $m$ such that every such pair appears infinitely often.

By the assumptions on $(A_n)_{n \in \omega}$, we can choose a strictly increasing sequence $(a_n)_{n \in \omega}$ such that for all $n \in \omega$,

$$a_{2n}, a_{2n+1} \in \bigcap_{k \in S_n} A_k \setminus \left( \bigcup_{k \in T_n} A_k \right).$$

Now the set $A = \{a_{2n} : n \in \omega\}$ is independent over $\{A_n : n < m\}$. Namely, let $S, T$ be disjoint finite subsets of $m$. Let $n \in \omega$ be such that $S = S_n$ and $T = T_n$. Now by the choice of $a_{2n}$,

$$a_{2n} \in \left( A \cap \bigcap_{k \in S_n} A_k \right) \setminus \left( \bigcup_{k \in T_n} A_k \right).$$
On the other hand,

\[ a_{2n+1} \in \bigcap_{k \in S_n} A_k \setminus \left( A \cup \bigcup_{k \in T_n} A_k \right). \]

Since there are infinitely many \( n \in \omega \) with \( (S, T) = (S_n, T_n) \), it follows that the sets

\[ \left( A \cap \bigcap_{k \in S_n} A_k \right) \setminus \left( \bigcup_{k \in T_n} A_k \right) \]

and

\[ \bigcap_{k \in S_n} A_k \setminus \left( A \cup \bigcup_{k \in T_n} A_k \right) \]

are both infinite.

Another straightforward application of Zorn's Lemma yields:

**Lemma 3.2.** Every independent family of subsets of \( \omega \) is contained in a maximal independent family of subsets of \( \omega \).

**Corollary 3.3.** Every infinite maximal independent family is uncountable. In particular, there is an uncountable independent family of subsets of \( \omega \).

*Proof.* By Lemma 3.2, there is a maximal independent family. By Lemma 3.1 such a family cannot be finite or countably infinite.

As in the case of almost disjoint families, this corollary only guarantees the existence of independent families of size \( \aleph_1 \). But Fichtenholz and Kantorovich showed that there are independent families on \( \omega \) of size continuum.

**Theorem 3.4.** There is an independent family of subsets of \( \omega \) of size \( 2^{\aleph_0} \).

In the following proofs of this theorem, we will replace the countable set \( \omega \) by other countable sets with a more suitable structure. Let us start with the original proof by Fichtenholz and Kantorovich [3] that was brought to my attention by Andreas Blass.

**First proof.** Let \( C \) be the countable set of all finite subsets of \( \mathbb{Q} \). For each \( r \in \mathbb{R} \) let

\[ A_r = \{ a \in C : a \cap (-\infty, r] \text{ is even} \}. \]

Now the family \( \{ A_r : r \in \mathbb{R} \} \) is an independent family of subsets of \( C \).

Let \( S \) and \( T \) be finite disjoint subsets of \( \mathbb{R} \). A set \( a \in C \) is an element of

\[ \bigcap_{r \in S} A_r \setminus \left( C \setminus \bigcup_{r \in T} A_r \right) \]

if for all \( r \in S \), \( a \cap (-\infty, r] \) is odd and for all \( r \in T \), \( a \cap (-\infty, r] \) is even. But it is easy to see that there are infinitely many finite sets \( a \) of rational numbers that satisfy these requirements.

\[ \square \]
The following proof is due to Hausdorff and generalizes to higher cardinals [4]. We will discuss this generalization in Section 4.

Second proof. Let
\[ I = \{(n, A) : n \in \omega \land A \subseteq \mathcal{P}(n)\} \]
For all \( X \subseteq \omega \) let \( X' = \{(n, A) \in I : X \cap n \in A\} \). We show that \( \{X' : X \in \mathcal{P}(\omega)\} \) is an independent family of subsets of \( I \).

Let \( S \) and \( T \) be finite disjoint subsets of \( \mathcal{P}(\omega) \). A pair \((n, A) \in I\) is in
\[ \bigcap_{X \in S} X' \cap \left( I \setminus \bigcup_{X \in T} X' \right) \]
if for all \( X \in S, X \cap n \in A \) and for all \( X \in T, X \cap n \notin A \). Since \( S \) and \( T \) are finite, there is \( n \in \omega \) such that for any two distinct \( X, Y \in S \cup T, X \cap n \neq Y \cap n \). Let \( A = \{X \cap n : X \in S\} \). Now
\[ (n, A) \in \bigcap_{X \in S} X' \cap \left( I \setminus \bigcup_{X \in T} X' \right). \]
Since there are infinitely many \( n \) such that for any two distinct \( X, Y \in S \cup T, X \cap n \neq Y \cap n \), this shows that
\[ \bigcap_{X \in S} X' \cap \left( I \setminus \bigcup_{X \in T} X' \right) \]
is infinite. \( \square \)

A combinatorially simple, topological proof of the existence of large independent families can be obtained using the Hewitt-Marczewski-Pondiczery theorem which says that the product space \( 2^R \) is separable ([5, 9, 10], also see [2]). This is the first topological proof.

Third proof. For each \( r \in R \) let \( B_r = \{f \in 2^R : f(r) = 0\} \). Now whenever \( S \) and \( T \) are finite disjoint subsets of \( \mathbb{R} \),
\[ \bigcap_{r \in S} B_r \cap \left( 2^R \setminus \bigcup_{r \in T} B_r \right) \]
is a nonempty clopen subset of \( 2^R \).

The family \((B_r)_{r \in R}\) is the prototypical example of an independent family of size continuum on any set. A striking fact about the space \( 2^R \) is that it is separable. Namely, let \( D \) denote the collection of all functions \( f : \mathbb{R} \to 2 \) such that there are rational numbers \( q_0 < q_1 < \cdots < q_{2n-1} \) such that for all \( x \in \mathbb{R}, \)
\[ f(x) = 1 \iff x \in \bigcup_{i \leq n} (q_{2i}, q_{2i+1}). \]
\( D \) is a countable dense subset of \( 2^R \).
For each \( r \in \mathbb{R} \) let \( A_r = B_r \cap D \). Now for all pairs \( S, T \) of finite disjoint subsets of \( \mathbb{R} \),
\[
\bigcap_{r \in S} A_r \cap \left( D \setminus \bigcup_{r \in T} A_r \right) = D \cap \bigcap_{r \in S} B_r \cap \left( 2^\mathbb{R} \setminus \bigcup_{r \in T} B_r \right)
\]
is infinite, being the intersection of a dense subset with a nonempty open subset of a topological space without isolated points. It follows that \( (A_r)_{r \in \mathbb{R}} \) is an independent family of size continuum on the countable set \( D \).

The second topological proof of Theorem 3.4 was pointed out by Ramiro de la Vega.

**Fourth proof.** Let \( \mathcal{B} \) be a countable base for the topology on \( \mathbb{R} \) that is closed under finite unions. Now for each \( r \in \mathbb{R} \) consider the set \( A_r = \{ B \in \mathcal{B} : r \in B \} \). Then \( (A_r)_{r \in \mathbb{R}} \) is an independent family of subsets of the countable \( \mathcal{B} \).

Namely, let \( S \) and \( T \) be disjoint finite subsets of \( \mathbb{R} \). The set \( \mathbb{R} \setminus T \) is open and hence there are open sets \( U_s \in \mathcal{B}, \ s \in S, \) such that each \( U_s \) contains \( s \) and is disjoint from \( T \). Since \( \mathcal{B} \) is closed under finite unions, \( U = \bigcup_{s \in S} U_s \in \mathcal{B} \). Clearly, there are actually infinitely many possible choices of a set \( U \in \mathcal{B} \) such that \( S \subseteq U \) and \( T \cap U = \emptyset \). This shows that \( \bigcap_{r \in S} A_r \setminus (\bigcup_{r \in T} A_r) \) is infinite.

A variant of the Hewitt-Marczewski-Pondiczery argument was mentioned by Martin Goldstern who claims to have heard it from Menachem Kojman.

**Fifth proof.** Let \( P \) be the set of all polynomials with rational coefficients. For each \( r \in \mathbb{R} \) let \( A_r = \{ p \in P : p(r) > 0 \} \). If \( S, T \subseteq \mathbb{R} \) are finite and disjoint, then there is a polynomial in \( P \) such that \( p(r) > 0 \) for all \( r \in A \) and \( p(r) \leq 0 \) for all \( r \in T \). All positive multiples of \( p \) satisfy the same inequalities. It follows that \( (A_r)_{r \in \mathbb{R}} \) is an independent family of size \( 2^\aleph_0 \) over the countable set \( P \).

The next proof was pointed out by Tim Gowers. This is the dynamical proof.

**Sixth proof.** Let \( X \) be a set of irrationals that is linearly independent over \( \mathbb{Q} \). Kronecker's theorem states that for every finite set \( \{r_1, \ldots, r_k\} \subseteq X \) with pairwise distinct \( r_i \), the closure of the set \( \{(nr_1, \ldots, nr_k) : n \in \mathbb{Z} \} \) is all of the \( k \)-dimensional torus \( \mathbb{R}^k/\mathbb{Z}^k \) ([7], also see [1]).

For each \( r \in X \) let \( A_r \) be the set of all \( n \in \mathbb{Z} \) such that the integer part of \( n \cdot r \) is even. Then \( (A_r : r \in X) \) is an independent family of size continuum. To see this, let \( S, T \subseteq X \) be finite and disjoint. By Kronecker's theorem there are infinitely many \( n \in \mathbb{Z} \) such that for all \( r \in S \), the integer part of \( n \cdot r \) is even and for all \( r \in T \), the integer part of \( n \cdot r \) is odd. For all such \( n \),
\[
n \in \bigcap_{r \in S} A_r \cap \bigcap_{r \in T} \mathbb{Z} \setminus A_r.
\]

The following proof was mentioned by KP Hart. Let us call it the almost disjoint proof.
Seventh proof. Let \( \mathcal{F} \) be an almost disjoint family on \( \omega \) of size continuum. To each \( A \in \mathcal{F} \) we assign the collection \( A' \) of all finite subsets of \( \omega \) that intersect \( A \). Now \( \{A' : A \in \mathcal{F}\} \) is an independent family of size continuum.

Given disjoint finite sets \( S, T \subseteq \mathcal{F} \), by the almost disjointness of \( \mathcal{F} \), each \( A \in S \) is almost disjoint from \( \bigcup T \). It follows that there are infinitely many finite subsets of \( \omega \) that intersect all \( A \in S \) but do not intersect any \( A \in T \). Hence

\[
\bigcap_{A \in S} A' \cap \left( \omega \setminus \bigcup_{A \in T} A' \right)
\]

is infinite.

The last proof was communicated by Peter Komjáth. This is the proof by finite approximation.

Eighth proof. First observe that for all \( n \in \omega \) there is a family \( (X_k)_{k \leq n} \) of subsets of \( 2^n \) such that for any two disjoint sets \( S, T \subseteq n \),

\[
\bigcap_{k \in S} X_k^n \cap \left( 2^n \setminus \bigcup_{k \in T} X_k \right)
\]

is nonempty. Namely, let \( X_k = \{f \in 2^n : f(k) = 0\} \).

Now choose, for every \( n \in \omega \), a family \( (X^n_s)_{s \in 2^n} \) of subsets of a finite set \( Y_n \) such that for disjoint sets \( S, T \subseteq 2^n \),

\[
\bigcap_{s \in S} X^n_s \cap \left( 2^n \setminus \bigcup_{s \in T} X^n_s \right)
\]

is nonempty. We may assume that the \( Y_n, n \in \omega \), are pairwise disjoint.

For each \( \sigma \in 2^\omega \) let \( X_\sigma = \bigcup_{n \in \omega} X^n_\sigma \). Now \( \{X_\sigma : \sigma \in 2^\omega\} \) is an independent family of size \( 2^{\aleph_0} \) on the countable set \( \bigcup_{n \in \omega} Y_n \).  

4. INDEPENDENT FAMILIES ON LARGER SETS

We briefly point out that for every cardinal \( \kappa \) there is an independent family of size \( 2^\kappa \) of subsets of \( \kappa \). We start with a corollary of the Hewitt-Marczewski-Pondiczery Theorem higher cardinalities.

Lemma 4.1. Let \( \kappa \) be an infinite cardinal. Then \( 2^{2^\kappa} \) has a dense subset \( D \) such that for every nonempty clopen subset \( A \) of \( 2^{2^\kappa} \), \( D \cap A \) is of size \( \kappa \). In particular, \( 2^{2^\kappa} \) has a dense subset of size \( \kappa \).

Proof. For each finite partial function \( s \) from \( \kappa \) to 2 let \([s]\) denote the set \( \{f \in 2^\kappa : s \subseteq f\} \). The product topology on \( 2^\kappa \) is generated by all sets of the form \([s]\). Every clopen subset of \( 2^\kappa \) is compact and therefore the union of finitely many sets of the form \([s]\). It follows that \( 2^\kappa \) has exactly \( \kappa \) clopen subsets. The continuous functions from \( 2^\kappa \) to 2 are just the characteristic functions of clopen sets. Hence there are only \( \kappa \) continuous functions from \( 2^\kappa \) to 2. Let \( D \) denote the set of all continuous functions from \( 2^\kappa \) to 2.
Since finitely many points in $2^\kappa$ can be separated simultaneously by pairwise disjoint clopen sets, every finite partial function from $2^\kappa$ to 2 extends to a continuous functions defined on all of $2^\kappa$. It follows that $D$ is a dense subset of $2^{2^\kappa}$ of size $\kappa$.

Now, if $A$ is a nonempty clopen subset of $2^{2^\kappa}$, then there is a finite partial function $s$ from $2^\kappa$ to 2 such that $[s] \subseteq A$. Cleary, the number of continuous extensions of $s$ to all of $2^\kappa$ is $\kappa$. Hence $D \cap A$ is of size $\kappa$. 

As in the case of independent families on $\omega$, from the previous lemma we can derive the existence of large independent families of subsets of $\kappa$.

**Theorem 4.2.** For every infinite cardinal cardinal $\kappa$, there is a family $\mathcal{F}$ of size $2^\kappa$ such that for all disjoint finite sets $A, B \subseteq \mathcal{F}$, the set

$$\left( \bigcap A \right) \setminus \bigcup B$$

is of size $\kappa$.

**First proof.** Let $D \subseteq 2^{2^\kappa}$ be as in Lemma 4.1. For each $x \in 2^\kappa$ let $B_x = \{f \in 2^{2^\kappa} : f(x) = 0\}$ and $A_x = D \cap B_x$. Whenever $S$ and $T$ are disjoint finite subsets of $2^\kappa$, then

$$\left( \bigcap_{x \in S} B_x \right) \setminus \bigcup_{x \in T} B_x$$

is a nonempty clopen subset of $2^{2^\kappa}$. It follows that

$$\left( \bigcap_{x \in S} A_x \right) \setminus \bigcup_{x \in T} A_x = D \cap \left( \left( \bigcap_{x \in S} B_x \right) \setminus \bigcup_{x \in T} B_x \right)$$

is of size $\kappa$. It follows that $\mathcal{F} = \{A_x : x \in 2^\kappa\}$ is as desired. \hfill \Box

We can translate this topological proof into combinatorics as follows:

The continuous functions from $2^\kappa$ to 2 are just characteristic functions of clopen sets. The basic clopen sets are of the form $[s]$, where $s$ is a finite partial function from $\kappa$ to 2. All clopen sets are finite unions of sets of the form $[s]$. Hence we can code clopen subsets of $2^\kappa$ in a natural way by finite sets of finite partial functions from $\kappa$ to 2. We formulate the previous proof in this combinatorial setting. The following proof is just a generalization of our second proof of Theorem 3.4. This is essentially Hausdorff’s proof of the existence large independent families in higher cardinalities.

**Second proof.** Let $D$ be the collection of all finite sets of finite partial functions from $\kappa$ to 2. For each $f : 2^\kappa \to 2$ let $A_f$ be the collection of all $a \in D$ such that for all $s \in a$ and all $x : \kappa \to 2$ with $s \subseteq x$ we have $f(x) = 1$.

**Claim 4.3.** For any two disjoint finite sets $S, T \subseteq 2^\kappa$ the set

$$\left( \bigcap_{x \in S} A_x \right) \setminus \bigcup_{x \in T} A_x$$

is of size $\kappa$. 
For all \( x \in S \) and all \( y \in T \) there is \( \alpha \in \kappa \) such that \( x(\alpha) \neq y(\alpha) \). It follows that for every \( x \in S \) there is a finite partial function \( s \) from \( \kappa \) to \( 2 \) such that \( s \subseteq x \) and for all \( y \in T \), \( s \not\subseteq T \). Hence there is a finite set \( a \) of finite partial functions from \( \kappa \) to \( 2 \) such that all \( x \in S \) are extensions of some \( s \in a \) and no \( y \in T \) extends any \( s \in a \). Now \( a \in (\bigcap_{x \in S} A_x) \setminus \bigcup_{x \in T} A_x \). But for every \( \alpha < \kappa \) we can build the set \( a \) in such a way that \( \alpha \) is in the domain of some \( s \in a \). It follows that there are in fact \( \kappa \) many distinct sets \( a \in (\bigcap_{x \in S} A_x) \setminus \bigcup_{x \in T} A_x \).

**REFERENCES**


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