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High order asymptotic expansion for the heat equation with a nonlinear boundary condition

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1 Introduction and Main Theorem

This is a survey article of the forthcoming paper [12].

We consider the heat equation in the half space of $\mathbb{R}^N$ with a nonlinear boundary condition,

$$
\begin{cases}
\partial_t u = \Delta u & \text{in} \quad \Omega \times (0,\infty), \\
\partial_\nu u = u^p & \text{on} \quad \partial\Omega \times (0,\infty), \\
u(x,0) = \varphi(x) \geq 0 & \text{in} \quad \Omega,
\end{cases}
\tag{1.1}
$$

where $\Omega = \{x = (x', x_N) \in \mathbb{R}^N : x_N > 0\}$, $N \geq 2$, $\partial_t = \partial/\partial t$, $\partial_\nu = -\partial/\partial x_N$, $p > 1$, and

$$
\varphi \in X_K := L^\infty(\Omega) \cap \left\{ f \in L^1(\Omega) : \int_\Omega (1+|x|)^K |f(x)| dx < \infty \right\}
\tag{1.2}
$$

for some $K \geq 0$. The nonlinear boundary value problem (1.1) can be physically interpreted as a nonlinear radiation law, and has been studied in many papers (see [1]–[5], [7], [10]–[13], and references therein). For this problem, it is well known that if $1 < p \leq 1 + 1/N$, then the problem (1.1) does not have any positive global in time solutions, and if $p > 1 + 1/N$, then, for some initial datum φ, the problem (1.1) has a positive global in time solution (see, for example, [3] and [5]). In particular, for the case where $\varphi \in X_0$ and $p > 1 + 1/N$, in [10], the author of this paper proved that, if $\|\varphi\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(\Omega)}^{N(p-1)-1}$ is sufficiently small, then the solution u of (1.1) exists globally in time, and satisfies

$$
\sup_{t>0} (1+t)^{\frac{N}{2}(1-\frac{1}{q})} \left[\|u(t)\|_{L^q(\Omega)} + t^{\frac{1}{2q}} \|u(t)\|_{L^q(\partial\Omega)} \right] < \infty
\tag{1.3}
$$

for any $q \in [1,\infty]$. Furthermore he prove that, if the solution u satisfies (1.3), then the solution u behaves like the Gauss kernel as $t \to \infty$, that is,

$$
\begin{cases}
\displaystyle\int_\Omega u(x,t)dx \text{ converges to a constant } M \text{ as } t \to \infty \text{ and} \\
\displaystyle\lim_{t\to\infty} \|u(t) - 2MG(1+t)\|_{L^q(\Omega)} / \|G(1+t)\|_{L^q(\Omega)} = 0 \text{ for any } q \in [1,\infty],
\end{cases}
$$

where

$$G(x,t) = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right). \tag{1.4}$$

This result gives the first term of the asymptotic expansion of the solution u of (1.1) satisfying (1.3). In general, the large time behavior of the solutions for nonlinear parabolic problem like the problem (1.1) is influenced by the behavior of the initial datum at the spatial infinity, and it is an interesting and important problem to study the relation between the large time behavior of the solutions and the behavior of the initial datum. For the problem (1.1) with $p > 1 + 1/N$ and $(N-2)p < N$, the author of this paper and Ishige in [7] gave a classification of the large time behaviors of the global solutions under condition $\varphi \in L^\infty(\Omega) \cap L^2(\Omega, e^{|x|^2/4}dx)$. In particular, they studied the decay rate of the $L^q(\Omega)$-norm of the remainder term $R(x,t) := u(x,t) - 2MG(x, 1+t)$, and proved that, for any $q \in [1, \infty]$,

$$t^{\frac{N}{2}(1-\frac{1}{q})} \|R(t)\|_{L^q(\Omega)} = O(t^{-\frac{1}{2}}) + O(t^{-\frac{N}{2}(p-1-\frac{1}{N})}) \tag{1.5}$$

as $t \to \infty$. Furthermore, applying the entropy dissipation method, the author of this paper in [11] proved that, if $\varphi \in X_2$, then, for any $q \in [1, \infty]$,

$$t^{\frac{N}{2}(1-\frac{1}{q})} \|R(t)\|_{L^q(\Omega)} = \begin{cases} O(t^{-\frac{1}{2}}) + O(t^{-\frac{N}{4}(p-1-\frac{1}{N})}) & \text{if} \quad p \neq 1 + \frac{3}{N}, \\ O(t^{-\frac{1}{2}}(\log t)^{\frac{1}{2}}) & \text{if} \quad p = 1 + \frac{3}{N}, \end{cases} \tag{1.6}$$

as $t \to \infty$. By these estimates (1.5) and (1.6) it seems that if the initial datum φ belongs to some suitable spaces like X_2 or $L^\infty(\Omega) \cap L^2(\Omega, e^{|x|^2/4}dx)$, then we can obtain the precise estimate on the difference between the solution and its asymptotic profiles. However we can not obtain the relationship between the decay rate of $\|R(t)\|_{L^q(\Omega)}$ and the decay rate of the initial datum φ at the spatial infinity. Furthermore, as far as we know, there are no results treating higher order asymptotic expansions of the solution of (1.1) even if $\varphi \in C_0^\infty(\Omega)$.

In this paper, under conditions (1.2) and $p > 1 + 1/N$, we consider the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0, \infty), \\ \partial_\nu u = \kappa |u|^{p-1} u & \text{on} \quad \partial\Omega \times (0, \infty), \\ u(x, 0) = \varphi(x) & \text{in} \quad \Omega, \end{cases} \tag{1.7}$$

where $\kappa \in \mathbb{R}$, which includes problem (1.1), and study the large time behavior of the solutions satisfying (1.3). In particular, improving the arguments in [8] and [9], we give higher order asymptotic expansion of the solution u of (1.7). Throughout this paper we write $A_p := N(p-1)/2$ for simplicity. We recall that $A_p > 1/2$ under condition $p > 1 + 1/N$.

We first introduce some notation. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $k \in \mathbb{R}$, let $[k]$ be an integer such that $k - 1 < [k] \le k$. For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_{N-1}) \in \mathbb{N}_0^{N-1}$ and $\lambda \in \mathbb{N}_0$, we put

$$|\alpha| := \sum_{i=1}^{N-1} |\alpha_i|, \quad \alpha! := \prod_{i=1}^{N-1} \alpha_i!, \quad x^\alpha := \prod_{i=1}^{N-1} x_i^{\alpha_i}, \quad \partial_{x'}^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_{N-1}^{\alpha_{N-1}}},$$
$$J(\alpha) := \{\rho = (\rho_1, \cdots, \rho_{N-1}) \in \mathbb{N}_0^{N-1} \setminus \{\alpha\} : \rho_i \le \alpha_i \text{ for all } i = 1, \cdots, N-1\},$$
$$g_{\alpha,2\lambda}(x,t) := \frac{(-1)^{|\alpha|+2\lambda}}{\alpha!(2\lambda)!} (\partial_{x'}^\alpha \partial_{x_N}^{2\lambda} G)(x, 1+t),$$

where $G(x,t)$ is the function given in (1.4). In particular, we write $g(x,t) = g_{0,0}(x,t)$ for simplicity. We denote by δ_{x_N} the Dirac delta distribution with respect to x_N-direction. Furthermore we denote by $S(t)\varphi$ the unique bounded solution of the heat equation on Ω with the homogeneous Neumann boundary condition and the initial datum φ, that is,

$$(S(t)\varphi)(x) := \int_\Omega \Gamma(x,y,t)\varphi(y)dy. \tag{1.8}$$

Here $\Gamma = \Gamma(x,y,t)$ is the Green function for the heat equation on Ω with the homogeneous Neumann boundary condition, that is,

$$\Gamma(x,y,t) = G(x-y,t) + G(x-y_*,t), \quad x,y \in \Omega,\ t > 0, \tag{1.9}$$

where $y_* = (y', -y_N)$ for $y = (y', y_N) \in \Omega$. For any two nonnegative functions f_1 and f_2 defined in a subset D of $[0,\infty)$, we say $f_1(t) \preceq f_2(t)$ for all $t \in D$ if there exists a positive constant C such that $f_1(t) \le C f_2(t)$ for all $t \in D$. For any $m \ge 0$, we denote by L^1_m the function spaces $L^1(\Omega, (1+|x|)^m dx)$. In what follows, we write

$$\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}, \qquad |||\cdot|||_m = \|\cdot\|_{L^1(\Omega,(1+|x|)^m dx)},$$
$$\|\cdot\|_{q,\partial\Omega} = \|\cdot\|_{L^q(\partial\Omega)}, \qquad |||\cdot|||_{m,\partial\Omega} = \|\cdot\|_{L^1(\partial\Omega,(1+|x|)^m d\sigma)},$$

for simplicity, where $q \in [1,\infty]$ and $m \ge 0$.

Let $k \in \mathbb{N}_0$ and $t > 0$. Then, modifying [6] and [9], we introduce a linear operator $P_k(t)$ on L^1_k by

$$[P_k(t)f](x) := f(x) - 2 \sum_{|\alpha|+2\lambda \le k} M_{\alpha,2\lambda}(f,t) g_{\alpha,2\lambda}(x,t), \tag{1.10}$$

where $f \in L^1_k$. Here $M_{\alpha,2\lambda}(f,t)$ is a constant defined inductively (in α and λ) by

$$M_{\alpha,2\lambda}(f,t) := \int_\Omega (x')^\alpha x_N^{2\lambda} f(x)dx - 2\sum_{j=1}^{3} \sum_{J_j} M_{\rho,2\mu}(f,t) \int_\Omega (x')^\alpha x_N^{2\lambda} g_{\rho,2\mu}(x,t)dx, \tag{1.11}$$

where $J_1 = \{\rho \in J(\alpha), \mu < \lambda\}$, $J_2 = \{\rho = \alpha, \mu < \lambda\}$, and $J_3 = \{\rho \in J(\alpha), \mu = \lambda\}$. Especially, if $|\alpha| + 2\lambda \leq 1$, then

$$M_{0,0}(f,t) := \int_\Omega f(x)dx, \qquad M_{\alpha,0}(f,t) := \int_\Omega (x')^\alpha f(x)dx \quad \text{with} \quad |\alpha| = 1. \tag{1.12}$$

It is easy to see that the operator $P_k(t)$ satisfies

$$\int_\Omega (x')^\alpha x_N^{2\lambda}[P_k(t)f](x)dx = 0, \qquad |\alpha| + 2\lambda \leq k, \tag{1.13}$$

for all $t > 0$. This operator is key of our proof, in particular (1.13) is crucial property in our analysis. Furthermore, if $\varphi \in X_K$ with $K \geq 0$, then we have

$$M_{\alpha,2\lambda}(S(t)\varphi, t) = M_{\alpha,2\lambda}(\varphi, 0), \qquad t \geq 0,$$

for all $|\alpha| + 2\lambda \leq K$ (see Lemma 2.4). This together with (1.13) yields

$$t^{\frac{N}{2}(1-\frac{1}{q})} \left\| u(t) - 2 \sum_{|\alpha|+2\lambda \leq K} M_{\alpha,2\lambda}(\varphi,0) g_{\alpha,2\lambda}(t) \right\|_q = \begin{cases} o(t^{-\frac{K}{2}}) & \text{if } K = [K], \\ O(t^{-\frac{K}{2}}) & \text{if } K > [K], \end{cases} \tag{1.14}$$

as $t \to \infty$, for any $q \in [1, \infty]$, which gives higher order asymptotic expansion of $S(t)\varphi$ (See also [12, Proposition 2.1]).

Next we give the definition of the solution of (1.7).

Definition 1.1 *Let $\varphi \in X_0$. Then the function $u \in C(\overline{\Omega} \times (0,\infty)) \cap L^\infty(0,\infty : L^\infty(\Omega))$ is said to be a solution of* (1.7) *if*

$$u(x,t) = \int_\Omega \Gamma(x,y,t)\varphi(y)dy + \kappa \int_0^t \int_{\partial\Omega} \Gamma(x,y,t-s)|u(y,s)|^{p-1}u(y,s)d\sigma_y ds$$

holds for all $(x,t) \in \Omega \times (0,\infty)$. Here Γ is the Green function given by (1.9) *and $d\sigma_y$ is the $(N-1)$-dimensional Lebesgue measure on $\partial\Omega = \mathbb{R}^{N-1}$.*

It is known that, under the above definition, for any nontrivial initial datum $\varphi \in X_0$, the problem (1.7) has a unique classical solution (see, for example, [7]). By using approximate solutions of (1.7) we have

$$\sup_{0<t<\infty} (1+t)^{-\frac{l}{2}} \left(|||u(t)|||_l + t^{\frac{1}{2}} |||u(t)|||_{l,\partial\Omega} \right) < \infty \tag{1.15}$$

for any $l \in [0, K]$. Therefore, for any $|\alpha| + 2\lambda \leq K$, we can define $M_{\alpha,2\lambda}(u(t),t)$ for all $t \geq 0$ (see (1.11) and (1.12)). For any $n = 1, 2, \ldots$, modifying [8] and [9], we introduce the function $U_n = U_n(x,t)$ defined inductively by

$$U_0(x,t) := 2 \sum_{|\alpha|+2\lambda \le K} M_{\alpha,2\lambda}(u(t),t) g_{\alpha,2\lambda}(x,t), \tag{1.16}$$

$$U_n(x,t) := U_0(x,t) + \int_0^t S(t-s)[P_K(s)F_{n-1}(s)\delta_{x_N}]ds, \quad n = 1,2,\dots, \tag{1.17}$$

where $F_{n-1}(x,t) = \kappa |U_{n-1}(x,t)|^{p-1} U_{n-1}(x,t)$.

Now we are ready to state the main theorem of this note.

Theorem 1.1 *Consider the initial-boundary value problem* (1.7) *under conditions* $A_p > 1/2$ *and* (1.2) *for some* $K \ge 0$. *Let* u *be a unique solution of* (1.7) *satisfying* (1.3), *and let* $n = 0,1,2,\dots$. *Then there holds the following:*

(i) *The function* U_n *defined by* (1.16) *and* (1.17) *satisfies*

$$\sup_{t>0} (1+t)^{\frac{N}{2}(1-\frac{1}{q})} \left[\|U_n(t)\|_q + t^{\frac{1}{2q}} \|U_n(t)\|_{q,\partial\Omega} \right] < \infty, \tag{1.18}$$

$$\sup_{t>0} (1+t)^{-\frac{l}{2}} \left[|||U_n(t)|||_l + t^{\frac{1}{2}} |||U_n(t)|||_{l,\partial\Omega} \right] < \infty, \tag{1.19}$$

for any $q \in [1,\infty]$ *and* $l \in [0,K]$;

(ii) *For any* $q \in [1,\infty]$,

$$\begin{aligned} & t^{\frac{N}{2}(1-\frac{1}{q})} \left[\|u(t) - U_n(t)\|_q + t^{\frac{1}{2q}} \|u(t) - U_n(t)\|_{q,\partial\Omega} \right] \\ & \preceq \begin{cases} (1+t)^{-\frac{K}{2}} + (1+t)^{-(n+1)(A_p - \frac{1}{2})} & \text{if } (n+1)(2A_p - 1) \ne K, \\ (1+t)^{-\frac{K}{2}} \log(2+t) & \text{if } (n+1)(2A_p - 1) = K, \end{cases} \end{aligned} \tag{1.20}$$

for all $t > 0$;

(iii) *If* $(n+1)(2A_p - 1) > K$, *then, for any* $q \in [1,\infty]$,

$$t^{\frac{N}{2}(1-\frac{1}{q})} \left[\|u(t) - U_n(t)\|_q + t^{\frac{1}{2q}} \|u(t) - U_n(t)\|_{q,\partial\Omega} \right] = \begin{cases} o(t^{-\frac{K}{2}}) & \text{if } K = [K], \\ O(t^{-\frac{K}{2}}) & \text{if } K > [K], \end{cases} \tag{1.21}$$

as $t \to \infty$;

(iv) *For any* $l \in [0,K]$ *and* $\sigma > 0$,

$$\begin{aligned} & (1+t)^{-\frac{l}{2}} \left[|||u(t) - U_n(t)|||_l + t^{\frac{1}{2}} |||u(t) - U_n(t)|||_{l,\partial\Omega} \right] \\ & \preceq (1+t)^{-\frac{K}{2}+\sigma} + (1+t)^{-(n+1)(A_p - \frac{1}{2})}, \qquad t > 0. \end{aligned} \tag{1.22}$$

By Theorem 1.1 we see that the functions U_0 and $\{U_n\}_{n=1}^{\infty}$ give a linear approximation and nonlinear approximation to the solution u, respectively.

Remark 1.1 (i) U_0 *is represented as a linear combination of* $\{g_{\alpha,2\lambda}(x,t)\}_{|\alpha|+2\lambda\le K}$, *and plays a role of projection of the solution onto the space spanned by* $\{g_{\alpha,2\lambda}(x,t)\}_{|\alpha|+2\lambda\le K}$.
(ii) *If* $(n+1)(2A_p-1)>K$, *then the decay estimate of* $\|u(t)-U_n(t)\|_q$ *as* $t\to\infty$ *in* (1.21) *is the same as in* (1.14).
(iii) U_n $(n=1,2,\dots)$ *gives the* $([K]+2)$*-th order asymptotic expansion of the solution* u *and is determined systematically by the function* U_0.

Our method of this paper is based on the arguments in [6], [8], and [9]. The arguments of these papers are useful and applied to the large class of the nonlinear parabolic equations in the whole space. However, since we consider the problem (1.7) in the half space of $\mathbb{R}^N$ which has a nonlinearity on the boundary $\partial\Omega$, we can not apply these argument directly. Therefore, modifying [6] and [9], we introduce the key operator $P_i(t)$ given in (1.10), which has not same form but same nice properties. Furthermore, by using the property of the half space and the representation formula of the solution (see Definition 1.1), we can establish the method of obtaining higher order asymptotic expansions of the solution of (1.7), and give decay estimates of the difference between the solution and its asymptotic expansions.

The rest of this paper is organized as follows. In Section 2 we give some properties of $S(t)\varphi$ and $P_i(t)$. Section 3 is devoted to the proof of theorem.

2 Preliminaries

Im this section, modifying [6] and [9], we give some preliminary results on the behavior of $S(t)\varphi$ given in (1.8) and the operator $P_k(t)$ defined by (1.10).

For any $\alpha\in\mathbb{N}_0^{N-1}$ and $\lambda\in\mathbb{N}_0$, let $g_{\alpha,2\lambda}$ be the function given in Section 1. Then there exists a constant C_1 such that

$$|\partial_{x'}^{\alpha}\partial_{x_N}^{\lambda}G(x,t)|\le C_1 t^{-\frac{N+|\alpha|+\lambda}{2}}\left[1+\left(\frac{|x|}{t^{1/2}}\right)\right]^{|\alpha|+\lambda}\exp\left(-\frac{|x|^2}{4t}\right)$$

for all $(x,t)\in\overline{\Omega}\times(0,\infty)$. This inequality yields the inequalities

$$\begin{aligned}&\|G_{\alpha,\lambda}(t)\|_q+t^{\frac{1}{2q}}\|G_{\alpha,\lambda}(t)\|_{q,\partial\Omega}\preceq t^{-\frac{N}{2}(1-\frac{1}{q})-\frac{|\alpha|+\lambda}{2}},\\&\int_\Omega|x|^m|\partial_{x'}^{\alpha}\partial_{x_N}^{\lambda}G(x,t)|dx+\int_{\partial\Omega}|x|^m|\partial_{x'}^{\alpha}\partial_{x_N}^{\lambda}G(x,t)|d\sigma\preceq t^{\frac{m-|\alpha|-\lambda}{2}},\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&\|g_{\alpha,2\lambda}(t)\|_q+(1+t)^{\frac{1}{2q}}\|g_{\alpha,2\lambda}(t)\|_{q,\partial\Omega}\preceq(1+t)^{-\frac{N}{2}(1-\frac{1}{q})-\frac{|\alpha|+2\lambda}{2}},\\&\int_\Omega|x|^m|g_{\alpha,2\lambda}(x,t)|dx+\int_{\partial\Omega}|x|^m|g_{\alpha,2\lambda}(x,t)|d\sigma\preceq(1+t)^{\frac{m-|\alpha|-\lambda-1}{2}},\end{aligned}\tag{2.2}$$

for all $t>0$ and any $q\in[1,\infty]$ and $m\ge 0$. Then, since

$$\Gamma(x,y,t)=G(x-y,t)+G(x-y_*,t),$$

by (1.8) and (2.1) we see that there exists a constant C_2 such that, for any $1\le p\le q\le\infty$,

$$\|S(t)\varphi\|_q\le C_2t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|\varphi\|_p,\quad \|S(t)\varphi\|_{q,\partial\Omega}\le C_2t^{-\frac{N}{2}(1-\frac{1}{q})-\frac{1}{2q}}\|\varphi\|_1, \tag{2.3}$$

for all $t>0$. Furthermore we give the following lemmas on the estimates of $S(t)\varphi$.

Lemma 2.1 *Let $\varphi\in X_k$ with $k\ge 0$. Then, for any $\epsilon>0$ and $l\in[0,k]$, there exists a constant C such that*

$$|||S(t)\varphi|||_l+t^{\frac{1}{2}}|||S(t)\varphi|||_{l,\partial\Omega}\le(1+\epsilon)|||\varphi|||_l+C(1+t^{\frac{l}{2}})\|\varphi\|_1 \tag{2.4}$$

for all $t>0$.

Lemma 2.2 *Let $\varphi\in L^1_k$ with $k\ge 0$ and assume*

$$\int_\Omega (x')^\alpha x_N^{2\lambda}\varphi(x)dx=0,\qquad |\alpha|+2\lambda\le m,$$

for some integer $m\in\{0,\dots,[k]\}$. Then there holds the following:
(i) *If $0\le m\le[k]-1$, for any $l\in[0,k-m-1]$, there exists a constant C_1 such that*

$$\begin{aligned}&\int_\Omega |x|^l\,|(S(t)\varphi)(x)|\,dx\\ &\le C_1t^{-\frac{m+1}{2}}\left[\int_\Omega |x|^{m+l+1}|\varphi(x)|dx+t^{\frac{l}{2}}\int_\Omega |x|^{m+1}|\varphi(x)|dx\right]\end{aligned}$$

for all $t>0$;
(ii) *If $m=[k]$, for any $l\in[0,k-[k]]$, there exists a constant C_2 such that*

$$\int_\Omega |x|^l\,|(S(t)\varphi)(x)|\,dx\le C_2t^{-\frac{k-l}{2}}\int_\Omega |x|^k|\varphi(x)|dx$$

for all $t>0$. In particular, if $k=[k]$, then $\lim_{t\to\infty}t^{\frac{k}{2}}\|S(t)\varphi\|_1=0$.

Next we give the two lemmas on the operator $P_k(t)$ (see also [6, Lemma 2.3] and [9, Lemma 2.3]).

Lemma 2.3 *Let $k\ge 0$ and $f=f(x,t)\in C(\overline{\Omega}\times(0,\infty))$ be a bounded function such that*

$$\sup_{0<\tau<t}\left[|||f(\tau)|||_k+t^{\frac{1}{2}}|||f(\tau)|||_{k,\partial\Omega}\right]<\infty$$

for all $t>0$. *Then there holds the following:*
(i) *Assume that there exists a constant* $\gamma \geq 0$ *such that*

$$\sup_{t>0} (1+t)^{-\frac{l}{2}+\gamma} t^{\frac{1}{2}} |||f(t)|||_{l,\partial\Omega} < \infty$$

for all $l \in [0,k]$. *Then, for any* $|\alpha| + 2\lambda \leq k$, *there exists a constant* C_1 *such that*

$$|M_{\alpha,2\lambda}(f(t)\delta_{x_N}, t)| \leq C_1 (1+t)^{\frac{|\alpha|+2\lambda}{2} - \gamma} t^{-\frac{1}{2}}$$

for all $t>0$. *Furthermore*

$$\sup_{t>0} (1+t)^{-\frac{l}{2}+\gamma} t^{\frac{1}{2}} |||P_K(t)[f(t)\delta_{x_N}]|||_l < \infty$$

for any $l \in [0,K]$ *and* $q \in [1,\infty]$;
(ii) *If there exists a constant* $\gamma' \geq 0$ *such that*

$$\sup_{t>0} \left[t^{\frac{N}{2}(1-\frac{1}{q})+\gamma'+\frac{1}{2q}} \|f(t)\|_{q,\partial\Omega} + (1+t)^{-\frac{l}{2}+\gamma'} t^{\frac{1}{2}} |||f(t)|||_{l,\partial\Omega} \right] < \infty$$

for all $l \in [0,K]$ *and* $q \in [1,\infty]$, *then*

$$t^{\frac{N}{2}(1-\frac{1}{q})} \left\| \int_0^t S(t-s) P_K(s)[f(s)\delta_{x_N}] ds \right\|_q \preceq t^{-\frac{K}{2}} \int_0^t (1+s)^{\frac{K}{2}-\gamma'} s^{-\frac{1}{2}} ds$$

for all $t>0$. *Furthermore, for any* $q \in [1,\infty]$,

$$t^{\frac{N}{2}(1-\frac{1}{q})+\frac{1}{2q}} \left\| \int_0^t S(t-s) P_K(s)[f(s)\delta_{x_N}] ds \right\|_{q,\partial\Omega} \preceq t^{-\frac{K}{2}} \int_0^t (1+s)^{\frac{K}{2}-\gamma'} s^{-\frac{1}{2}} ds$$

for all $t>0$.

Lemma 2.4 *Assume the same conditions as in Lemma 2.3. Let* u *be a solution of the initial-boundary value problem*

$$\partial_t u = \Delta u \ \ in \ \ \Omega \times (0,\infty), \quad \partial_\nu u = f(x,t) \ \ on \ \ \partial\Omega \times (0,\infty), \quad u(x,0) = \varphi(x) \ \ in \ \ \Omega,$$

where $\varphi \in X_k$. *Then there holds the following:*
(i) *The function* $v = [P_k(t)u(t)](x)$ *satisfies*

$$\begin{cases} \partial_t v = \Delta v - 2 \displaystyle\sum_{|\alpha|+2\lambda \leq k} M_{\alpha,2\lambda}(f(t)\delta_{x_N}, t) g_{\alpha,2\lambda}(x,t) & in \quad \Omega \times (0,\infty), \\ \partial_\nu v = f(x,t) & on \quad \partial\Omega \times (0,\infty), \\ v(x,0) = (P_k(0)u(0))(x) & in \quad \Omega; \end{cases}$$

(ii) *For any* $|\alpha| + 2\lambda \leq k$,

$$\frac{d}{dt} M_{\alpha,2\lambda}(u(t),t) = M_{\alpha,2\lambda}(f(t)\delta_{x_N}, t)$$

for all $t>0$.

3 Proof of Main Theorem

In this section we prove Theorem 1.1. We first prove assertions (i), (ii), and (iv) of Theorem 1.1.

Proof of assertions (i), (ii), and (iv) of Theorem 1.1. By (1.15) we can apply Lemma 2.3 (i) with $\gamma = 0$ to the function U_0 (see (1.16)), and obtain

$$|U_0(x,t)| \le 2 \sum_{|\alpha|+2\lambda \le K} |M_{\alpha,2\lambda}(u(t),t)||g_{\alpha,2\lambda}(x,t)| \preceq \sum_{|\alpha|+2\lambda \le K} (1+t)^{\frac{|\alpha|+2\lambda}{2}} |g_{\alpha,2\lambda}(x,t)|$$

for all $(x,t) \in \overline{\Omega} \times (0,\infty)$. This inequality together with (2.2) yield (1.18) and (1.19) for the case $n=0$, and assertion (i) follows for the case $n=0$.

Let $n = -1, 0, 1, 2, \dots$. We assume, without loss of generality, that $\sigma \in (0, A_p - 1/2)$. Put

$$\sigma_n = \begin{cases} \sigma & \text{if} \quad n(2A_p - 1) \ge K, \\ (K/2) - n(A_p - 1/2) & \text{if} \quad n(2A_p - 1) < K, \end{cases} \qquad \gamma_n = A_p + \frac{K}{2} - \sigma_n.$$

Let $U_{-1} \equiv 0$ in $\overline{\Omega} \times (0,\infty)$. Then (1.17) holds for $n = 0, 1, 2, \dots$. Furthermore, since the solution u satisfies (1.3) and (1.15), assertions (i), (ii), and (iv) hold with $n = -1$ and $\sigma = \sigma_0$.

Assume that there exists a number $n_* \in \{-1, 0, 1, 2, \cdots\}$ such that assertions (i), (ii), and (iv) hold with $n = n_*$ and $\sigma = \sigma_{n_*+1}$. We first prove assertion (i) for $n = n_* + 1$. Since assertion (i) holds with $n = n_*$ and $F_n(x,t) = \kappa |U_n(x,t)|^{p-1} U_n(x,t)$, we obtain

$$\sup_{t>0} (1+t)^{A_p} \left[t^{\frac{N}{2}(1-\frac{1}{q})+\frac{1}{2q}} \|F_{n_*}(t)\|_{q,\partial\Omega} + (1+t)^{-\frac{l}{2}} t^{\frac{1}{2}} |||F_{n_*}(t)|||_{l,\partial\Omega} \right] < \infty, \tag{3.1}$$

for any $q \in [1,\infty]$ and $l \in [0,K]$. This together with Lemma 2.3 (i) implies that

$$\sup_{t>0} (1+t)^{A_p - \frac{l}{2}} t^{\frac{1}{2}} |||P_K(t)[F_{n_*}(t)\delta_{x_N}]|||_l < \infty \tag{3.2}$$

for any $l \in [0,K]$. Since $A_p > 1/2$, by (1.17), (1.19) with $n=0$, (2.4), and (3.2) we have

$$\begin{aligned} |||U_{n_*+1}(t)|||_l &\le |||U_0(t)|||_l + \left|\left|\left| \int_0^t S(t-s) P_K(s)[F_{n_*}(s)\delta_{x_N}] ds \right|\right|\right|_l \\ &\preceq (1+t)^{\frac{l}{2}} + \int_0^t |||P_K(s)[F_{n_*}(s)\delta_{x_N}]|||_l ds \\ &\qquad + \int_0^t (1+(t-s)^{\frac{l}{2}}) \|P_K(s)[F_{n_*}(s)\delta_{x_N}]\|_1 ds \\ &\preceq (1+t)^{\frac{l}{2}} + \int_0^t (1+s)^{-A_p+\frac{l}{2}} s^{-\frac{1}{2}} ds + \int_0^t (1+(t-s)^{\frac{l}{2}})(1+s)^{-A_p} s^{-\frac{1}{2}} ds \preceq (1+t)^{\frac{l}{2}} \end{aligned} \tag{3.3}$$

for all $t > 0$. Furthermore, applying similar argument as above, we obtain

$$t^{\frac{1}{2}}|||U_{n_*+1}(t)|||_{l,\partial\Omega} \preceq (1+t)^{\frac{l}{2}} \tag{3.4}$$

for all $t > 0$. On the other hand, by (1.19) with $n = 0$, (2.2), (3.1), and Lemma 2.3 (i) we have

$$\begin{aligned}
|U_{n_*+1}(x,t)| &\le |U_0(x,t)| + 2\sum_{|\alpha|+2\lambda\le K}\int_0^t M_{\alpha,2\lambda}(F_{n_*}(s)\delta_{x_N},s)ds|g_{\alpha,2\lambda}(x,t)| \\
&\qquad + \left|\int_0^t\int_{\partial\Omega}\Gamma(x,y,t-s)F_{n_*}(y,s)d\sigma_y ds\right| \\
&\preceq t^{-\frac{N}{2}} + t^{-\frac{N}{2}}\sum_{|\alpha|+2\lambda\le K} t^{-\frac{|\alpha|+2\lambda}{2}}\int_0^t (1+s)^{\frac{|\alpha|+2\lambda}{2}-A_p}s^{-\frac{1}{2}}ds \\
&\qquad + \int_0^{t/2}(t-s)^{-\frac{N}{2}}\|F_{n_*}(s)\|_{1,\partial\Omega}ds + \int_{t/2}^t (t-s)^{-\frac{1}{2}}\|F_{n_*}(s)\|_{\infty,\partial\Omega}ds \\
&\preceq t^{-\frac{N}{2}}\left(1 + \int_0^{t/2}(1+s)^{-A_p}s^{-\frac{1}{2}}ds + \int_{t/2}^t (t-s)^{-\frac{1}{2}}(1+s)^{-A_p}ds\right) \preceq t^{-\frac{N}{2}}
\end{aligned} \tag{3.5}$$

for all $(x,t) \in \overline{\Omega}\times[1,\infty)$. Furthermore, applying same argument as above, we obtain

$$\sup_{0<t\le 1}\sup_{x\in\overline{\Omega}}|U_{n_*+1}(x,t)| < \infty.$$

This together with (3.3)–(3.5) implies that assertion (i) holds with $n = n_* + 1$.

Next we prove that assertions (ii) and (iv) hold with $n = n_* + 1$ and $\sigma = \sigma_{n_*+2}$. Since the solution u satisfies (1.3) and (1.15), due to assertion (i) with $n = n_* + 1$, it suffices to prove that (1.20) and (1.22) hold with $n = n_* + 1$ and $\sigma = \sigma_{n_*+2}$ for all sufficiently large t. Put $z(t) := u(t) - U_{n_*+1}(t)$. Then, by (1.10) and (1.17) we have

$$z(x,t) = P_K(t)u(t) - \int_0^t S(t-s)P_K(s)[F_{n_*}(s)\delta_{x_N}]ds.$$

Then, by (1.11) and Lemma 2.4 (i) we obtain

$$\begin{cases}
\partial_t z = \Delta z - 2\displaystyle\sum_{|\alpha|+2\lambda\le K} M_{\alpha,2\lambda}((F(t)-F_{n_*}(t))\delta_{x_N},t)g_{\alpha,2\lambda}(x,t) & \text{in} \quad \Omega\times(0,\infty), \\
\partial_\nu z = F(x,t) - F_{n_*}(x,t) & \text{on} \quad \partial\Omega\times(0,\infty), \\
z(x,0) = (P_K(0)u(0))(x) & \text{in} \quad \Omega,
\end{cases}$$

This implies that

$$z(t) = S(t-t_0)z(t_0) + \int_{t_0}^t S(t-s)P_K(s)[(F(s)-F_{n_*}(s))\delta_{x_N}]ds \tag{3.6}$$

for all $t > t_0 \geq 0$. Let $q \in [1,\infty]$. By (2.3) we have

$$t^{\frac{N}{2}(1-\frac{1}{q})}\|S(t)z(0)\|_q = t^{\frac{N}{2}(1-\frac{1}{q})}\|S(t/2)S(t/2)z(0)\|_q \preceq \|S(t/2)z(0)\|_1 \tag{3.7}$$

for all $t > 0$. Similarly we have

$$t^{\frac{N}{2}(1-\frac{1}{q})+\frac{1}{2q}}\|S(t)z(0)\|_{q,\partial\Omega} = t^{\frac{N}{2}(1-\frac{1}{q})+\frac{1}{2q}}\|S(t/2)S(t/2)z(0)\|_{q,\partial\Omega} \preceq \|S(t/2)z(0)\|_1 \tag{3.8}$$

for all $t > 0$. Furthermore, since it follows from (1.13) that

$$\int_\Omega (x')^\alpha x_N^{2\lambda} z(x,0)dx = \int_\Omega (x')^\alpha x_N^{2\lambda}(P_K(0)u(0))(x)dx = 0, \qquad |\alpha| + 2\lambda \leq K,$$

we can apply Lemma 2.2 (ii) to have

$$\|S(t/2)z(0)\|_1 \preceq t^{-\frac{K}{2}}$$

for all $t > 0$. This together with (3.7) and (3.8) implies

$$t^{\frac{N}{2}(1-\frac{1}{q})}\left(\|S(t)z(0)\|_q + t^{\frac{1}{2q}}\|S(t)z(0)\|_{q,\partial\Omega}\right) \preceq t^{-\frac{K}{2}} \tag{3.9}$$

for all $t > 0$. On the other hand, by (1.3) and (1.18) with $n = n_*$ we have

$$|F(x,t) - F_{n_*}(x,t)| \preceq (1+t)^{-A_p}|u(x,t) - U_{n_*}(x,t)| \tag{3.10}$$

for all $(x,t) \in \overline{\Omega} \times (0,\infty)$. Then, since assertions (ii) and (iv) hold with $n = n_*$ and $\sigma = \sigma_{n_*+1}$, by (3.10) we obtain

$$\begin{aligned} &\sup_{t>0} t^{\frac{N}{2}(1-\frac{1}{q})+\frac{1}{2q}+\gamma_{n_*+1}}\|F(t) - F_{n_*}(t)\|_{q,\partial\Omega} \\ &\qquad + \sup_{t>0}(1+t)^{-\frac{l}{2}+\gamma_{n_*+1}}t^{\frac{1}{2}}|||F(t) - F_{n_*}(t)|||_{l,\partial\Omega} < \infty \end{aligned} \tag{3.11}$$

for any $q \in [1,\infty]$ and $l \in [0,K]$. This together with Lemma 2.3 (i) implies that

$$\sup_{t>0}(1+t)^{-\frac{l}{2}+\gamma_{n_*+1}}t^{\frac{1}{2}}|||P_K(t)[(F(t) - F_{n_*}(t))\delta_{x_N}]|||_l < \infty \tag{3.12}$$

for any $l \in [0,K]$. Furthermore, by (3.11) we can apply Lemma 2.3 (ii) with $\gamma' = \gamma_{n_*+1}$, and obtain

$$\begin{aligned} &t^{\frac{N}{2}(1-\frac{1}{q})}\left\|\int_0^t S(t-s)P_K(s)[(F(s) - F_{n_*}(s))\delta_{x_N}]ds\right\|_q \\ &\preceq t^{-\frac{K}{2}}\int_0^t (1+s)^{\frac{K}{2}-\gamma_{n_*+1}}s^{-\frac{1}{2}}ds = t^{-\frac{K}{2}}\int_0^t (1+s)^{-A_p+\sigma_{n_*+1}}s^{-\frac{1}{2}}ds \\ &\preceq \begin{cases} t^{-\frac{K}{2}} & \text{if} \quad (n_*+2)(2A_p - 1) > K, \\ t^{-\frac{K}{2}}\log t & \text{if} \quad (n_*+2)(2A_p - 1) = K, \\ t^{-(n_*+2)(A_p-\frac{1}{2})} & \text{if} \quad (n_*+2)(2A_p - 1) < K, \end{cases} \end{aligned} \tag{3.13}$$

for all sufficiently large t. Similarly, we have

$$t^{\frac{N}{2}(1-\frac{1}{q})+\frac{1}{2q}}\left\|\int_0^t S(t-s)P_K(s)[(F(s)-F_{n_*}(s))\delta_{x_N}]ds\right\|_{q,\partial\Omega} \preceq \begin{cases} t^{-\frac{K}{2}} & \text{if} \quad (n_*+2)(2A_p-1)>K, \\ t^{-\frac{K}{2}}\log t & \text{if} \quad (n_*+2)(2A_p-1)=K, \\ t^{-(n_*+2)(A_p-\frac{1}{2})} & \text{if} \quad (n_*+2)(2A_p-1)<K, \end{cases} \tag{3.14}$$

for all sufficiently large t. Therefore we apply (3.9), (3.13), and (3.14) to (3.6) with $t_0=0$, and obtain inequality (1.20) with $n=n_*+1$ for any sufficiently large t. Thus assertion (ii) holds with $n=n_*+1$.

On the other hand, for any $l\in[0,K]$, we have

$$\begin{aligned}&(1+t)^{-\frac{l}{2}}\left(|||z(t)|||_l+t^{\frac{1}{2}}|||z(t)|||_{l,\partial\Omega}\right)\\ &\preceq \|z(t)\|_1+t^{\frac{1}{2}}\|z(t)\|_{1,\partial\Omega}+(1+t)^{-\frac{K}{2}}\left(|||z(t)|||_K+t^{\frac{1}{2}}|||z(t)|||_{K,\partial\Omega}\right)\end{aligned}$$

for all $t>0$. Then, by (1.20) with $q=1$ and $n=n_*+1$ we see that, if there holds (1.22) with $l=K$, then we have (1.22) for $l\in[0,K]$. Thus it suffices to prove (1.22) with $l=K$, $n=n_*+1$, and $\sigma=\sigma_{n_*+2}$. Put $Z(t)=|||z(t)|||_K$. By (3.6) we have

$$Z(2t)\le |||S(t)z(t)|||_K+\int_t^{2t}|||S(2t-s)P_K(s)[(F(s)-F_{n_*}(s))\delta_{x_N}]|||_K ds \tag{3.15}$$

for all $t>0$. Let $\delta>0$. Then, by (2.4) and (1.20) with $n=n_*+1$ we have

$$|||S(t)z(t)|||_K\le(1+\delta)|||z(t)|||_K+C_1(1+t^{\frac{K}{2}})\|z(t)\|_1\le(1+\delta)Z(t)+C_2t^{\sigma_{n_*+2}} \tag{3.16}$$

for all $t\ge 1/2$, where C_1 and C_2 constants. Furthermore, by (2.4) and (3.12) we have

$$\begin{aligned}&\int_t^{2t}|||S(2t-s)P_K(s)[(F(s)-F_{n_*}(s))\delta_{x_N}]|||_K ds\\ &\preceq \int_t^{2t}|||P_K(s)[(F(s)-F_{n_*}(s))\delta_{x_N}]|||_K ds\\ &\qquad+\int_t^{2t}\left[1+(2t-s)^{\frac{K}{2}}\right]\|P_K(s)[(F(s)-F_{n_*}(s))\delta_{x_N}]\|_1 ds\\ &\preceq \int_t^{2t}(1+s)^{\frac{K}{2}-\gamma_{n_*+1}}s^{-\frac{1}{2}}ds+\int_t^{2t}\left[1+(2t-s)^{\frac{K}{2}}\right](1+s)^{-\gamma_{n_*+1}}s^{-\frac{1}{2}}ds\\ &\preceq t^{\frac{K}{2}-\gamma_{n_*+1}+\frac{1}{2}}=t^{-(A_p-\frac{1}{2})+\sigma_{n_*+1}}\preceq t^{\sigma_{n_*+2}}\end{aligned} \tag{3.17}$$

for all $t \geq 1/2$. Therefore, by (3.15)–(3.17) we can find a constant C_3 satisfying

$$Z(2t) \leq (1+\delta)Z(t) + C_3 t^{\sigma_{n_*+2}}, \qquad t \geq 1/2. \tag{3.18}$$

Furthermore, since it follows from (1.15) and (1.19) with $n = n_* + 1$ that $\sup_{0<t<1} Z(t) < \infty$, applying the same argument as in the proof of Lemma 3.2 in [6] with the inequality (3.18), we obtain

$$Z(t) \preceq t^{\sigma_{n_*+2}} \tag{3.19}$$

for all $t \geq 1$. On the other hand, by (2.4), (1.20) with $n = n_* + 1$, and (3.19) we have

$$t^{\frac{1}{2}} |||S(t)z(t)|||_{K,\partial\Omega} \preceq Z(t) + (1 + t^{\frac{K}{2}})\|z(t)\|_1 \preceq t^{\sigma_{n_*+2}} \tag{3.20}$$

for all $t \geq 1$. Furthermore, applying similar argument as in (3.17), we obtain

$$t^{\frac{1}{2}} \int_t^{2t} |||S(2t-s)P_K(s)[(F(s) - F_{n_*}(s))\delta_{x_N}]|||_{K,\partial\Omega} ds \preceq t^{\sigma_{n_*+2}}$$

for all $t \geq 1$. This together with (3.6) and (3.20) implies that

$$t^{\frac{1}{2}} |||z(t)|||_{K,\partial\Omega} \preceq t^{\sigma_{n_*+2}} \tag{3.21}$$

for all $t \geq 1$. By (3.19) and (3.21) we have inequality (1.22) with $n = n_*+1$ with $\sigma = \sigma_{n_*+2}$ for any sufficiently large t. Therefore assertions (ii) and (iv) hold with $n = n_* + 1$ for all $t > 0$. Thus, by induction we see that (1.19), (1.20) and (1.22) hold with $\sigma = \sigma_{n+1}$ for all $n = 0, 1, 2, \dots$, and assertions (i), (ii), and (iv) of Theorem 1.1 follow. $\square$

We complete the proof of Theorem 1.1.

Proof of Theorem 1.1. It suffices to prove assertion (iii) of Theorem 1.1. Since there holds (1.21) for the case $K > [K]$ by Theorem 1.1 (ii), it suffices to prove (1.21) for the case $K = [K]$. Let $K = [K]$. Let $n \in \{0, 1, 2, \dots\}$ be such that

$$(n+1)(2A_p - 1) > K.$$

Then we can take a positive constant σ so that

$$K - n(2A_p - 1) < 2\sigma < 2A_p - 1. \tag{3.22}$$

Put $\tilde{F}_{n-1}(t) = F(t) - F_{n-1}(t)$, $\tilde{U}_{n-1}(t) = u(t) - U_{n-1}(t)$, and $2\epsilon := 2A_p - 1 - 2\sigma > 0$. Then, by (3.22) we apply Theorem 1.1 (ii) and (iv) to obtain

$$\begin{aligned}
& t^{\frac{N}{2}(1-\frac{1}{q})+\frac{1}{2q}} \|\tilde{F}_{n-1}(t)\|_{q,\partial\Omega} + (1+t)^{-\frac{l}{2}} t^{\frac{1}{2}} |||\tilde{F}_{n-1}(t)|||_{l,\partial\Omega} \\
&\preceq (1+t)^{-A_p} \left\{ t^{\frac{N}{2}(1-\frac{1}{q})+\frac{1}{2q}} \left\|\tilde{U}_{n-1}(t)\right\|_{q,\partial\Omega} + (1+t)^{-\frac{l}{2}} t^{\frac{1}{2}} |||\tilde{U}_{n-1}(t)|||_{l,\partial\Omega} \right\} \\
&\preceq (1+t)^{-A_p} [(1+t)^{-\frac{K}{2}+\sigma} + (1+t)^{-n(A_p-\frac{1}{2})}] \\
&\preceq (1+t)^{-A_p-\frac{K}{2}+\sigma} \preceq (1+t)^{-\frac{K}{2}-\frac{1}{2}-\epsilon}, \qquad t > 0,
\end{aligned} \tag{3.23}$$

where $q \in [1, \infty]$ and $l \in [0, K]$. Furthermore, by (3.23) and Lemma 2.3 (i) we have

$$(1+t)^{-\frac{l}{2}} |||P_K(t)[\tilde{F}_{n-1}(t)\delta_{x_N}]|||_l \preceq (1+t)^{-\frac{K}{2}-\frac{1}{2}-\epsilon} t^{-\frac{1}{2}} \tag{3.24}$$

for all $t > 0$. Put $z_n(t) = u(t) - U_n(t)$. By (3.6), for any $L > 0$, we have

$$\begin{aligned} z_n(t) &= S(t)z_n(0) + \int_0^t S(t-s)P_K(s)[\tilde{F}_{n-1}(s)\delta_{x_N}]ds \\ &= S(t)z_n(0) + \left(\int_{t/2}^t + \int_L^{t/2} + \int_0^L \right) S(t-s)P_K(s)[\tilde{F}_{n-1}(s)\delta_{x_N}]ds \\ &=: S(t)z_n(0) + I_1(t) + I_2(t) + I_3(t) \end{aligned} \tag{3.25}$$

for $t \geq 2L$. Since $z_n(0) = P_K(0)u(0)$, by (1.13) we have

$$\int_{\mathbf{R}^N} x^\alpha z_n(0)dx = 0, \qquad |\alpha| \leq [K] = K,$$

and by (2.3) and Lemma 2.2 (ii) we obtain

$$\lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{q})+\frac{K}{2}} \|S(t)z_n(0)\|_q \preceq \lim_{t\to\infty} t^{\frac{K}{2}} \|S(t/2)z_n(0)\|_1 = 0. \tag{3.26}$$

$$\lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{q})+\frac{1}{2q}+\frac{K}{2}} \|S(t)z_n(0)\|_{q,\partial\Omega} \preceq \lim_{t\to\infty} t^{\frac{K}{2}} \|S(t/2)z_n(0)\|_1 = 0. \tag{3.27}$$

We first give the estimate for $I_1(t)$. By (3.23) and Lemma 2.3 (i) we obtain

$$|M_{\alpha,2\lambda}(\tilde{F}_{n-1}(t)\delta_{x_N}, t)| \preceq (1+t)^{\frac{|\alpha|+2\lambda}{2}-\frac{K}{2}-\frac{1}{2}-\epsilon} t^{-\frac{1}{2}} \tag{3.28}$$

for all $t > 0$. Since $S(t-s)g_{\alpha,2\lambda}(s) = g_{\alpha,2\lambda}(t)$ for $t > s \geq 0$, by (1.10) we see that

$$\begin{aligned} I_1(x,t) = &\int_{t/2}^t \int_{\partial\Omega} \Gamma(x,y,t-s)\tilde{F}_{n-1}(y,s)d\sigma_y ds \\ &-2g_{\alpha,2\lambda}(x,t) \int_{t/2}^t \sum_{|\alpha|+2\lambda\leq K} M_{\alpha,2\lambda}(\tilde{F}_{n-1}(s)\delta_{x_N}, s)ds \end{aligned}$$

for all $(x,t) \in \Omega \times (0,\infty)$. Therefore, by (2.2), (3.23), and (3.28) we have

$$\begin{aligned} t^{\frac{N}{2}}|I_1(x,t)| &\preceq t^{\frac{N}{2}} \int_{t/2}^t (t-s)^{-\frac{1}{2}} \|\tilde{F}_{n-1}(t)\|_{\infty,\partial\Omega} ds \\ &\quad + t^{\frac{N}{2}} \sum_{|\alpha|+2\lambda\leq K} \int_{t/2}^t |M_{\alpha,2\lambda}(\tilde{F}_{n-1}(t)\delta_{x_N}, t)| \|g_{\alpha,2\lambda}\|_\infty ds \\ &\preceq t^{-\frac{K}{2}-\frac{1}{2}-\epsilon} \int_{t/2}^t (t-s)^{-\frac{1}{2}} ds + t^{-\frac{K}{2}-\epsilon} \preceq t^{-\frac{K}{2}-\epsilon} \end{aligned}$$

for all $(x,t) \in \overline{\Omega} \times (0,\infty)$. This implies that

$$t^{\frac{N}{2}}(\|I_1(t)\|_\infty + \|I_1\|_{\infty,\partial\Omega}) = o(t^{-\frac{K}{2}}) \tag{3.29}$$

as $t \to \infty$. Furthermore, by (2.3) and (3.24) with $l = 0$ we obtain

$$\|I_1(t)\|_1 \le \int_{t/2}^t \|P_K(s)[\tilde{F}_{n-1}(t)\delta_{x_N}]\|_1 ds \preceq \int_{t/2}^t s^{-\frac{K}{2}-1-\epsilon} ds \preceq t^{-\frac{K}{2}-\epsilon} = o(t^{-\frac{K}{2}}) \tag{3.30}$$

as $t \to \infty$. Similarly we have

$$t^{\frac{1}{2}}\|I_1(t)\|_{1,\partial\Omega} = o(t^{-\frac{K}{2}})$$

as $t \to \infty$. This together with (3.29) and (3.30) yields

$$t^{\frac{N}{2}(1-\frac{1}{q})}\left(\|I_1(t)\|_q + t^{\frac{1}{2q}}\|I_1(t)\|_{q,\partial\Omega}\right) = o(t^{-\frac{K}{2}}) \quad \text{as} \quad t \to \infty. \tag{3.31}$$

Next we give the estimates for $I_2(t)$ and $I_3(t)$. By Lemma 2.2 (ii), (2.3) and (3.24) we have

$$\begin{aligned} & t^{\frac{N}{2}(1-\frac{1}{q})}\left(\|I_2(t)\|_q + t^{\frac{1}{2q}}\|I_2(t)\|_{q,\partial\Omega}\right) \\ & \preceq \int_L^{t/2} \left\| S\left(\frac{t-s}{2}\right) P_K(s)[\tilde{F}_{n-1}(s)\delta_{x_N}]\right\|_1 ds \\ & \preceq \int_L^{t/2} (t-s)^{-\frac{K}{2}} \||P_K(s)[\tilde{F}_{n-1}(s)\delta_{x_N}]\||_K ds \preceq t^{-\frac{K}{2}} \int_L^{t/2} s^{-1-\epsilon} ds \preceq t^{-\frac{K}{2}} L^{-\epsilon} \end{aligned} \tag{3.32}$$

for all sufficiently large t. Similarly, by (2.3) we obtain

$$t^{\frac{N}{2}(1-\frac{1}{q})}\left(\|I_3(t)\|_q + t^{\frac{1}{2q}}\|I_3(t)\|_{q,\partial\Omega}\right) \preceq \int_0^L \left\| S\left(\frac{t-s}{2}\right) P_K(s)[\tilde{F}_{n-1}(s)\delta_{x_N}]\right\|_1 ds \tag{3.33}$$

for all $t > 0$. On the other hand, by Lemma 2.2 (ii), (1.13), and (3.24) we have

$$\begin{aligned} & \lim_{t\to\infty} t^{\frac{K}{2}} \left\| S\left(\frac{t-s}{2}\right) P_K(s)[\tilde{F}_{n-1}(s)\delta_{x_N}]\right\|_1 \\ & = \lim_{t\to\infty} (t-s)^{\frac{K}{2}} \left\| S\left(\frac{t-s}{2}\right) P_K(s)[\tilde{F}_{n-1}(s)\delta_{x_N}]\right\|_1 = 0, \end{aligned} \tag{3.34}$$

$$\begin{aligned} & \left\| S\left(\frac{t-s}{2}\right) P_K(s)[\tilde{F}_{n-1}(s)\delta_{x_N}]\right\|_1 \\ & \preceq (t-s)^{-\frac{K}{2}} \||P_K(s)[\tilde{F}_{n-1}(s)\delta_{x_N}]\||_K \preceq t^{-\frac{K}{2}} s^{-\frac{1}{2}}, \quad t \ge 2L, \end{aligned} \tag{3.35}$$

for all $s \in (0, L)$. By (3.34) and (3.35) we apply the Lebesgue dominated convergence theorem to (3.33), and obtain

$$t^{\frac{N}{2}(1-\frac{1}{q})}\|I_3(t)\|_q = o(t^{-\frac{K}{2}}) \tag{3.36}$$

as $t \to \infty$. Therefore, combining (3.25)-(3.27), (3.31), (3.32), and (3.36), we see that there exists a constant C_4 such that

$$\limsup_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{q})+\frac{K}{2}} \left(\|z_n(t)\|_q + t^{\frac{1}{2q}}\|z(t)\|_{q,\partial\Omega}\right) \le C_4 L^{-\epsilon}.$$

Then, since L is arbitrary, we have

$$\lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{q})+\frac{K}{2}} \left(\|z_n(t)\|_q + t^{\frac{1}{2q}}\|z(t)\|_{q,\partial\Omega}\right) = 0.$$

Thus we have (1.21) for the case $K = [K]$, and the proof of Theorem 1.1 is complete. □

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