数理解析研究所講究録
第 1790 巻 2012 年 16-30

Descriptive set theory of complete quasi-metric spaces

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Abstract

We give a summary of results from our investigation into extending classical descriptive set theory to the entire class of countably based T_0-spaces. Polish spaces play a central role in the descriptive set theory of metrizable spaces, and we suggest that countably based completely quasi-metrizable spaces, which we refer to as quasi-Polish spaces, play the central role in the extended theory. The class of quasi-Polish spaces is general enough to include both Polish spaces and ω-continuous domains, which have many applications in theoretical computer science. We show that quasi-Polish spaces are a very natural generalization of Polish spaces in terms of their topological characterizations and their completeness properties. In particular, a metrizable space is quasi-Polish if and only if it is Polish, and many classical theorems concerning Polish spaces, such as the Hausdorff-Kuratowski theorem, generalize to all quasi-Polish spaces.

1. Introduction

Descriptive set theory has proven to be an invaluable tool for the study of separable metrizable spaces, and the techniques and results have been applied to many fields such as functional analysis, topological group theory, and mathematical logic. Separable completely metrizable spaces, called Polish spaces, play a central role in classical descriptive set theory. These spaces include the space of natural numbers with the discrete topology, the real numbers with the Euclidean topology, as well as the separable Hilbert and Banach spaces.

Somewhat more recently, however, there has been growing interest in non-metrizable spaces, in particular the continuous lattices and domains of domain theory [5]. These spaces generally fail to satisfy even the T_1-separation axiom, but naturally occur in the general theory of computation, the analysis of function spaces, as well as in algebra and logic. Continuous domains are also characterized by a kind of completeness property, which at first glance seems rather different than the completeness property of a metric.

Despite the great success of descriptive set theory with the analysis of metrizable spaces, the extension of this approach to more general spaces seems to have

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been largely overlooked by the mathematical community. Notable exceptions are work by Dana Scott [14], A. Tang [19, 20], and Victor Selivanov (see [17] for an overview) on the descriptive set theory of ω-continuous domains.

In this paper, we will give a summary of results for a class of spaces, which we call quasi-Polish spaces, which we propose as the proper generalization of Polish spaces for extending descriptive set theory to all countably based T_0-spaces. Quasi-Polish spaces are defined as the countably based spaces which admit a (Smyth)-complete quasi-metric, and have many properties that are natural generalizations of Polish spaces. For example, a subspace of a quasi-Polish space is quasi-Polish if and only if it is a $\mathbf{\Pi}^0_2$ subset, and quasi-Polish spaces have a game-theoretic characterization in terms of a simple modification of the strong Choquet game. The class of quasi-Polish spaces is general enough to contain both the Polish spaces and the countably based locally compact sober spaces, hence all ω-continuous domains, but is not too general as demonstrated by the fact that every quasi-Polish space is sober and every metrizable quasi-Polish space is Polish.

The majority of this paper will be dedicated to showing the naturalness of extending the descriptive set theory of Polish spaces to the class of quasi-Polish spaces. We will see that many classical results on Polish spaces apply to all quasi-Polish spaces. For example, quasi-Polish topologies can be extended to finer quasi-Polish topologies in a manner similar to the case for Polish spaces, and the Hausdorff-Kuratowski theorem extends to all quasi-Polish spaces. The naturalness of quasi-Polish spaces will also be demonstrated by showing that they are precisely the spaces that are homeomorphic to the subspace of non-compact elements of an ω-continuous domain, and that they are precisely the countably based spaces that have a total admissible representation in the sense of Type 2 Theory of Effectivity.

Our results show that the techniques and results of classical descriptive set theory naturally generalize to all countably based T_0-spaces. This offers new opportunities for an exchange of results and ideas between the fields of descriptive set theory, domain theory, and the theory of generalized metrics.

Definitions and background for classical descriptive set theory can be found in [7]. Definitions and background for domain theory can be found in [5].

2. Borel Hierarchy

It is common for non-Hausdorff spaces to have open sets that are not F_σ (i.e., countable unions of closed sets) and closed sets that are not G_δ (i.e., countable intersections of open sets). The Sierpsinski space, which has $\{\bot, \top\}$ as an underlying set and the singleton $\{\top\}$ open but not closed, is perhaps the simplest example of this phenomenon. This implies that the classical definition of the Borel hierarchy, which defines level $\mathbf{\Sigma}^0_2$ as the F_σ-sets and $\mathbf{\Pi}^0_2$ as the G_δ-sets, is not appropriate in the general setting. We can overcome this problem by modifying the classical definition of the Borel hierachy as follows.

Definition 1. *Let (X, τ) be a topological space. For each ordinal α ($1 \le \alpha < \omega_1$) we define $\mathbf{\Sigma}^0_\alpha(X, \tau)$ inductively as follows.*

1. $\mathbf{\Sigma}^0_1(X, \tau) = \tau$.
2. *For $\alpha > 1$, $\mathbf{\Sigma}^0_\alpha(X, \tau)$ is the set of all subsets A of X which can be expressed in the form*
$$A = \bigcup_{i \in \omega} B_i \setminus B'_i,$$
where for each i, B_i and B'_i are in $\mathbf{\Sigma}^0_{\beta_i}(X, \tau)$ for some $\beta_i < \alpha$.

We define $\mathbf{\Pi}^0_\alpha(X, \tau) = \{X \setminus A \mid A \in \mathbf{\Sigma}^0_\alpha(X, \tau)\}$ and $\mathbf{\Delta}^0_\alpha(X, \tau) = \mathbf{\Sigma}^0_\alpha(X, \tau) \cap \mathbf{\Pi}^0_\alpha(X, \tau)$. Finally, we define $\mathbf{B}(X, \tau) = \bigcup_{\alpha < \omega_1} \mathbf{\Sigma}^0_\alpha(X, \tau)$ to be the Borel *subsets of (X, τ).* □

When the topology is clear from context, we will usually write $\mathbf{\Sigma}^0_\alpha(X)$ instead of $\mathbf{\Sigma}^0_\alpha(X, \tau)$.

The definition above is equivalent to the classical definition of the Borel hierarchy on metrizable spaces, but differs in general. The above hierarchy has been investigated by D. Scott [14] and A. Tang [19] for $X = \mathcal{P}(\omega)$, and much more generally by V. Selivanov (see [17] for an overview of results).

In general, singleton subsets and the diagonal of countably based T_0-spaces are not closed. However, the following two propositions show that they are always $\mathbf{\Pi}^0_2$ in the Borel hierarchy.

Proposition 2. *If X is a countably based T_0-space then every singleton set $\{x\} \subseteq X$ is in $\mathbf{\Pi}^0_2(X)$.* □

Proposition 3. *If X is a countably based T_0-space then the diagonal of X (i.e., $\{\langle x, y\rangle \in X \times X \mid x = y\}$) is in $\mathbf{\Pi}^0_2(X \times X)$.* □

3. Quasi-metric spaces

Quasi-metrics are a generalization of metrics where the axiom of symmetry is dropped. These provide a useful way to generalize results from the theory of metric spaces to more general topological spaces.

Definition 4. *A* quasi-metric *on a set X is a function $d\colon X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:*

1. $x = y \iff d(x, y) = d(y, x) = 0$
2. $d(x, z) \le d(x, y) + d(y, z)$.

A quasi-metric space *is a pair (X, d) where d is a quasi-metric on X.* □

A quasi-metric d on X induces a T_0 topology τ_d on X generated by basic open balls of the form $B_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ for $x \in X$ and real number $\varepsilon > 0$.

If (X, d) is a quasi-metric space, then $(X, \widehat{d})$ is a metric space, where $\widehat{d}$ is defined as $\widehat{d}(x, y) = \max\{d(x, y), d(y, x)\}$. The metric topology induced by $\widehat{d}$ will be denoted $\tau_{\widehat{d}}$.

Proposition 5 (H.-P. A. Künzi [8]). *A quasi-metric space (X, d) is countably based if and only if $(X, \widehat{d})$ is separable.* □

Theorem 6. *If (X, d) is a countably based quasi-metric space, then the metric topology $\tau_{\widehat{d}}$ is a subset of $\mathbf{\Sigma}^0_2(X, \tau_d)$. In particular, $\mathbf{\Sigma}^0_\alpha(X, \tau_d) = \mathbf{\Sigma}^0_\alpha(X, \tau_{\widehat{d}})$ for all $\alpha \geq \omega$, and $\mathbf{B}(X, \tau_d) = \mathbf{B}(X, \tau_{\widehat{d}})$.* □

The power set of the natural numbers, denoted $\mathcal{P}(\omega)$, with the Scott-topology has a compatible quasi-metric d defined as

$$d(X, Y) = \sup\{2^{-n} \mid n \in X \setminus Y\}$$

for $X, Y \subseteq \omega$, where we define the supremum of the empty set to be zero. In other words, $d(X, Y) = 2^{-n}$ where n is the least element in X and not in Y if such an element exists, and $d(X, Y) = 0$ if X is a subset of Y. Then $\widehat{d}$ is the usual complete metric on 2^ω if we identify elements of $\mathcal{P}(\omega)$ with their characteristic function. Selivanov [17] has shown that in this case $\mathbf{\Sigma}^0_n(\mathcal{P}(\omega), \tau_d) \not\subseteq \mathbf{\Pi}^0_n(\mathcal{P}(\omega), \tau_{\widehat{d}})$ and $\mathbf{\Pi}^0_n(\mathcal{P}(\omega), \tau_{\widehat{d}}) \not\subseteq \mathbf{\Sigma}^0_{n+1}(\mathcal{P}(\omega), \tau_d)$ for all $n < \omega$.

4. Complete quasi-metric spaces

In the literature on quasi-metric spaces there are many competing definitions of "Cauchy sequence" and "completeness". The definition of "Cauchy" that we will adopt is sometimes called "left K-Cauchy" and our definition of completeness is sometimes called "Smyth-complete" (see [9]). The main goal of this section is to characterize the countably based spaces which have topologies induced by a complete quasi-metric.

Definition 7. *A sequence $(x_n)_{n\in\omega}$ in a quasi-metric space (X, d) is* Cauchy *if and only if for each real number $\varepsilon > 0$ there exists $n_0 \in \omega$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n \geq n_0$. (X, d) is a* complete quasi-metric space *if and only if every Cauchy sequence in X converges with respect to the metric topology $\tau_{\widehat{d}}$.* □

We will say that a topological space (X, τ) is *completely quasi-metrizable* if and only if there is a complete quasi-metric d on X such that $\tau = \tau_d$.

Definition 8. *A topological space is* quasi-Polish *if and only if it is countably based and completely quasi-metrizable.* □

If (X, d) is a countably based complete quasi-metric space, then $(X, \widehat{d})$ is separable by Proposition 5 and $\widehat{d}$ is complete because any sequence that is Cauchy with respect to $\widehat{d}$ is Cauchy with respect to d. Therefore, $(X, \widehat{d})$ has a Polish topology. We can immediately use this connection between quasi-Polish spaces and Polish spaces to make a few simple observations.

Proposition 9. *Every uncountable quasi-Polish space has cardinality $2^{\aleph_0}$.* □

We can also show that the fact that the Borel hierarchy on uncountable Polish spaces does not collapse (see, for example, Theorem 22.4 in [7]) generalizes to uncountable quasi-Polish spaces.

Theorem 10. *If X is an uncountable quasi-Polish space, then the Borel hierarchy on X does not collapse.* □

V. Selivanov [17] has shown that the Borel hierarchy does not collapse for some uncountable ω-continuous domains, including $\mathcal{P}(\omega)$. We will see later that every ω-continuous domain is quasi-Polish, so the hierarchy does not collapse on any uncountable ω-continuous domain.

Quasi-Polish subspaces of quasi-Polish spaces have the following simple characterization.

Theorem 11. *A subspace of a quasi-Polish space is quasi-Polish if and only if it is $\mathbf{\Pi}^0_2$.* □

$\mathcal{P}(\omega)$ is complete with respect to the quasi-metric d in the previous section. Since every countably based space can be embedded into $\mathcal{P}(\omega)$, we obtain the following.

Corollary 12. *A space is quasi-Polish if and only if it is homeomorphic to a $\mathbf{\Pi}^0_2$-subset of $\mathcal{P}(\omega)$.* □

Finally, we mention that the class of quasi-Polish spaces is closed under retracts. Recall that a topological space X is a *retract* of Y if and only if there exist continuous functions $s\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ s$ is the identity on X.

Corollary 13. *Any retract of a quasi-Polish space is quasi-Polish.* □

Similar results concerning retracts can be found in [14] and [19]. Retracts also play an important role in V. Selivanov's [17] development of descriptive set theory for domains.

5. Open continuous surjections from quasi-Polish spaces

In this section we characterize quasi-Polish spaces as precisely the images of Polish spaces under continuous open functions. Recall that a function is *open* if and only if the image of every open set is open.

Theorem 14. *A non-empty T_0-space X is quasi-Polish if and only if there exists a continuous open surjection from ω^ω to X.* □

Theorem 15. *If X is quasi-Polish, Y is a T_0-space, and $f\colon X \to Y$ is an open continuous surjection, then Y is quasi-Polish.* □

It is well known (see, for example, Theorem 8.19 in [7]) that if X is Polish, Y is a separable metrizable space, and $f\colon X \to Y$ is a continuous open surjection, then Y is Polish.

Corollary 16. *A metrizable space is quasi-Polish if and only if it is Polish.* □

The next corollary follows by taking products (or disjoint unions) of suitable continuous open surjections.

Corollary 17. *Every countable product of quasi-Polish spaces is quasi-Polish, and every countable disjoint union of quasi-Polish spaces is quasi-Polish.* □

6. Countably based locally compact sober spaces

In this section we show that every countably based locally compact sober space is quasi-Polish. This implies, in particular, that every ω-continuous domain is quasi-Polish.

A closed set is *irreducible* if it is not the union of two proper closed subsets. A space is *sober* if and only if every irreducible closed set equals the closure of a unique point.

Theorem 18. *Every quasi-Polish space is sober.* □

A topological space X is *locally compact* if and only if for every $x \in X$ and open U containing x, there is an open set V and compact set K such that $x \in V \subseteq K \subseteq U$. Given open sets U and V of a topological space X, we write $V \ll U$ to denote that V is relatively compact in U (i.e., every open cover of U admits a finite subcover of V). As shown in [6], a sober space X is locally compact if and only if for every $x \in X$ and open U containing x, there is open V such that $x \in V \ll U$. Equivalently, a sober space is locally compact if and only if every open set is equal to the union of its relatively compact open subsets.

Theorem 19. *Every countably based locally compact sober space is quasi-Polish.* □

Every continuous domain is locally compact and sober (see Proposition III-3.7 in [5]). Therefore, we immediately obtain the following.

Corollary 20. *Every ω-continuous domain is quasi-Polish.* □

7. Admissible representations of quasi-Polish spaces

In this section we characterize quasi-Polish spaces as precisely the countably based spaces that have an admissible representation with Polish domain. Equivalently, quasi-Polish spaces are precisely the countably based spaces with total admissible representations defined on all of ω^ω. Admissible representations of topological spaces are fundamental to the development of computable analysis under the Type 2 Theory of Effectivity (see [22]).

Definition 21 (K. Weihrauch [22], M. Schröder [16]). *A partial continuous function $\rho\colon \subseteq \omega^\omega \to X$ is an* admissible representation *of X if and only if for every partial continuous $f\colon \subseteq \omega^\omega \to X$ there exists a partial continuous $g\colon \subseteq \omega^\omega \to \omega^\omega$ such that $f = \rho \circ g$.* □

A characterization of the topological spaces which have admissible representations has been given by M. Schröder [16]. Every space which has an admissible representation satisfies the T_0-axiom.

The major importance of admissible representations is due to the following fact. If X and Y are countably based spaces[1], and $\rho_X : \subseteq \omega^\omega \to X$ and $\rho_Y : \subseteq \omega^\omega \to Y$ are admissible representations, then a function $f\colon X \to Y$ is continuous if and only if there exists a continuous partial function $g\colon \subseteq \omega^\omega \to \omega^\omega$ such that $f \circ \rho_X = \rho_Y \circ g$. This reduces the analysis of continuous functions between represented spaces to the analysis of (partial) continuous functions on ω^ω, which are usually better understood and carry a natural definition of computability.

The next theorem shows that quasi-Polish spaces have an important role in the theory of admissible representations.

Theorem 22. *A countably based space X is quasi-Polish if and only if there is an admissible representation $\rho\colon \subseteq \omega^\omega \to X$ of X such that $dom(\rho)$ is Polish.* □

V. Brattka has shown (Corollary 4.4.12 in [2]) that every Polish space X has a total admissible representation $\rho\colon \omega^\omega \to X$. By composing representations we obtain the following corollary.

Corollary 23. *A countably based space X is quasi-Polish if and only if there is a total admissible representation $\rho\colon \omega^\omega \to X$ of X.* □

The requirement that X be countably based in the above theorem and corollary can not be dropped. In Example 3 of [16], an admissible representation is constructed for a countable Hausdorff space which is not first-countable (hence not quasi-metrizable). It is easy to see that the domain of the representation in this example is Polish, which implies that the space has a total admissible representation. An interesting question is whether or not the completeness properties of quasi-Polish spaces generalizes in some way to all spaces with total admissible representations.

8. A game theoretic characterization of quasi-Polish spaces

In this section we give a game theoretic characterization of quasi-Polish spaces by a simple modification of the strong Choquet game (see [7]).

[1] The statement still holds for non-countably based X and Y if we either require X and Y to be sequential spaces or we relax the continuity requirement of f to sequential continuity (see [16] for details).

Definition 24. *Given a non-empty topological space (X,τ), the game $\mathcal{G}(X,\tau)$ is defined as follows.*

$$\begin{array}{lccccc} \textit{Player I:} & x_0, U_0 & & x_1, U_1 & & \dots \\ \textit{Player II:} & & V_0 & & V_1 & \dots \end{array}$$

Players I and II take turns playing non-empty open subsets of X such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq \ldots$, but additionally Player I is required to play any point $x_n \in U_n$ and II must then play $V_n \subseteq U_n$ with $x_n \in V_n$.

Player II wins the game $\mathcal{G}(X,\tau)$ if and only if $\{V_i \mid i \in \omega\}$ is a neighborhood basis of some $x \in X$ (i.e., for any open $U \subseteq X$ containing x, there is $i \in \omega$ such that $x \in V_i \subseteq U$). Equivalently, Player II wins if and only if $\{U_i \mid i \in \omega\}$ is a neighborhood basis of some $x \in X$. □

If the topology of X is clear from context, then we write $\mathcal{G}(X)$ instead of $\mathcal{G}(X,\tau)$. The *strong Choquet game* for a topological space X is played with the same rules as $\mathcal{G}(X)$, but with the exception that Player II wins if and only if $\bigcap_{n\in\omega} U_n$ is non-empty. A topological space X is a *strong Choquet space* if and only if Player II has a winning strategy[2] for the strong Choquet game on X. It immediately follows that if Player II has a winning strategy in the game $\mathcal{G}(X)$, then X is a strong Choquet space.

Theorem 25. *If X is a non-empty countably based T_0-space, then Player II has a winning strategy in the game $\mathcal{G}(X)$ if and only if X is a quasi-Polish space.* □

It follows that every quasi-Polish space is strong Choquet. Also, every strong Choquet space is a Baire space (i.e., countable intersections of dense open sets are dense), thus we obtain the following.

Corollary 26. *Every quasi-Polish space is a Baire space.* □

9. Embedding quasi-Polish spaces into ω-continuous domains

In this section we give a domain-theoretic characterization of quasi-Polish spaces. We then show some applications to modeling spaces as the maximal elements of a domain.

Theorem 27. *The following are equivalent for a topological space X:*

1. *X is a quasi-Polish space,*
2. *X is homeomorphic to the set of non-compact elements of some ω-continuous domain,*
3. *X is homeomorphic to the set of non-compact elements of some ω-algebraic domain.*

[2]A precise definition for the term "winning strategy" can be found in [7].

□

Given a topological space X, let $Max(X)$ denote the set of maximal elements of X with respect to the specialization order. Every quasi-Polish space is a dcpo with respect to the specialization order by virtue of being sober, hence $Max(X)$ is non-empty when X is a non-empty quasi-Polish space.

Recently there has been interest in characterizing the spaces that are homeomorphic to the set of maximal elements of some continuous domain (see [5]). An ω-ideal domain [11] is defined as an ω-algebraic domain in which every element is compact or maximal with respect to the specialization order. Furthermore, an ω-ideal model of a topological space X is defined to be an ω-ideal domain D in which X is homeomorphic to $Max(D)$. The spaces with ω-ideal models have the following simple characterization, which is a corollary of the proof of Theorem 27.

Corollary 28. *A topological space has an ω-ideal model if and only if it is quasi-Polish and satisfies the T_1-separation axiom.* □

To characterize the complexity of $Max(X)$ for arbitrary quasi-Polish X, we begin with a lemma. Below we let π_X denote the projection from $X \times Y$ onto X.

Lemma 29. *The following are equivalent for a subset A of a quasi-Polish space X:*

1. $A = \pi_X(F)$ *for some* $\mathbf{\Pi}^0_2$ *subset* $F \subseteq X \times \omega^\omega$.
2. $A = \pi_X(F)$ *for some quasi-Polish* Y *and* $\mathbf{\Pi}^0_2$ *subset* $F \subseteq X \times Y$.
3. $A = \pi_X(B)$ *for some quasi-Polish* Y *and Borel subset* $B \subseteq X \times Y$.
4. $A = f(\omega^\omega)$ *for some continuous* $f\colon \omega^\omega \to Y$.
5. $A = f(Y)$ *for some quasi-Polish* Y *and continuous* $f\colon Y \to X$.

□

This equivalence allows us to extend the definition of analytic sets to quasi-Polish spaces.

Definition 30. *Let X be quasi-Polish. A subset $A \subseteq X$ is called* analytic *if and only if it satisfies one of the equivalent conditions of Lemma 29. A subset is* co-analytic *if and only if its complement is analytic. A subset is* bi-analytic *if and only if it is both analytic and co-analytic. The analytic, co-analytic, and bi-analytic subsets of X will be denoted $\mathbf{\Sigma}^1_1(X)$, $\mathbf{\Pi}^1_1(X)$, and $\mathbf{\Delta}^1_1(X)$, respectively.*
□

If (X, d) is a countably based complete quasi-metric space, then $(X, \widehat{d})$ is Polish and $\mathbf{B}(X, \tau_d) = \mathbf{B}(X, \tau_{\widehat{d}})$. Therefore, most of the known properties of analytic sets in Polish spaces carry directly over to quasi-Polish spaces. For example, we have the following generalization of Souslin's Theorem.

Theorem 31. *If X is quasi-Polish, then $\mathbf{B}(X) = \mathbf{\Delta}^1_1(X)$.* □

The above observation has already been made by D. Scott and V. Selivanov for the case of $\mathcal{P}(\omega)$.

We now give an upper bound on the complexity of the maximal elements of a quasi-Polish space.

Theorem 32. *If X is quasi-Polish then $Max(X) \in \mathbf{\Pi}^1_1(X)$.* □

It turns out that this is the best lower bound possible in general. C. Mummert has shown (Theorem 2.8 in [12]) that any co-analytic subset of ω^ω can be embedded into a relatively closed subset of the maximal elements of an ω-continuous domain. If we choose a co-analytic set that is not analytic, then the maximal elements of such a domain can not be Borel. C. Mummert and F. Stephan [13] have shown that the spaces that are homeomorphic to the maximal elements of some ω-continuous domain are precisely the countably based strong Choquet spaces that satisfy the T_1-separation axiom (K. Martin had previously shown that the maximal elements are strong Choquet).

Corollary 33. *A topological space X is homeomorphic to $Max(Y)$ for some quasi-Polish space Y if and only if X is a countably based strong Choquet space satisfying the T_1-axiom.* □

10. Scattered spaces

In this section we show that scattered countably based T_0-spaces are quasi-Polish, which extends the known result that scattered metrizable spaces are Polish. Non-metrizable countably based scattered spaces naturally occur in the field of inductive inference as precisely those spaces that can be identified in the limit (relative to some oracle) with an ordinal mind change bound [10, 4].

Definition 34. *A point x of a topological space is* isolated *if and only if $\{x\}$ is open. If x is not isolated, then it is a* limit point. *A space is* perfect *if all of its points are limit points.* □

Definition 35. *A topological space X is* scattered *if and only if every subspace of X contains an isolated point.* □

It is not difficult to see that if X is countably based and scattered then X has at most countably many points.

The following is a separation axiom proposed by C. E. Aull and W. J. Thron [1] that is strictly between the T_0 and T_1 axioms. Recall that a subset of a topological space is *locally closed* if and only if it is equal to the intersection of an open set and a closed set.

Definition 36. *A topological space X satisfies the T_D-separation axiom if and only if $\{x\}$ is locally closed for every $x \in X$.* □

Clearly, every scattered space satisfies the T_D-axiom, although the converse does not hold in general.

Theorem 37. *A countably based space is scattered if and only if it is a countable quasi-Polish space satisfying the T_D-axiom.* □

Corollary 38. *Every non-empty perfect quasi-Polish space satisfying the T_D-axiom has cardinality $2^{\aleph_0}$.*

The T_D-axiom is necessary in the above corollary, because the ordinal $\omega+1$ with the Scott-topology is a countable perfect quasi-Polish space which does not satisfy the T_D-axiom.

11. Generalized Hausdorff-Kuratowski Theorem

In this section, we show that all levels of the difference hierarchy on countably based T_0-spaces are preserved under admissible representations. This result is then used to prove a generalization of the Hausdorff-Kuratowski Theorem for quasi-Polish spaces. The difference hierarchy on Polish spaces is well understood [7], and recently V. Selivanov [18] has extended many of these results to ω-continuous domains.

Definition 39. *Any ordinal α can be expressed as $\alpha = \beta + n$, where β is a limit ordinal or 0, and $n < \omega$. We say that α is* even *if n is even, and* odd*, otherwise. For any ordinal α, let $r(\alpha) = 0$ if α is even, and $r(\alpha) = 1$, otherwise. For any ordinal α, define*

$$D_\alpha(\{A_\beta\}_{\beta<\alpha}) = \bigcup\{A_\beta \setminus \bigcup_{\gamma<\beta} A_\gamma \mid \beta < \alpha,\, r(\beta) \neq r(\alpha)\},$$

where $\{A_\beta\}_{\beta<\alpha}$ is a sequence of sets such that $A_\gamma \subseteq A_\beta$ for all $\gamma < \beta < \alpha$.

For any topological space X and countable ordinals α and β, define $D_\alpha(\mathbf{\Sigma}^0_\beta(X))$ to be the class of all sets $D_\alpha(\{A_\gamma\}_{\gamma<\alpha})$, where $\{A_\gamma\}_{\gamma<\alpha}$ is an increasing sequence of elements of $\mathbf{\Sigma}^0_\beta(X)$. □

The proof of the following theorem depends on a result by J. Saint Raymond (Lemma 17 in [15]) that is closely related to the Vaught transform [21].

Theorem 40. *Let X be a countably based T_0 space and $\rho\colon\subseteq \omega^\omega \to X$ an admissible representation. For any countable ordinals $\alpha, \theta > 0$ and $S \subseteq X$,*

$$S \in D_\alpha(\mathbf{\Sigma}^0_\theta(X)) \iff \rho^{-1}(S) \in D_\alpha(\mathbf{\Sigma}^0_\theta(dom(\rho))).$$

□

Since $D_1(\mathbf{\Sigma}^0_\alpha(X)) = \mathbf{\Sigma}^0_\alpha(X)$, we obtain the following result from [3].

Corollary 41. *Let X be a countably based T_0-space and $\rho\colon\subseteq \omega^\omega \to X$ an admissible representation. For any $S \subseteq X$ and $1 \leq \alpha < \omega_1$, $S \in \mathbf{\Sigma}^0_\alpha(X)$ if and only if $\rho^{-1}(S) \in \mathbf{\Sigma}^0_\alpha(dom(\rho))$.* □

The following is a generalization of the Hausdorff-Kuratowski Theorem. The case for $\theta = 1$ was proven by V. Selivanov [18] for all ω-continuous domains, but $\theta > 1$ was left open.

Theorem 42. *If X is a quasi-Polish space and $1 \leq \theta < \omega_1$, then*

$$\mathbf{\Delta}^0_{\theta+1}(X) = \bigcup_{1 \leq \alpha < \omega_1} D_\alpha(\mathbf{\Sigma}^0_\theta(X)).$$

□

12. Extending quasi-Polish topologies

In this section we show that classic results concerning the extension of Polish topologies naturally generalize to the quasi-Polish case. An important new result is that any (separable) metrizable extension of a quasi-Polish topology by $\mathbf{\Sigma}^0_2$-sets results in a Polish topology. As corollaries, we obtain that the metric topology induced by an arbitrary (compatible) quasi-metric on a quasi-Polish space is Polish, and that the Lawson topology on an ω-continuous domain is Polish.

Theorem 43. *Let X be a quasi-Polish space and $A_n \in \mathbf{\Delta}^0_2(X)$ for $n \in \omega$. Then the topology on X generated by adding $\{A_n\}_{n\in\omega}$ as open sets is quasi-Polish.* □

If X is Polish and $B \subseteq X$ is closed, then the topology on X generated by adding B as an open set is also Polish (see Lemma 13.2 in [7]). However, if $B \in \mathbf{\Delta}^0_2(X)$ is not closed then the resulting topology might fail to be metrizable. For a simple example, let $\mathbb{R}$ be the real numbers with the usual topology, $K = \{1/n \mid n \in \omega, n \geq 1\}$, and $B = \mathbb{R} \setminus K$. Then $K \in \mathbf{\Delta}^0_2(\mathbb{R})$ because it is countable and Polish, hence $B \in \mathbf{\Delta}^0_2(\mathbb{R})$. The topology on $\mathbb{R}$ generated by adding B as an open set, sometimes called the K-topology on $\mathbb{R}$, is quasi-Polish by Theorem 43 and clearly Hausdorff, but it is not regular, hence not Polish, because 0 and the closed set K do not have disjoint neighborhoods.

We also easily obtain the following generalization of a theorem by K. Kuratowski (see Theorem 22.18 in [7]).

Theorem 44. *Let (X, τ) be quasi-Polish and $A_n \in \mathbf{\Sigma}^0_\alpha(X, \tau)$ for $n \in \omega$. Then there is a quasi-Polish topology $\tau' \subseteq \mathbf{\Sigma}^0_\alpha(X, \tau)$ extending τ such that A_n is open in (X, τ') for all $n \in \omega$.* □

We conclude with an important result concerning metrizable extensions of quasi-Polish topologies.

Theorem 45. *Let τ and τ' be topologies on X such that (X, τ) is quasi-Polish, (X, τ') is separable metrizable, and $\tau \subseteq \tau' \subseteq \mathbf{\Sigma}^0_2(X, \tau)$. Then (X, τ') is Polish.*

□

Corollary 46. *If X is quasi-Polish and d is any quasi-metric compatible with the topology on X, then $(X, \widehat{d})$ is Polish.* □

Note that the above corollary *does not* claim that $(X, \widehat{d})$ is a complete metric space, which is false in general. It only means that the topology on $(X, \widehat{d})$ is compatible with some complete metric, possibly different than $\widehat{d}$.

For another simple application of Theorem 45, let X be an ω-continuous domain and let τ be the Scott-topology on X. Let $\{B_i\}_{i\in\omega}$ be an enumeration of all subsets of X of the form $\twoheaduparrow b_0 \setminus (\uparrow b_1 \cup \cdots \cup \uparrow b_n)$, where $b_0, b_1, \ldots, b_n$ are elements of some fixed countable basis (in the domain theoretic sense) for X. The topology λ generated by $\{B_i\}_{i\in\omega}$ is called the *Lawson topology* on X, and is known to be separable and metrizable for ω-continuous domains (see Theorem III-4.5 and Corollary III-4.6 in [5]). Since $\twoheaduparrow b$ is open and $\uparrow b$ is G_δ with respect to the Scott-topology, it is clear that $\tau \subseteq \lambda \subseteq \mathbf{\Sigma}^0_2(X, \tau)$. Theorem 45 therefore provides an alternative proof of the known fact that the Lawson topology on an ω-continuous domain is Polish (compare with the proof of Proposition V-5.17 in [5]).

13. Conclusions

We have seen that the quasi-Polish spaces provide a nice common ground for the development of descriptive set theory for both Polish spaces and ω-continuous domains. Our results also suggest that much can be gained by a further integration of the fields of descriptive set theory, domain theory, and generalized metrics.

Theorem 40 and Corollary 41 show that the Borel complexity of a subset of an admissibly represented countably based space is precisely the Borel complexity (relative to the domain of the representation) of the set of elements of ω^ω representing the subset. This provides additional evidence that the modification of the Borel hierarchy that we have adopted in this paper is the "correct" definition for generalizing descriptive set theory to all countably based T_0-spaces. This equivalence between the complexity of subsets and their representations can serve as a basic guideline for further extending the techniques of descriptive set theory to the entire class of admissibly represented spaces. This is an important task because the admissibly represented spaces form a cartesian closed category [16], whereas the countably based spaces do not. An important first step in this direction is to determine whether or not Corollary 41 can be extended in a natural way to all admissibly represented spaces.

Acknowledgments

I would like to thank V. Selivanov, H.-P. A. Künzi, V. Brattka, and D. Spreen for valuable comments on earlier versions of this paper and for useful references. I am also indebted to the Kyoto Computable Analysis group, in particular H. Tsuiki, and also to K. Keimel for interesting discussions during his stay in Kyoto.

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