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Life span of positive solutions for a semilinear heat equation with non-decaying initial data

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1 Introduction

We consider the Cauchy problem for the following semilinear heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}=\Delta u+u^{p}, & (x,t)\in\mathbf{R}^{n}\times(0,\infty), \\ u(x,0)=\phi(x)\geq 0, & x\in\mathbf{R}^{n}, \end{cases} \tag{1}$$

where Δ is the n-dimensional Laplacian, $n\in\mathbf{N}$, $p>1$, and ϕ is a bounded continuous function on $\mathbf{R}^{n}$.

Existence and nonexistence results for time-global solutions of (1) are well-known. Here, we put $p_{F}=1+2/n$.

- Let $p\in(1,p_{F}]$. Then every nontrivial solution of (1) blows up in finite time.

- Let $p \in (p_F, \infty)$. Then (1) has a time-global classical solution for small initial data ϕ, and has a blowing up solution for large or slowly decaying initial data ϕ.

- Let $p \in (0,1)$. Then the solution of (1) exists globally .

For slowly decaying initial data, in [7] Lee and Ni showed a sufficient condition for finite time blow up on the decay order of initial data.

Theorem 1.1 ([7]). *The solution of the equation* (1) *blows up in finite time if*

$$\liminf_{x\to\infty} |x|^{2/(p-1)}\phi(x) > \mu_1^{1/(p-1)},$$

where μ_R *is the first Dirichlet eigenvalue of* $-\Delta$ *in the ball* B_R.

We put $\Omega = \{(r,\omega) \in (0,\infty) \times \mathbf{S}^{n-1};\ r > R, d(\omega,\omega_0) < cr^{-\mu}\}$ for some $R > 0$, $c > 0$, $\omega_0 \in \mathbf{S}^{n-1}$, and $0 \le \mu < 1$, where $d(\cdot,\cdot)$ denotes the usual distance on the unit sphere $\mathbf{S}^{n-1}$. Mizoguchi and Yanagida [8] showed a sufficient condition for finite time blow up on the decay order of initial data in Ω.

Theorem 1.2 ([8]). *Assume that initial data* ϕ *is nonnegative. Suppose that* $\phi \in L^\infty(\mathbf{R}^n)$ *satisfies*

$$\phi \ge K_1 r^{-\alpha} \text{ in } \Omega \text{ for some } \alpha > 0 \text{ and } K_1 > 0,$$

with $0 < \alpha < 2(1-\mu)/(p-1)$. *Then the solution of* (1) *blows up in finite time.*

We remark that from the theorem, in particular, for nondecaying initial data the solution of (1) blows up in finite time, and that the slow decay of initial data in all directions is not necessary for finite time blow up.

2 Known results for life span

In this section, we introduce several known results for the life span of solutions for (1). Here, we define the life span $T_{\max}$ as

$T_{\max} := \sup\{T > 0|$ The problem possesses a unique classical solution in $\mathbf{R}^n \times [0, T)\}$.

First we introduce the results for the life span for the equation with large or small initial data. we consider the following Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + u^p, & (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \lambda\psi(x) \geq 0 & x \in \mathbf{R}^n, \end{cases} \tag{2}$$

where $n \in \mathbf{N}$, $p > 1$. Let ψ be a bounded continuous function on $\mathbf{R}^n$ and λ be a positive parameter.

In [7], Lee and Ni showed the asymptotic behavior of the life span $T_{\max}(\lambda)$ for (2) as large or small λ.

Theorem 2.1 ([7]). *Assume that ψ is nonnegative.*
(i) *There exist constants $C_1 > 0$ and $C_2 > 0$ such that $C_1\lambda^{1-p} \leq T_{\max}(\lambda) \leq C_2\lambda^{1-p}$ for large λ.*
(ii) *If $\liminf_{|x|\to\infty} \psi(x) > 0$, then there exist constants $C_1 > 0$ and $C_2 > 0$ such that $C_1\lambda^{1-p} \leq T_{\max}(\lambda) \leq C_2\lambda^{1-p}$ for small λ.*

In [6], Gui and Wang obtained more detailed information of the asymptotics for (2). The following result indicates that for large λ the supremum of initial data ϕ is dominant in the asymptotics, and that for small λ the limiting value of ϕ at space infinity is dominant.

Theorem 2.2 ([6]). *Assume that ψ is nonnegative.*
(i) *We have*

$$\lim_{\lambda\to\infty} T_{\max}(\lambda) \cdot \lambda^{p-1} = \frac{1}{p-1} \|\psi\|_{L^\infty(\mathbf{R}^n)}^{1-p}$$

(ii) *If* $\lim_{|x|\to\infty}\psi(x)=\psi_\infty>0$, *then*

$$\lim_{\lambda\to 0} T_{\max}(\lambda)\cdot\lambda^{p-1}=\frac{1}{p-1}\psi_\infty^{1-p}.$$

The proof of the theorem is based on Kaplan's method, and the assumption $\lim_{|x|\to\infty}\psi(x)=\psi_\infty$ plays an important role in the proof.

Next, we discuss the life span for the equation with large diffusion. We shall consider the following Cauchy problem:

$$\begin{cases}\dfrac{\partial u}{\partial t}=D\Delta u+|u|^{p-1}u, & (x,t)\in\mathbf{R}^n\times(0,\infty),\\ u(x,0)=\lambda+\phi(x) & x\in\mathbf{R}^n,\end{cases}\tag{3}$$

where $D>0$, $p>1$, $n\geq 3$, $\lambda>0$, and $\phi\in L^\infty(\mathbf{R}^n)\cap L^1(\mathbf{R}^n,(1+|x|)^2dx)$.

In [1, 2] Fujishima and Ishige obtained the asymptotics of the life span $T_{\max}(D)$ of the solution of (3) as $D\to\infty$. We prepare the following notation:

$$M(\phi):=\int_{\mathbf{R}^n}\phi(x)dx,\ \Xi(\phi):=\int_{\mathbf{R}^n}x\phi(x)dx,\ S_\lambda:=\frac{\lambda^{1-p}}{p-1}.$$

Theorem 2.3 ([1, 2]). (i) *Assume* $M(\phi)>0$. *Then* $T_{\max}(D)\leq S_\lambda$ *for any* $D>0$, *and*

$$S_\lambda-T_{\max}(D)=(4\pi S_\lambda)^{-n/2}\lambda^{-p}D^{-n/2}[M(\phi)+O(D^{-1})]$$

as $D\to\infty$.

(ii) *Assume* $M(\phi)=0$. *Then* $T_{\max}(D)\leq S_\lambda$ *for any* $D>0$, *and*

$$S_\lambda-T_{\max}(D)=\frac{(4\pi S_\lambda)^{-n/2}}{\lambda^p\sqrt{2eS_\lambda}}D^{(-n-1)/2}[\Xi(\phi)+O(D^{-1/2})]$$

as $D\to\infty$.

(iii) *Assume* $M(\phi)<0$. *Then* $T_{\max}(D)\leq S_\lambda$ *for any* $D>0$, *and*

$$S_\lambda-T_{\max}(D)=O(D^{-\frac{n}{2}-1})$$

as $D\to\infty$.

We remark that the problem with large diffusion is equivalent to the equation with small initial data by changing variable.

At last, we discuss the life span for the following parabolic equations (cf. [3, 4, 5, 10, 11]):

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u), & (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \phi(x) \geq 0, & x \in \mathbf{R}^n, \end{cases} \tag{4}$$

where ϕ is a bounded continuous function on $\mathbf{R}^n$. Suppose that

$$\begin{cases} f \text{ is locally Lipschitz function in } [0,\infty), \\ f(\xi) > 0 \quad (\xi > 0), \\ \displaystyle\int_1^\infty \frac{d\xi}{f(\xi)} < \infty. \end{cases}$$

From the comparison principle to (4), we easily see

$$T_{\max} \geq \int_{\|\phi\|_{L^\infty(\mathbf{R}^n)}}^\infty \frac{d\xi}{f(\xi)}.$$

When $f(u) = u^p$, we always have

$$T_{\max} \geq \frac{1}{p-1} \|\phi\|_{L^\infty(\mathbf{R}^n)}^{1-p}.$$

A solution u to (4) with initial data ϕ is said to blow up at minimal blow-up time provided that

$$T_{\max} = \int_{\|\phi\|_{L^\infty(\mathbf{R}^n)}}^\infty \frac{d\xi}{f(\xi)}.$$

We put $\rho(x) := e^{-|x|}/(\int_{\mathbf{R}^n} e^{-|y|}dy)$ and $A_\rho(x;\phi) := \int_{\mathbf{R}^n} \rho(y-x)\phi(y)dy$. In [3], Giga, Seki and Umeda obtained the necessary and sufficient conditions of initial data ϕ for blowing up at minimal blow-up time.

Theorem 2.4 ([3]). *Let u be a solution of* (4). *Assume that there exist constants $\xi_0 > 0$ and $p > 1$ such that $f(\xi)/\xi^p$ is nondecreasing for $\xi \geq \xi_0$. Then u blows up at minimal blow-up time iff one of the following two conditions for initial data ϕ holds:*

There exists a sequence $\{x_n\} \subset \mathbf{R}^n$ such that
$|x_n| \to \infty$ and $\phi(x + x_n) \to \|\phi\|_{L^\infty(\mathbf{R}^n)}$ a.e. in $\mathbf{R}^n$ as $n \to \infty$;

$$\sup_{x \in \mathbf{R}^n} A_\rho(x; \phi) = \|\phi\|_{L^\infty(\mathbf{R}^n)}.$$

3 Main results

In this section, we shall show an upper bound of the life span of positive solutions of the Cauchy problem for a semilinear heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u), & (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \phi(x) \geq 0, & x \in \mathbf{R}^n, \end{cases} \tag{5}$$

where $n \in \mathbf{N}$ and ϕ is a bounded continuous function on $\mathbf{R}^n$. We assume that $F(u)$ satisfies

$$f(u) \geq u^p \quad \text{for } u \geq 0,$$

with $p > 1$.

We prepare several notations. For $\xi' \in \mathbf{S}^{n-1}$, and $\delta \in (0, \sqrt{2})$, we set neighborhood $S_{\xi'}(\delta)$:

$$S_{\xi'}(\delta) := \left\{\eta' \in \mathbf{S}^{n-1};\ |\eta' - \xi'| < \delta\right\}.$$

Define

$$M_\infty := \sup_{\xi' \in \mathbf{S}^{n-1}, \delta > 0} \left\{ \operatorname*{ess.inf}_{x' \in S_{\xi'}(\delta)} \left(\liminf_{r \to +\infty} \phi(rx') \right) \right\}.$$

Theorem 3.1 ([9, 12, 13]). (i) *Let $n \geq 2$. Assume that $M_\infty > 0$. Then the classical solution for* (5) *blows up in finite time, and the blow up time is estimated as follows:*

$$T_{\max} \leq \frac{1}{p-1} M_\infty^{1-p}.$$

(ii) *Let $n = 1$. Assume that*

$$\max\left\{\liminf_{x \to -\infty} \phi(x), \liminf_{x \to +\infty} \phi(x)\right\} > 0.$$

Then the classical solution for (5) *blows up in finite time, and the blow up time is estimated as follows:*

$$T_{\max} \leq \frac{1}{p-1}\left(\max\left\{\liminf_{x \to -\infty} \phi(x), \liminf_{x \to +\infty} \phi(x)\right\}\right)^{1-p}.$$

Corollary 3.1. (i) *Let $n \geq 2$. Suppose that*

$$M_\infty = \|\phi\|_{L^\infty(\mathbf{R}^n)}.$$

Then the solution u blows up at minimal blow-up time.

(ii) *Let $n = 1$. Suppose that*

$$\max\left\{\liminf_{x \to -\infty} \phi(x), \liminf_{x \to +\infty} \phi(x)\right\} = \|\phi\|_{L^\infty(\mathbf{R})}.$$

Then the solution u blows up at minimal blow-up time.

Idea of the proof of Theorem 3.1 (i). For $\xi' \in \mathbf{S}^{n-1}$ and $\delta > 0$, we first determine the sequences $\{a_j\} \subset \mathbf{R}^n$ and $\{R_j\} \subset (0, \infty)$ as follows:

- $|a_j| \to \infty$ as $j \to \infty$,
- $a_j/|a_j| = \xi'$ for any $j \in \mathbf{N}$,
- $R_j = (\delta\sqrt{4-\delta^2}/2)|a_j|$.

For $R_j > 0$, let ρ_{R_j} be the first eigenfunction of $-\Delta$ on $B_{R_j}(0) = \{x \in \mathbf{R}^n;\ |x| < R_j\}$ with zero Dirichlet boundary condition under the normalization $\int_{B_{R_j}(0)} \rho_{R_j}(x)dx = 1$. Let μ_{R_j} be the corresponding first eigenvalue. For the solutions for (1), we define

$$w_j(t) := \int_{B_{R_j}(0)} u(x + a_j, t)\rho_{R_j}(x)dx.$$

Now we introduce the following two lemmas for the life span of w_j and for the asymptotics for $w_j(0)$.

Lemma 3.1 ([12]). *The blow up time of w_j is estimated from above as follows:*

$$T^*_{w_j} \le \frac{\log\left(1 - \mu_{R_j} w_j^{1-p}(0)\right)}{-(p-1)\mu_{R_j}}$$

for large j.

Lemma 3.2 ([12]). (i) *We have*

$$\liminf_{j\to+\infty} w_j(0) \ge \operatorname*{ess.inf}_{x'\in S_{\xi'}(\delta)} \phi_\infty(x').$$

(ii) *We have*

$$\lim_{j\to+\infty} \frac{\log\left(1 - \mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} = 1.$$

From the definition of $w_j(t)$, $T_{\max} \le T^*_{w_j}$ holds for large j. Hence, we obtain

$$\begin{aligned}
T_{\max} &\le \limsup_{j+\to\infty} T^*_{w_j} \\
&\le \frac{1}{p-1} \lim_{j\to+\infty} \frac{\log\left(1 - \mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} \cdot \left(\liminf_{j\to+\infty} w_j(0)\right)^{1-p} \\
&\le \frac{1}{p-1} M_\infty^{1-p}. \Box
\end{aligned}$$

References

[1] Y. Fujishima, K. Ishige, *Blow-up for a semilinear parabolic equation with large diffusion on R^N*, J. Differential Equations **250** (2011), 2508–2543.

[2] Y. Fujishima, K. Ishige, *Blow-up for a semilinear parabolic equation with large diffusion on R^N. II*, J. Differential Equation, to appear.

[3] Y. Giga, Y. Seki, N. Umeda, *Blow-up at space infinity for nonlinear heat equations*, Recent Advances in Nonlinear Analysis, World Scientific Publishing (2009), 77–94.

[4] Y. Giga, N. Umeda, *On blow-up at space infinity for semilinear heat equations*, J. Math. Anal. Appl. **316** (2006), 538-555.

[5] Y. Giga, N. Umeda, *Blow-up directions at space infinity for semilinear heat equations*, Bol. Soc. Parana. Mat. **23** (2005), 9-28.

[6] C. Gui, X. Wang, *Life span of solutions on the Cauchy problem for a semilinear heat equation*, J. Differential Equations **115** (1995), 166-172.

[7] T.Y. Lee, W.M. Ni, *Global existence, large time behavior and life span of solutions on a semilinear parabolic Cauchy problem*, Trans. Amer. Math. Soc. **333** (1992), 365–378.

[8] N. Mizoguchi, E. Yanagida, *Blowup and life span of solutions for a semilinear parabolic equation*, SIAM J. Math. Anal. **29** (1998), 1434-1446.

[9] T. Ozawa, Y. Yamauchi, *Life span of positive solutions for a semilinear heat equation with general non-decaying initial data*, J. Math. Anal. Appl., **379** (2011), 518–523.

[10] Y. Seki, *On directional blow-up for quasilinear parabolic equations with fast diffusion*, J. Math. Anal. Appl. **338** (2008), 572-587.

[11] Y. Seki, N. Umeda, R. Suzuki, *Blow-up directions for quasilinear parabolic equations*, Proc. R. Soc. Edinb. A **138** (2008), 379-405.

[12] Y. Yamauchi, *Life span of solutions for a semilinear heat equation with initial data having positive limit inferior at infinity*, Nonlinear Anal., **74** (2011), 5008–5014.

[13] M. Yamaguchi, Y. Yamauchi, *Life span of positive solutions for a semilinear heat equation with non-decaying initial data*, Differential Integral Equations, **23** (2010), 1151–1157.