数理解析研究所講究録
第 1792 巻 2012 年 57-81

Weak variational formulation for the constrained Navier-Stokes equations

京都教育大学 教育学部 深尾 武史 (Takeshi FUKAO)†
Department of Mathematics, Kyoto University of Education

佛教大学 教育学部 剣持 信幸 (Nobuyuki KENMOCHI)
School of Education (Mathematics), Bukkyo University

Abstract

In this paper, the well-posedness of the variational inequality of Navier-Stokes type is considered in 3-dimensional space. The absolute value of the velocity field is constrained by a given smooth function depends on time. The abstract theory of nonlinear evolution equations governed by subdifferentials of a time-dependent convex functional is useful in constructing approximate solutions. In the proof of the main theorem, the crucial point is to specify the closure of the class of convex functionals, which satisfy a weak time-dependence condition.

1 Introduction

In this paper we consider Navier-Stokes equations with a time-dependent velocity constraint of the form

$$|\boldsymbol{v}| := \sqrt{\sum_{j=1}^{3}(v_j)^2} \le \psi, \quad \boldsymbol{v} := (v_1, v_2, v_3),$$

where ψ is a time-dependent given constraint function, which is continuous and strictly positive. In particular, the initial boundary value problem for the constrained partial differential equation is considered. This kind of problem can be treated from various mathematical perspectives, not only for the heat equation, but also systems between the fluid dynamics. It is well known that the constraint is the surplus condition so the problem is interpreted as a variational inequality under the suitable constraint set, and the abstract theory of evolution equations governed by the subdifferential is useful for showing the well-posedness. Our objective is to specify a wider class of weak solutions treated by Lions [25] and Brézis, [9], it is called *weak variational formulation.* Under an intricate assumption, we treated the same problem in [14], finding that the solution

†Supported by a Grant-in-Aid for Encouragement of Young Scientists (B) (No.21740130), JSPS

satisfies the *strong variational formulation.* We specify the class of convex functionals that satisfy a weak time-dependence condition, in order to discuss the well-posedness of our problem.

The motivation of this work is as follows. Our problem comes from an initial boundary value problem for a thermohydraulics model that is related to the solid-liquid phase transition. The solid-liquid phase transition is one of the most interesting phenomena in the material science. From the view point of partial differential equations, it is a sort of free boundary problems. When we take account of the influence of fluid flow in the material Ω, it is natural that the fluid dynamics are considered only in the liquid region. However, the liquid region is unknown and is determined as a part of solution. In the enthalpy formulation of the Stefan problem, an idea was proposed by Rodrigues [33] and Rodrigues and Urbano [35, 36] for using the penalty method. They considered a coupled system consisting of a heat equation and a variational formulation of the Navier-Stokes equations in Ω, having test functions whose compact supports are included in the unknown liquid region Ω_ℓ. To establish the variational formulation on Ω_ℓ, we need at least continuity of the temperature field in Ω. However, in the case of 3-dimensional space, it is difficult to observe this property, because the corresponding heat equation includes a convective term due to fluid flow, and the velocity field is not enough smooth (see Remark 1 and 2 in [33]). We recall now the terminology for our constrained problem. If we suppose that the critical temperature for the phase transition is 0 and the constraint function ψ vanishes when the temperature is negative, then we can realize that the velocity $\boldsymbol{v}$ is 0 in the negative temperature region, namely in the solid region. However, in this case ψ is not strictly positive and the convex constraint set $K(t)$ depends on the unknown function. This kind of problem is called a *quasi-variational inequality* and arises in various mathematical models of nonlinear phenomena. Many papers, for example Baiocchi and Capelo [3] and Mignot and Puel [26], treat the classical concept, and others [1, 2, 11, 23, 27, 34] deal with various concrete problems such as the system of nonlinear parabolic partial differential equations with an unknown dependent constraint. We shall discuss the details and an application to the variational inequality for the Navier-Stokes type with a temperature-dependent constraint in our forthcoming paper [15].

In Section 2, we present the main theorem and known results. In Section 3, we prove the main theorem using an auxiliary proposition and some lemmas. The outline of the proof is as follows: First, approximating the constraint function, namely approximating the convex functional by a smooth one, we construct approximate solutions by applying the abstract theory of time-dependent subdifferentials, and then obtain uniform estimates. Second, from these uniform estimates we observe the strong convergence to a candidate for the solution, which satisfies the definition of our solution. The uniqueness is guaranteed by the constraint imposed on the velocity fields. In the last section, we prove the auxiliary proposition and lemmas which are used in Section 3. To prove the auxiliary proposition for the variational inequality for the Navier-Stokes type, we use a similar idea to that of Kano, Kenmochi and Murase [16] (see also [14]).

2 Definition and main theorem

In this section, we state the main theorem concerning the well-posedness. First we note some definitions and recall the basic concepts under consideration.

2.1 Definitions and notation

Let $0 < T < +\infty$, $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary, $Q := (0,T)\times\Omega$. Let $H := L^2(\Omega)$, denoting by $|\cdot|_H$ the usual norm on H, and let $V := W_0^{1,2}(\Omega)$ and V^* be the dual space $W^{-1,2}(\Omega)$ of V. H is a Hilbert space with the usual inner product $(\cdot,\cdot)_H$. Then $V \hookrightarrow H \hookrightarrow V^*$ holds with continuous and compact imbeddings. In terms of vector-valued function spaces, $\boldsymbol{\mathcal{D}}_\sigma(\Omega) := \{\boldsymbol{u} \in \boldsymbol{C}_0^\infty(\Omega) := (C_0^\infty(\Omega))^3; \mathrm{div}\boldsymbol{u} = 0 \text{ in } \Omega\}$, $\boldsymbol{H} := \boldsymbol{L}_\sigma^2(\Omega)$, $\boldsymbol{V} := \boldsymbol{W}_{0,\sigma}^{1,2}(\Omega)$ with the usual norms, where $\boldsymbol{L}_\sigma^2(\Omega)$ and $\boldsymbol{W}_{0,\sigma}^{1,2}(\Omega)$ are the closures of $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in the spaces $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{W}_0^{1,2}(\Omega)$, respectively. $\boldsymbol{H}$ is a Hilbert space with inner product $(\cdot,\cdot)_{\boldsymbol{H}}$, which is induced from $\boldsymbol{L}^2(\Omega)$, and $\boldsymbol{V} \hookrightarrow \boldsymbol{H} \hookrightarrow \boldsymbol{V}^*$ holds. We work in the standard framework for the Navier-Stokes equations (see, [32]). Accordingly, we define the bilinear functional $a(\cdot,\cdot) : \boldsymbol{V}\times\boldsymbol{V} \to \mathbb{R}$ and the trilinear functional $b(\cdot,\cdot,\cdot) : \boldsymbol{V}\times\boldsymbol{V}\times\boldsymbol{V} \to \mathbb{R}$ by

$$a(\boldsymbol{u},\boldsymbol{w}) := \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_i}(x)\frac{\partial w_j}{\partial x_i}(x)dx,$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \sum_{i,j=1}^{3} \int_\Omega u_i(x)\frac{\partial v_j}{\partial x_i}(x)w_j(x)dx \quad \text{for all } \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V},$$

noting that $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})$ and $b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{w}) = 0$ for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}$. Moreover, we define $\|\boldsymbol{u}\| := a(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}$ for all $\boldsymbol{u} \in \boldsymbol{V}$, which is the equivalent norm of $|\boldsymbol{u}|_{\boldsymbol{V}}$.

2.2 Main theorem and basic concept

The unknown function $\boldsymbol{v} := \boldsymbol{v}(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$ is the velocity field. We now define the convex constraint set $K(t)$, which depends on time $t \in [0,T]$ and plays an important role in this paper:

$$K(t) := \left\{ \boldsymbol{z} \in \boldsymbol{V}; |\boldsymbol{z}(x)| := \sqrt{\sum_{j=1}^{3}(z_j(x))^2} \le \psi(t,x) \quad \text{for a.a. } x \in \Omega \right\} \quad \text{for all } t \in [0,T],$$

where $\psi : \overline{Q} \to \mathbb{R}$ is a given time-dependent constraint function satisfying:

(A1) $\psi \in C(\overline{Q})$, and there exist positive constants c_0, $c_\psi > 0$ such that $0 < c_0 \le \psi \le c_\psi$ in $\overline{Q}$.

Using this, we define variational formulations for the constrained Navier-Stokes inequality:

Definition 2.1. *The vector function* $\boldsymbol{v} \in W^{1,2}(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V})$ *is called a solution of the strong variational formulation if it satisfies*

$$\boldsymbol{v}(t) \in K(t) \quad \text{for a.a. } t \in (0,T), \tag{1}$$

$$\begin{aligned} &(\boldsymbol{v}'(t), \boldsymbol{v}(t) - \boldsymbol{z})_{\boldsymbol{H}} + a(\boldsymbol{v}(t), \boldsymbol{v}(t) - \boldsymbol{z}) + b(\boldsymbol{v}(t), \boldsymbol{v}(t), \boldsymbol{v}(t) - \boldsymbol{z}) \\ &\quad \le (\boldsymbol{g}(t), \boldsymbol{v}(t) - \boldsymbol{z})_{\boldsymbol{H}} \quad \text{for all } \boldsymbol{z} \in K(t) \text{ and for a.a. } t \in (0,T), \end{aligned} \tag{2}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in } \boldsymbol{H}. \tag{3}$$

Definition 2.2. *The vector function* $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H})$ *is called a solution of the weak variational formulation if it satisfies*

$$\boldsymbol{v}(t) \in K(t) \quad \text{for a.a. } t \in (0,T), \tag{4}$$

$$\begin{aligned} &\int_0^T (\boldsymbol{\eta}'(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau + \int_0^T a(\boldsymbol{v}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau)) d\tau \\ &\quad + \int_0^T b(\boldsymbol{v}(\tau), \boldsymbol{v}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau)) d\tau \\ &\le \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\eta}(0)|_{\boldsymbol{H}}^2 + \int_0^T (\boldsymbol{g}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau \quad \text{for all } \boldsymbol{\eta} \in \mathcal{K}, \end{aligned} \tag{5}$$

where

$$\mathcal{K} := \left\{ \boldsymbol{\eta} \in L^2(0,T;\boldsymbol{V}); \boldsymbol{\eta}' \in L^2(0,T;\boldsymbol{H}), \boldsymbol{\eta}(t) \in K(t) \quad \text{for a.a. } t \in [0,T] \right\}.$$

Remark. In the definition of the weak variational formulation we do not specify that the solution $\boldsymbol{v}$ satisfies the initial condition $\boldsymbol{v}(0) = \boldsymbol{v}_0$. If additionally $\boldsymbol{v} \in C([0,T];\boldsymbol{H})$, then we expect that $\boldsymbol{v}(0) = \boldsymbol{v}_0$. Actually, in the case of time-independent constraint, Theorem 2 of [9] shows this additional property in 2-dimensional case. See also Theorem 6.2, Chapter 3 of [25]. We also obtain

$$\begin{aligned} &\frac{1}{2}|\boldsymbol{v}(T) - \boldsymbol{\eta}(T)|_{\boldsymbol{H}}^2 + \int_0^T (\boldsymbol{\eta}'(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau \\ &\quad + \int_0^T a(\boldsymbol{v}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau)) d\tau + \int_0^T b(\boldsymbol{v}(\tau), \boldsymbol{v}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau)) d\tau \\ &\le \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\eta}(0)|_{\boldsymbol{H}}^2 + \int_0^T (\boldsymbol{g}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau \quad \text{for all } \boldsymbol{\eta} \in \mathcal{K}, \end{aligned} \tag{6}$$

in place of (5). The original definition of $\mathcal{K}$ in [25, 9] is slightly different from ours, namely require $\boldsymbol{\eta}' \in L^2(0,T;\boldsymbol{V}^*)$, but it is essentially the same.

The main theorem is concerned with the well-posedness of the variational inequality of the Navier-Stokes type with a time-dependent constraint:

Theorem 2.1. *Assume $\boldsymbol{g} \in L^2(0,T;\boldsymbol{H})$, $\boldsymbol{v}_0 \in K(0)$ and (A1). Then there exists at least one function $\boldsymbol{v} \in C([0,T];\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}) \cap \boldsymbol{L}^\infty(Q)$ such that $\boldsymbol{v}$ is a solution of the weak variational formulation. Additionally, $\boldsymbol{v}$ satisfies $\boldsymbol{v}(t) \in K(t)$ for all $t \in [0,T]$, $\boldsymbol{v}(0) = \boldsymbol{v}_0$ in $\boldsymbol{H}$ and (6).*

Let $\boldsymbol{g}, \tilde{\boldsymbol{g}} \in L^2(0,T;\boldsymbol{H})$ and $\boldsymbol{v}_0, \tilde{\boldsymbol{v}}_0 \in K(0)$, and let the functions $\boldsymbol{v}$, $\tilde{\boldsymbol{v}}$ be solutions obtained in Theorem 2.1 corresponding to the data $\{\boldsymbol{g}, \boldsymbol{v}_0\}$, $\{\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{v}}_0\}$, respectively. Then we have the following continuous dependence of $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ on the data.

Theorem 2.2. *The solutions $\boldsymbol{v}$, $\tilde{\boldsymbol{v}}$ satisfy the following estimate:*

$$|\boldsymbol{v}(t)-\tilde{\boldsymbol{v}}(t)|_{\boldsymbol{H}}^2 \leq \left(|\boldsymbol{v}_0 - \tilde{\boldsymbol{v}}_0|_{\boldsymbol{H}}^2 + \int_0^T |\boldsymbol{g}(\tau) - \tilde{\boldsymbol{g}}(\tau)|_{\boldsymbol{H}}^2 d\tau\right) \exp((3c_\psi^2+1)T) \quad \textit{for all } t \in [0,T].$$

Remark. Theorem 2.2 with $\boldsymbol{g} = \tilde{\boldsymbol{g}}$ and $\boldsymbol{v}_0 = \tilde{\boldsymbol{v}}_0$ implies the uniqueness of the solution. This is an advantage of the constraint imposed on the velocity field. In spite of the 3-dimensional domain and the weak variational formulation, the continuous dependence namely the uniqueness can be obtained. This is a point of emphasis in this paper.

2.3 Known results

We first discuss the Cauchy problem of the evolution equation for the variational inequality of the Navier-Stokes type. In the case of a time-independent constraint function, the problem is treated in Prouse [31] for a constraint on the ball, which is the same as $\psi(t,x) := c_\psi$ for all $(t,x) \in Q$ in our setting. It was an extension of the 2-dimensional abstract results by Lions [25] and Biroli [6]. For the other kinds of constraints, Biroli [7] treated a problem with time-dependent gradient constraint, Barbu and Sritharan [5] treated a bilateral problem as an example of the abstract evolution equations in dual spaces, where constraints were not functions. See also [13] for a time-dependent unilateral problem of the Stokes equations, which can be formulated as a variational inequality. Also there are some related applications of the abstract theory of evolution equations governed by subdifferentials, see [4, 10]. For a proper, lower semi-continuous, convex functional $\phi : \boldsymbol{H} \to \mathbb{R} \cup \{+\infty\}$, the subdifferential of ϕ is a possibly multi-valued operator in $\boldsymbol{H}$, and is defined by $\boldsymbol{u}^* \in \partial\phi(\boldsymbol{u})$ if and only if $\boldsymbol{u} \in D(\phi) = \{\boldsymbol{z} \in \boldsymbol{H}; \phi(\boldsymbol{z}) < +\infty\}$, $\boldsymbol{u}^* \in \boldsymbol{H}$ and

$$(\boldsymbol{u}^*, \boldsymbol{z} - \boldsymbol{u})_{\boldsymbol{H}} \leq \phi(\boldsymbol{z}) - \phi(\boldsymbol{u}) \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H}. \tag{7}$$

Formally, taking the indicator function on a suitable convex constraint K as ϕ, we see that the variational inequality is compatible with the constrained problem, and the abstract existence results for evolution equations governed by the subdifferential are useful. We refer the readers to Brézis [10], Naumann [29] and Ôtani [30] for the abstract approach from the theory of evolution equations to the Navier-Stokes equations.

The related abstract theory of evolution equations governed by time-dependent subdifferentials, Brézis [10], Yamada [39], Kubo and Yamazaki [22] (see also [17, 18]). For the direction of Moreau's sweeping processes, see Rossi and Stefanelli [37], Stefanelli [38] and the references therein. The further developments recently made by Kano, Kenmochi and Murase [16], are useful in this paper. Based on the time-dependent theory developed in [18, 39], Ôtani [30] obtained an abstract result regarding existence and regularity for the following evolution equation:

$$\boldsymbol{v}'(t)+\partial\varphi^t(\boldsymbol{v}(t))+B(t,\boldsymbol{v}(t))\ni\boldsymbol{g}(t)\quad\text{in }\boldsymbol{H},$$

where $\varphi^t:\boldsymbol{H}\to\mathbb{R}$ is a time-dependent, proper, lower semi-continuous, convex functional, and $B(t,\cdot)$ is a non-monotone nonlinear term. Recently, a different approach was given in Barbu and Sritharan [5] and Lefter [24]. In [14], under an intricate assumption, the same problem for the Navier-Stokes equations with a time-dependent constraint was treated in the following abstract form essentially due to [16]:

$$\boldsymbol{v}'(t)+\partial\varphi^t(\boldsymbol{v};\boldsymbol{v}(t))\ni\boldsymbol{g}(t)\quad\text{in }\boldsymbol{H}.$$

See also Stefanelli [38], Kenmochi and Stefanelli [20] for related advanced topics. The first component $\boldsymbol{v}$ of $\varphi^t(\boldsymbol{v};\boldsymbol{v}(t))$ is a parameter which determines the convex functional $\varphi^t(\boldsymbol{v};\cdot)$, and we are required to seek for a parameter $\boldsymbol{v}$ that coincides with the solution $\boldsymbol{v}(t)$. This is called a quasi-variational evolution inequality.

3 Proof of the main theorem

In this section, we prove the main theorem using an auxiliary proposition and some lemmas, the proofs of which are given in the final section.

First, we define the following convex set K and the functional $\varphi_0:\boldsymbol{H}\to\mathbb{R}\cup\{+\infty\}$ as follows:

$$K:=\Big\{\boldsymbol{z}\in\boldsymbol{V};|\boldsymbol{z}(x)|\le c_\psi\quad\text{for a.a. }x\in\Omega\Big\},$$

$$\varphi_0(\boldsymbol{z}):=\begin{cases}\frac{1}{2}\|\boldsymbol{z}\|^2-c_\varphi\|\boldsymbol{z}\|+\frac{1}{2}c_\varphi^2 & \text{if }\ \boldsymbol{z}\in K,\\ +\infty & \text{if }\ \boldsymbol{z}\in\boldsymbol{H}\setminus K,\end{cases}$$

where $c_\varphi:=3c_\psi^2|\Omega|^{\frac{1}{2}}$ and $|\Omega|$ is the volume of Ω. Moreover, for a fixed constant $\delta_0>0$, we introduce a vector-valued function space:

$$\mathcal{V}(-\delta_0,t):=\Big\{\boldsymbol{u}\in W^{1,2}(-\delta_0,t;\boldsymbol{H})\cap L^\infty(-\delta_0,t;\boldsymbol{V});\boldsymbol{u}(s)\in K\quad\text{for all }s\in[-\delta_0,t]\Big\}.$$

3.1 Convex functionals and auxiliary problems

Under a suitable regularization of ψ, we have already seen in [14] that the strong variational formulation (1)–(3) can be solved by the usual fixed point argument. We shall apply this result, taking an approximate sequence $\{\psi_n\}_{n\in\mathbb{N}}\subset W^{1,2}(0,T;C(\overline{\Omega}))$ satisfying

$$0<c_0\le\psi_n\le c_\psi,\quad \psi_n\to\psi\quad\text{in }C(\overline{Q})\quad\text{as }n\to+\infty,\tag{8}$$

and an approximate sequence $\{\boldsymbol{v}_{0,n}\}_{n\in\mathbb{N}}$ satisfying

$$\boldsymbol{v}_{0,n} \in K_n(0) \quad \text{for all } n \in \mathbb{N}, \quad \boldsymbol{v}_{0,n} \to \boldsymbol{v}_0 \quad \text{in } \boldsymbol{V} \quad \text{as } n \to +\infty,$$

where

$$K_n(t) := \left\{ \boldsymbol{z} \in \boldsymbol{V}; |\boldsymbol{z}(x)| \leq \psi_n(t,x) \quad \text{for a.a. } x \in \Omega \right\} \quad \text{for all } t \in [0,T].$$

In fact, for sufficiently large $n \in \mathbb{N}$ we have $|\psi_n(0) - \psi(0)|_{C(\overline{\Omega})} < c_0$, since $\boldsymbol{v}_0 \in K(0)$ and $\psi(0,x)/c_0 > 1$, we can take

$$\boldsymbol{v}_{0,n} := \left(1 - \frac{1}{c_0}|\psi_n(0) - \psi(0)|_{C(\overline{\Omega})}\right)\boldsymbol{v}_0,$$

which satisfies

$$\begin{aligned}
|\boldsymbol{v}_{0,n}(x)| &= \left(1 - \frac{1}{c_0}|\psi_n(0) - \psi(0)|_{C(\overline{\Omega})}\right)|\boldsymbol{v}_0(x)| \\
&\leq \left(1 - \frac{1}{c_0}|\psi_n(0) - \psi(0)|_{C(\overline{\Omega})}\right)\psi(0,x) \\
&= \psi(0,x) - \frac{\psi(0,x)}{c_0}|\psi_n(0) - \psi(0)|_{C(\overline{\Omega})} \\
&\leq \psi(0,x) - \psi_n(0,x) + \psi_n(0,x) - |\psi_n(0) - \psi(0)|_{C(\overline{\Omega})} \\
&\leq \psi_n(0,x) \quad \text{for a.a. } x \in \Omega.
\end{aligned}$$

Now, $\mathrm{div}\boldsymbol{v}_{0,n} = 0$ implies that $\boldsymbol{v}_{0,n} \in K_n(0)$. For each $n \in \mathbb{N}$ and $t \in [0,T]$, we define the functional $\varphi_n^t : \mathcal{V}(-\delta_0, t) \times \boldsymbol{H} \to \mathbb{R} \cup \{+\infty\}$ by:

$$\varphi_n^t(\boldsymbol{u};\boldsymbol{z}) := \begin{cases} \frac{1}{2}\|\boldsymbol{z}\|^2 + b(\boldsymbol{u}(t),\boldsymbol{u}(t),\boldsymbol{z}) + \frac{1}{2}c_\varphi^2 & \text{if } \boldsymbol{z} \in K_n(t), \\ +\infty & \text{if } \boldsymbol{z} \in \boldsymbol{H} \setminus K_n(t), \end{cases} \quad \text{for all } \boldsymbol{u} \in \mathcal{V}(-\delta_0, t).$$

We remark that $\boldsymbol{0} \in K_n(t)$ and $\varphi_n^t(\boldsymbol{u};\boldsymbol{0}) = c_\varphi^2/2 = 9c_\psi^4|\Omega|/2 \geq 0$ for all $n \in \mathbb{N}$ and $t \in [0,T]$. Moreover the following lemma holds.

Lemma 3.1. *For each $n \in \mathbb{N}$, $t \in [0,T]$ and $\boldsymbol{u} \in \mathcal{V}(-\delta_0,t)$, the functionals $\varphi_n^t(\boldsymbol{u};\cdot)$ and φ_0 are proper, lower semi-continuous and convex on $\boldsymbol{H}$. Moreover, we have*

$$\varphi_n^t(\boldsymbol{u};\boldsymbol{z}) \geq \varphi_0(\boldsymbol{z}) \geq 0 \quad \text{for all } \boldsymbol{z} \in K_n(t) \subset K, \tag{9}$$

and the subdifferential $\partial\varphi_n^t(\boldsymbol{u};\cdot)$ is characterized by: $\boldsymbol{z}^* \in \partial\varphi_n^t(\boldsymbol{u};\boldsymbol{z})$ if and only if $\boldsymbol{z} \in K_n(t)$, $\boldsymbol{z}^* \in \boldsymbol{H}$ and

$$(\boldsymbol{z}^*, \tilde{\boldsymbol{z}} - \boldsymbol{z})_{\boldsymbol{H}} \leq a(\boldsymbol{z}, \tilde{\boldsymbol{z}} - \boldsymbol{z}) + b(\boldsymbol{u}(t), \boldsymbol{u}(t), \tilde{\boldsymbol{z}} - \boldsymbol{z}) \quad \text{for all } \tilde{\boldsymbol{z}} \in K_n(t). \tag{10}$$

For each $n \in \mathbb{N}$, the uniform continuity of ψ_n means that there exists $T_0^{(n)} \in (0,T]$ such that

$$|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})} < c_0 \quad \text{for all } s,t \in [0,T] \text{ with } |s-t| \leq T_0^{(n)}.$$

From this we verify the following time-dependence condition of the convex functionals:

Lemma 3.2. [[18], Proposition 3.2.2] *For each $n \in \mathbb{N}$, Let $\boldsymbol{u}$ be any function in $\mathcal{V}(-\delta_0, T)$. Then, for each $s, t \in [0, T]$ with $|s-t| \le T_0^{(n)}$ and $\boldsymbol{z} \in K_n(s)$, there exists $\tilde{\boldsymbol{z}} \in K_n(t)$ such that*

$$|\tilde{\boldsymbol{z}} - \boldsymbol{z}|_{\boldsymbol{H}} \le \frac{1}{c_0} |\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})} |\boldsymbol{z}|_{\boldsymbol{H}}, \tag{11}$$

$$\varphi_n^t(\boldsymbol{u}; \tilde{\boldsymbol{z}}) - \varphi_n^s(\boldsymbol{u}; \boldsymbol{z}) \le c_1 \Big\{ |\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})} + |\boldsymbol{u}(t) - \boldsymbol{u}(s)|_{\boldsymbol{H}} \Big\} (1 + \varphi_n^s(\boldsymbol{u}; \boldsymbol{z})), \tag{12}$$

where $c_1 > 0$ is a positive constant independent of $\boldsymbol{u} \in \mathcal{V}(-\delta_0, T)$ and $n \in \mathbb{N}$.

Using these settings, we now consider the following strong variational formulation:

$$\boldsymbol{v}_n(t) \in K_n(t) \quad \text{for all } t \in (0, T), \tag{13}$$

$$\begin{aligned} &(\boldsymbol{v}_n'(t), \boldsymbol{v}_n(t) - \boldsymbol{z})_{\boldsymbol{H}} + a(\boldsymbol{v}_n(t), \boldsymbol{v}_n(t) - \boldsymbol{z}) + b(\boldsymbol{v}_n(t), \boldsymbol{v}_n(t), \boldsymbol{v}_n(t) - \boldsymbol{z}) \\ &\le (\boldsymbol{g}(t), \boldsymbol{v}_n(t) - \boldsymbol{z})_{\boldsymbol{H}} \quad \text{for all } \boldsymbol{z} \in K_n(t) \text{ and for a.a. } t \in (0, T), \end{aligned} \tag{14}$$

$$\boldsymbol{v}_n(0) = \boldsymbol{v}_{0,n} \quad \text{in } \boldsymbol{H}. \tag{15}$$

Lemmas 3.1 and 3.2, along with the regularity of $\psi_n \in W^{1,2}(0, T; C(\overline{\Omega}))$, allows us to prove the solvability of (13)–(15). Actually, we have:

Proposition 3.1. [[14], Theorem 2.1, 2.2] *For each $n \in \mathbb{N}$, there exists a unique $\boldsymbol{v}_n \in W^{1,2}(0, T; \boldsymbol{H}) \cap L^\infty(0, T; \boldsymbol{V}) \cap \boldsymbol{L}^\infty(Q)$ such that*

$$\boldsymbol{v}_n'(t) + \partial \varphi_n^t(\boldsymbol{v}_n; \boldsymbol{v}_n(t)) \ni \boldsymbol{g}(t) \quad \text{in } \boldsymbol{H} \text{ for a.a. } t \in (0, T), \tag{16}$$

$$\boldsymbol{v}_n(t) = \boldsymbol{v}_{o,n}(t) := \boldsymbol{v}_{0,n} \quad \text{in } \boldsymbol{H} \text{ for all } t \in [-\delta_0, 0]. \tag{17}$$

Moreover, there exist positive constants M_1, M_2 independent of $n \in \mathbb{N}$ such that

$$|\boldsymbol{v}_n(t)|_{\boldsymbol{H}}^2 \le M_1, \tag{18}$$

$$\int_0^t \|\boldsymbol{v}_n(\tau)\|^2 d\tau \le M_2 \quad \text{for all } t \in [0, T]. \tag{19}$$

Lemma 3.1 implies that the Cauchy problem, expressed by (16) and (17), is equivalent to the strong variational formulation (13)–(15). The first component $\boldsymbol{v}_n$ of $\varphi_n^t(\boldsymbol{v}_n; \boldsymbol{z})$ is a parameter that determines the convex functional $\varphi_n^t(\boldsymbol{v}_n; \cdot)$. In (16), we are required to seek for the parameter $\boldsymbol{v}_n$ that coincides with the solution $\boldsymbol{v}_n(t)$. In this respect, we call (16) a quasi-variational evolution inequality.

3.2 Convergence and key lemma

From the uniform estimates (18) and (19), we see that there exists a subsequence $\{\boldsymbol{v}_{n_k}\}_{k \in \mathbb{N}}$ of $\{\boldsymbol{v}_n\}_{n \in \mathbb{N}}$ and $\boldsymbol{v} \in L^\infty(0, T; \boldsymbol{H}) \cap L^2(0, T; \boldsymbol{V})$ such that

$$\boldsymbol{v}_{n_k} \to \boldsymbol{v} \quad \text{weakly-}* \text{ in } L^\infty(0, T; \boldsymbol{H}),$$

$$\boldsymbol{v}_{n_k} \to \boldsymbol{v} \quad \text{weakly in } L^2(0, T; \boldsymbol{V}) \quad \text{as } k \to +\infty.$$

We should note that $\boldsymbol{v}_{n_k}$ satisfies strong variational formulations of the form (13)–(15) for all $k \in \mathbb{N}$, respectively, where (14) holds for each test function $\boldsymbol{z} \in K_{n_k}(t)$ (the test functions are dependent on $k \in \mathbb{N}$). Now we have:

Lemma 3.3. *For each $r \in (0,1)$ there exists $N_r \in \mathbb{N}$ such that*

$$r\boldsymbol{v}_{n_k}(t) \in K_{n_\ell}(t), \quad r\boldsymbol{v}_{n_\ell}(t) \in K_{n_k}(t) \quad \textit{for all } k,\ell \geq N_r \textit{ and } t \in [0,T].$$

Proof. The uniform convergence in (8) means that for each $r \in (0,1)$ there exists $N_r \in \mathbb{N}$ such that

$$|\psi_{n_k} - \psi_{n_\ell}|_{C(\overline{Q})} \leq c_0(1-r) \quad \text{for all } k,\ell \geq N_r.$$

Therefore, using (13) and $\psi_{n_k}(t,x)/c_0 \geq 1$, we see that

$$\begin{aligned} |r\boldsymbol{v}_{n_k}(t,x)| &\leq \left(1 - \frac{1}{c_0}|\psi_{n_k} - \psi_{n_\ell}|_{C(\overline{Q})}\right)\psi_{n_k}(t,x) \\ &\leq \psi_{n_k}(t,x) - \psi_{n_\ell}(t,x) + \psi_{n_\ell}(t,x) - |\psi_{n_k} - \psi_{n_\ell}|_{C(\overline{Q})} \\ &\leq \psi_{n_\ell}(t,x) \quad \text{for a.a. } x \in \Omega \quad \text{for all } k,\ell \geq N_r. \end{aligned}$$

Now, $\mathrm{div}\boldsymbol{v}_{n_k}(t) = 0$ implies that $r\boldsymbol{v}_{n_k}(t) \in K_{n_\ell}(t)$ for all $k,\ell \geq N_r$, $t \in [0,T]$. The same approach works for $r\boldsymbol{v}_{n_\ell}(t) \in K_{n_k}(t)$. □

3.3 Proof of main theorem

The essential idea is due to [19].

Proof of Theorem 2.1. Let $\{\boldsymbol{v}_{n_k}\}_{k\in\mathbb{N}} \subset W^{1,2}(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V}) \cap \boldsymbol{L}^\infty(Q)$ be the subsequence of approximate solutions which was constructed in Proposition 3.1. Now consider the strong variational formulations of $\boldsymbol{v}_{n_k}$ and $\boldsymbol{v}_{n_\ell}$ of the form (13)–(15) at $t=\tau$. Denote them by $(13)_k$, $(13)_\ell$, $(14)_k$, $\cdots$, respectively. Let $r \in (0,1)$. First, we show the convergence of the subsequence $\{\boldsymbol{v}_{n_k}\}_{k\in\mathbb{N}}$ in $C([0,T];\boldsymbol{H})$. In fact, letting $k,\ell \geq N_r$ and using Lemma 3.3, we can choose $r\boldsymbol{v}_{n_\ell}(\tau)$ as the test function $\boldsymbol{z} \in K_{n_k}(\tau)$ of $(14)_k$ at $t=\tau$ and $r\boldsymbol{v}_{n_k}(\tau)$ as the test function of $(14)_\ell$. Then we have

$$\begin{aligned} &(\boldsymbol{v}'_{n_k}(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}_{n_\ell}(\tau))_{\boldsymbol{H}} + (1-r)(\boldsymbol{v}'_{n_k}(\tau), \boldsymbol{v}_{n_\ell}(\tau))_{\boldsymbol{H}} \\ &\quad + a(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}_{n_\ell}(\tau)) + (1-r)a(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_\ell}(\tau)) - rb(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_\ell}(\tau), \boldsymbol{v}_{n_k}(\tau)) \\ &\leq (\boldsymbol{g}(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}_{n_\ell}(\tau))_{\boldsymbol{H}} + (1-r)(\boldsymbol{g}(\tau), \boldsymbol{v}_{n_k}(\tau))_{\boldsymbol{H}}, \end{aligned}$$

$$\begin{aligned} &(\boldsymbol{v}'_{n_\ell}(\tau), \boldsymbol{v}_{n_\ell}(\tau) - \boldsymbol{v}_{n_k}(\tau))_{\boldsymbol{H}} + (1-r)(\boldsymbol{v}'_{n_\ell}(\tau), \boldsymbol{v}_{n_k}(\tau))_{\boldsymbol{H}} \\ &\quad + a(\boldsymbol{v}_{n_\ell}(\tau), \boldsymbol{v}_{n_\ell}(\tau) - \boldsymbol{v}_{n_k}(\tau)) + (1-r)a(\boldsymbol{v}_{n_\ell}(\tau), \boldsymbol{v}_{n_k}(\tau)) - rb(\boldsymbol{v}_{n_\ell}(\tau), \boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_\ell}(\tau)) \\ &\leq (\boldsymbol{g}(\tau), \boldsymbol{v}_{n_\ell}(\tau) - \boldsymbol{v}_{n_k}(\tau))_{\boldsymbol{H}} + (1-r)(\boldsymbol{g}(\tau), \boldsymbol{v}_{n_\ell}(\tau))_{\boldsymbol{H}}, \end{aligned}$$

for a.a. $\tau \in [0,T]$ and $k,\ell \geq N_r$. Adding these and using $b(\boldsymbol{v}_{n_k}(\tau)-\boldsymbol{v}_{n_\ell}(\tau), \boldsymbol{v}_{n_\ell}(\tau), \boldsymbol{v}_{n_\ell}(\tau)) = 0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}|\boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}_{n_\ell}(\tau)|_{\boldsymbol{H}}^2 + \|\boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}_{n_\ell}(\tau)\|^2 \\
\leq\ & -(1-r)\frac{d}{d\tau}(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_\ell}(\tau))_{\boldsymbol{H}} - 2(1-r)a(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_\ell}(\tau)) \\
& + rb(\boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}_{n_\ell}(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}_{n_\ell}(\tau), \boldsymbol{v}_{n_\ell}(\tau)) + (1-r)(\boldsymbol{g}(\tau), \boldsymbol{v}_{n_k}(\tau) + \boldsymbol{v}_{n_\ell}(\tau))_{\boldsymbol{H}} \\
\leq\ & -(1-r)\frac{d}{d\tau}(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_\ell}(\tau))_{\boldsymbol{H}} + 2(1-r)\|\boldsymbol{v}_{n_k}(\tau)\|\ \|\boldsymbol{v}_{n_\ell}(\tau)\| + \frac{1}{2}\|\boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}_{n_\ell}(\tau)\|^2 \\
& + \frac{3}{2}c_\psi^2|\boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}_{n_\ell}(\tau)|_{\boldsymbol{H}}^2 + (1-r)|\boldsymbol{g}(\tau)|_{\boldsymbol{H}}|\boldsymbol{v}_{n_k}(\tau) + \boldsymbol{v}_{n_\ell}(\tau)|_{\boldsymbol{H}},
\end{aligned} \tag{20}$$

for a.a. $\tau \in (0,T)$ and for all $k,\ell \geq N_r$. The uniform estimates (18), (19) and Gronwall's inequality imply that

$$\begin{aligned}
&|\boldsymbol{v}_{n_k}(t) - \boldsymbol{v}_{n_\ell}(t)|_{\boldsymbol{H}}^2 \\
\leq\ & \Big\{|\boldsymbol{v}_{0,n_k} - \boldsymbol{v}_{0,n_\ell}|_{\boldsymbol{H}}^2 + 2(1-r)\Big(|\boldsymbol{v}_{0,n_k}|_{\boldsymbol{H}}|\boldsymbol{v}_{0,n_\ell}|_{\boldsymbol{H}} + |\boldsymbol{v}_{n_k}(t)|_{\boldsymbol{H}}|\boldsymbol{v}_{n_\ell}(t)|_{\boldsymbol{H}}\Big) \\
& + 4(1-r)\int_0^t \|\boldsymbol{v}_{n_k}(\tau)\|\ \|\boldsymbol{v}_{n_\ell}(\tau)\|d\tau \\
& + 2(1-r)\int_0^t |\boldsymbol{g}(\tau)|_{\boldsymbol{H}}\Big(|\boldsymbol{v}_{n_k}(\tau)|_{\boldsymbol{H}} + |\boldsymbol{v}_{n_\ell}(\tau)|_{\boldsymbol{H}}\Big)d\tau\Big\}\exp(3c_\psi^2 t) \\
\leq\ & \Big\{|\boldsymbol{v}_{0,n_k} - \boldsymbol{v}_{0,n_\ell}|_{\boldsymbol{H}}^2 + 4(1-r)(M_1+M_2) + 4(1-r)|\boldsymbol{g}|_{L^2(0,T;\boldsymbol{H})}M_1^{\frac{1}{2}}T^{\frac{1}{2}}\Big\}\exp(3c_\psi^2 T),
\end{aligned}$$

for all $t \in [0,T]$ and all $k,\ell \geq N_r$. Thus

$$\begin{aligned}
&\limsup_{k,\ell\to+\infty}|\boldsymbol{v}_{n_k}(t) - \boldsymbol{v}_{n_\ell}(t)|_{\boldsymbol{H}}^2 \\
&\leq\ 4(1-r)\Big\{M_1 + M_2 + |\boldsymbol{g}|_{L^2(0,T;\boldsymbol{H})}M_1^{\frac{1}{2}}T^{\frac{1}{2}}\Big\}\exp(3c_\psi^2 T) \quad \text{for all } t \in [0,T].
\end{aligned}$$

Letting $r \to 1$, we see that $\lim_{k,\ell\to+\infty}|\boldsymbol{v}_{n_k}(t) - \boldsymbol{v}_{n_\ell}(t)|_{\boldsymbol{H}} = 0$ for all $t \in [0,T]$, namely $\{\boldsymbol{v}_{n_k}\}_{k\in\mathbb{N}} \subset W^{1,2}(0,T;\boldsymbol{H})$ is a Cauchy sequence in $C([0,T];\boldsymbol{H})$. Thus $\boldsymbol{v} \in C([0,T];\boldsymbol{H})$, and

$$\boldsymbol{v}_{n_k} \to \boldsymbol{v} \quad \text{in } C([0,T];\boldsymbol{H}) \quad \text{as } k \to +\infty, \tag{21}$$

and hence $\boldsymbol{v}(0) = \boldsymbol{v}_0$ in $\boldsymbol{H}$. Moreover, for each $t \in [0,T]$, we can choose a subsequence $\{\boldsymbol{v}_{n_k}(t)\}_{k\in\mathbb{N}}$ satisfying

$$\boldsymbol{v}_{n_k}(t,x) \to \boldsymbol{v}(t,x) \quad \text{for a.a. } x \in \Omega \quad \text{as } k \to +\infty.$$

Then, using $\boldsymbol{v}_{n_k}(t) \in K_{n_k}(t)$, we obtain

$$\begin{aligned}
|\boldsymbol{v}(t,x)| &\leq |\boldsymbol{v}(t,x) - \boldsymbol{v}_{n_k}(t,x)| + \psi_{n_k}(t,x) \\
&\leq |\boldsymbol{v}(t,x) - \boldsymbol{v}_{n_k}(t,x)| + |\psi_{n_k}(t,x) - \psi(t,x)| + \psi(t,x) \quad \text{for a.a. } x \in \Omega,
\end{aligned}$$

so letting $k \to +\infty$, we see that $\boldsymbol{v}(t) \in K(t)$ for all $t \in [0,T]$. On the other hand, integrating (20) with respect to τ over $[0,T]$, and taking $\limsup_{k,\ell\to+\infty}$,

$$\limsup_{k,\ell\to+\infty} \int_0^T \|\boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}_{n_\ell}(\tau)\|^2 d\tau \leq 4(1-r)\left\{M_1 + M_2 + |\boldsymbol{g}|_{L^2(0,T;\boldsymbol{H})} M_1^{\frac{1}{2}} T^{\frac{1}{2}}\right\}.$$

Letting $r \to 1$, we see that

$$\boldsymbol{v}_{n_k} \to \boldsymbol{v} \quad \text{in } L^2(0,T;\boldsymbol{V}) \quad \text{as } k \to +\infty. \tag{22}$$

Finally, we show that $\boldsymbol{v}$ satisfies a weak variational formulation of the form (6). Let $\boldsymbol{\eta} \in \mathcal{K}$. For each $r \in (0,1)$, there exists $N_r^* \in \mathbb{N}$ such that

$$|\psi_{n_k} - \psi|_{C(\overline{Q})} \leq c_0(1-r) \quad \text{for all } k \geq N_r^*.$$

Then, using the same approach as in the proof of Lemma 3.3, we see that $r\boldsymbol{\eta}(t) \in K_{n_k}(t)$ for all $\boldsymbol{\eta} \in \mathcal{K}$, $k \geq N_r^*$ and $t \in [0,T]$. From Proposition 3.1, we see that $\boldsymbol{v}_{n_k}$ satisfies the strong variational formulation $(13)_k$–$(15)_k$. Now, choose $r\boldsymbol{\eta}(\tau)$ as the test function $\boldsymbol{z} \in K_{n_k}(\tau)$ of $(14)_k$ at $t = \tau$, to give

$$\begin{aligned}
&(\boldsymbol{v}'_{n_k}(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} + (1-r)(\boldsymbol{v}'_{n_k}(\tau), \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} \\
&\qquad + a(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{\eta}(\tau)) + (1-r)a(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{\eta}(\tau)) - rb(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_k}(\tau), \boldsymbol{\eta}(\tau)) \\
&\leq \; (\boldsymbol{g}(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} + (1-r)(\boldsymbol{g}(\tau), \boldsymbol{\eta}(\tau))_{\boldsymbol{H}},
\end{aligned}$$

for a.a. $\tau \in (0,T)$ and for all $k \geq N_r^*$. Integrating this with respect to τ over $[0,T]$, and using the fact that

$$\begin{aligned}
&\int_0^T (\boldsymbol{v}'_{n_k}(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau \\
&= \int_0^T (\boldsymbol{v}'_{n_k}(\tau) - \boldsymbol{\eta}'(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau + \int_0^T (\boldsymbol{\eta}'(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau \\
&= \frac{1}{2}|\boldsymbol{v}_{n_k}(T) - \boldsymbol{\eta}(T)|_{\boldsymbol{H}}^2 - \frac{1}{2}|\boldsymbol{v}_{0,n_k} - \boldsymbol{\eta}(0)|_{\boldsymbol{H}}^2 + \int_0^T (\boldsymbol{\eta}'(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}|\boldsymbol{v}_{n_k}(T) - \boldsymbol{\eta}(T)|_{\boldsymbol{H}}^2 + \int_0^T (\boldsymbol{\eta}'(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau \\
&\qquad + \int_0^T a(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{\eta}(\tau)) d\tau - r\int_0^T b(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_k}(\tau), \boldsymbol{\eta}(\tau)) d\tau \\
&\leq \frac{1}{2}|\boldsymbol{v}_{0,n_k} - \boldsymbol{\eta}(0)|_{\boldsymbol{H}}^2 - (1-r)\Big((\boldsymbol{v}_{n_k}(T), \boldsymbol{\eta}(T))_{\boldsymbol{H}} - (\boldsymbol{v}_{0,n_k}, \boldsymbol{\eta}(0))_{\boldsymbol{H}}\Big) \\
&\qquad + (1-r)\int_0^T (\boldsymbol{v}_{n_k}(\tau), \boldsymbol{\eta}'(\tau))_{\boldsymbol{H}} d\tau - (1-r)\int_0^T a(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{\eta}(\tau)) d\tau \\
&\qquad + \int_0^T (\boldsymbol{g}(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau + (1-r)\int_0^T (\boldsymbol{g}(\tau), \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau \quad \text{for all } k \geq N_r^*.
\end{aligned}$$

Letting $k \to +\infty$, we have from (21) and (22) that

$$\begin{aligned}
&\left|\int_0^T b(\boldsymbol{v}_{n_k}(\tau), \boldsymbol{v}_{n_k}(\tau), \boldsymbol{\eta}(\tau))d\tau - \int_0^T b(\boldsymbol{v}(\tau), \boldsymbol{v}(\tau), \boldsymbol{\eta}(\tau))d\tau\right| \\
\leq\ & \left|\int_0^T b(\boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}(\tau), \boldsymbol{\eta}(\tau), \boldsymbol{v}_{n_k}(\tau))d\tau\right| + \left|\int_0^T b(\boldsymbol{v}(\tau), \boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}(\tau), \boldsymbol{\eta}(\tau))d\tau\right| \\
\leq\ & \sqrt{3}c_\psi \int_0^T |\boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}(\tau)|_{\boldsymbol{H}} \|\boldsymbol{\eta}(\tau)\| d\tau + \sqrt{3}c_\psi \int_0^T \|\boldsymbol{v}_{n_k}(\tau) - \boldsymbol{v}(\tau)\|\ |\boldsymbol{\eta}(\tau)|_{\boldsymbol{H}} d\tau \\
\leq\ & \sqrt{3}c_\psi |\boldsymbol{v}_{n_k} - \boldsymbol{v}|_{C([0,T];\boldsymbol{H})} |\boldsymbol{\eta}|_{L^2(0,T;\boldsymbol{V})} T^{\frac{1}{2}} + \sqrt{3}c_\psi |\boldsymbol{v}_{n_k} - \boldsymbol{v}|_{L^2(0,T;\boldsymbol{V})} |\boldsymbol{\eta}|_{L^2(0,T;\boldsymbol{H})} \\
\to\ & 0 \quad \text{as } k \to +\infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}|\boldsymbol{v}(T) - \boldsymbol{\eta}(T)|_{\boldsymbol{H}}^2 + \int_0^T (\boldsymbol{\eta}'(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau \\
&\qquad + \int_0^T a(\boldsymbol{v}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau))d\tau + r\int_0^T b(\boldsymbol{v}(\tau), \boldsymbol{v}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau))d\tau \\
&\leq \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\eta}(0)|_{\boldsymbol{H}}^2 - (1-r)\Big((\boldsymbol{v}(T), \boldsymbol{\eta}(T))_{\boldsymbol{H}} - (\boldsymbol{v}_0, \boldsymbol{\eta}(0))_{\boldsymbol{H}}\Big) \\
&\qquad + (1-r)\int_0^T (\boldsymbol{v}(\tau), \boldsymbol{\eta}'(\tau))_{\boldsymbol{H}} d\tau - (1-r)\int_0^T a(\boldsymbol{v}(\tau), \boldsymbol{\eta}(\tau))d\tau \\
&\qquad + \int_0^T (\boldsymbol{g}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau + (1-r)\int_0^T (\boldsymbol{g}(\tau), \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau.
\end{aligned}$$

Letting $r \to 1$, we see that $\boldsymbol{v}$ satisfies (6). This accomplishes the proof. □

Proof of Theorem 2.2. Let $\boldsymbol{g}, \tilde{\boldsymbol{g}} \in L^2(0,T;\boldsymbol{H})$ and $\boldsymbol{v}_0, \tilde{\boldsymbol{v}}_0 \in K(0)$, and denote by $\boldsymbol{v}$ the solution constructed in the proof of Theorem 2.1 corresponding to the data $\{\boldsymbol{g}, \boldsymbol{v}_0\}$, and by $\tilde{\boldsymbol{v}}$ any solution corresponding to the data $\{\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{v}}_0\}$. That is, $\tilde{\boldsymbol{v}}$ is not necessarily the limit of approximate solutions for (16) and (17). Now we see that $\tilde{\boldsymbol{v}} \in C([0,T];\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V}) \cap \boldsymbol{L}^\infty(Q)$ satisfies $\tilde{\boldsymbol{v}}(t) \in K(t)$ for all $t \in [0,T]$, $\tilde{\boldsymbol{v}}(0) = \tilde{\boldsymbol{v}}_0$ in $\boldsymbol{H}$, and

$$\begin{aligned}
&\frac{1}{2}|\tilde{\boldsymbol{v}}(T) - \boldsymbol{\eta}(T)|_{\boldsymbol{H}}^2 + \int_0^T (\boldsymbol{\eta}'(\tau), \tilde{\boldsymbol{v}}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau \\
&\qquad + \int_0^T a(\tilde{\boldsymbol{v}}(\tau), \tilde{\boldsymbol{v}}(\tau) - \boldsymbol{\eta}(\tau))d\tau + \int_0^T b(\tilde{\boldsymbol{v}}(\tau), \tilde{\boldsymbol{v}}(\tau), \tilde{\boldsymbol{v}}(\tau) - \boldsymbol{\eta}(\tau))d\tau \\
&\leq \frac{1}{2}|\tilde{\boldsymbol{v}}_0 - \boldsymbol{\eta}(0)|_{\boldsymbol{H}}^2 + \int_0^T (\tilde{\boldsymbol{g}}(\tau), \tilde{\boldsymbol{v}}(\tau) - \boldsymbol{\eta}(\tau))_{\boldsymbol{H}} d\tau \quad \text{for all } \boldsymbol{\eta} \in \mathcal{K}. \qquad (23)
\end{aligned}$$

Now, $\boldsymbol{v}$ is constructed from approximate solutions, so there exists a sequence $\{\boldsymbol{v}_{n_k}\}_{k\in\mathbb{N}} \subset W^{1,2}(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V}) \cap \boldsymbol{L}^\infty(Q)$ such that $\boldsymbol{v}_{n_k}$ satisfies $(13)_k$–$(15)_k$. That is, $\boldsymbol{v}_{n_k}(t) \in K_{n_k}(t)$ for all $t \in [0,T]$ as in $(13)_k$,

$$\begin{aligned}
&(\boldsymbol{v}_{n_k}'(t), \boldsymbol{v}_{n_k}(t) - \boldsymbol{z})_{\boldsymbol{H}} + a(\boldsymbol{v}_{n_k}(t), \boldsymbol{v}_{n_k}(t) - \boldsymbol{z}) + b(\boldsymbol{v}_{n_k}(t), \boldsymbol{v}_{n_k}(t), \boldsymbol{v}_{n_k}(t) - \boldsymbol{z}) \\
&\leq (\boldsymbol{g}(t), \boldsymbol{v}_{n_k}(t) - \boldsymbol{z})_{\boldsymbol{H}} \quad \text{for all } \boldsymbol{z} \in K_{n_k}(t) \text{ and for a.a. } t \in (0,T),
\end{aligned}$$

as in $(14)_k$, and $\boldsymbol{v}_{n_k}(0)=\boldsymbol{v}_{0,n_k}$ in $\boldsymbol{H}$ as in $(15)_k$, for all $k\in\mathbb{N}$, with the strong convergences (21) and (22). Now using the same way as in the proofs of Lemma 3.3 and Theorem 2.1, for each $r\in(0,1)$ we see that $r\tilde{\boldsymbol{v}}(t)\in K_{n_k}(t)$ and $r\boldsymbol{v}_{n_k}(t)\in K(t)$ for all $k\geq N_r^*$ and $t\in[0,T]$. Now choose $r\boldsymbol{v}_{n_k}\in W^{1,2}(0,T;\boldsymbol{H})\cap L^2(0,T;\boldsymbol{V})$ as the test function $\boldsymbol{\eta}\in\mathcal{K}$ of the weak variational formulation (23), and $r\tilde{\boldsymbol{v}}(\tau)$ as the test function $\boldsymbol{z}\in K_{n_k}(\tau)$ of the strong formulation $(14)_k$ at $t=\tau$, respectively. Then we have

$$
\begin{aligned}
&\frac{1}{2}|\tilde{\boldsymbol{v}}(t)-r\boldsymbol{v}_{n_k}(t)|_{\boldsymbol{H}}^2+r\int_0^t(\boldsymbol{v}'_{n_k}(\tau),\tilde{\boldsymbol{v}}(\tau))_{\boldsymbol{H}}d\tau\\
&\quad -r^2\int_0^t(\boldsymbol{v}'_{n_k}(\tau),\boldsymbol{v}_{n_k}(\tau))_{\boldsymbol{H}}d\tau+\int_0^t a(\tilde{\boldsymbol{v}}(\tau),\tilde{\boldsymbol{v}}(\tau)-\boldsymbol{v}_{n_k}(\tau))d\tau\\
&\quad +(1-r)\int_0^t a(\tilde{\boldsymbol{v}}(\tau),\boldsymbol{v}_{n_k}(\tau))d\tau+r\int_0^t b(\tilde{\boldsymbol{v}}(\tau),\boldsymbol{v}_{n_k}(\tau),\tilde{\boldsymbol{v}}(\tau))d\tau\\
\leq\ &\frac{1}{2}|\tilde{\boldsymbol{v}}_0-r\boldsymbol{v}_{n_k}(0)|_{\boldsymbol{H}}^2+\int_0^t(\tilde{\boldsymbol{g}}(\tau),\tilde{\boldsymbol{v}}(\tau)-\boldsymbol{v}_{n_k}(\tau))_{\boldsymbol{H}}d\tau+(1-r)\int_0^t(\tilde{\boldsymbol{g}}(\tau),\boldsymbol{v}_{n_k}(\tau))_{\boldsymbol{H}}d\tau,
\end{aligned}
$$

for all $k\geq N_r^*$, and integrating $(14)_k$ with respect to τ over $[0,T]$, we have

$$
\begin{aligned}
&\int_0^t(\boldsymbol{v}'_{n_k}(\tau),\boldsymbol{v}_{n_k}(\tau))_{\boldsymbol{H}}d\tau-r\int_0^t(\boldsymbol{v}'_{n_k}(\tau),\tilde{\boldsymbol{v}}(\tau))_{\boldsymbol{H}}d\tau\\
&\quad +\int_0^t a(\boldsymbol{v}_{n_k}(\tau),\boldsymbol{v}_{n_k}(\tau)-\tilde{\boldsymbol{v}}(\tau))d\tau+(1-r)\int_0^t a(\boldsymbol{v}_{n_k}(\tau),\tilde{\boldsymbol{v}}(\tau))d\tau\\
&\quad -r\int_0^t b(\boldsymbol{v}_{n_k}(\tau),\boldsymbol{v}_{n_k}(\tau),\tilde{\boldsymbol{v}}(\tau))d\tau\\
\leq\ &\int_0^t(\boldsymbol{g}(\tau),\boldsymbol{v}_{n_k}(\tau)-\tilde{\boldsymbol{v}}(\tau))_{\boldsymbol{H}}d\tau+(1-r)\int_0^t(\boldsymbol{g}(\tau),\tilde{\boldsymbol{v}}(\tau))_{\boldsymbol{H}}d\tau\quad\text{for all } k\geq N_r^*.
\end{aligned}
$$

Adding these inequalities, we get

$$
\begin{aligned}
&\frac{1}{2}|\tilde{\boldsymbol{v}}(t)-r\boldsymbol{v}_{n_k}(t)|_{\boldsymbol{H}}^2+\int_0^t\|\tilde{\boldsymbol{v}}(\tau)-\boldsymbol{v}_{n_k}(\tau)\|^2d\tau\\
\leq\ &\frac{1}{2}|\tilde{\boldsymbol{v}}_0-r\boldsymbol{v}_{0,n_k}|_{\boldsymbol{H}}^2+\frac{r^2-1}{2}\Big(|\boldsymbol{v}_{n_k}(t)|_{\boldsymbol{H}}^2-|\boldsymbol{v}_{0,n_k}|_{\boldsymbol{H}}^2\Big)+2(r-1)\int_0^t\|\tilde{\boldsymbol{v}}(\tau)\|\,\|\boldsymbol{v}_{n_k}(\tau)\|d\tau\\
&+\frac{1}{2}\int_0^t\|\tilde{\boldsymbol{v}}(\tau)-\boldsymbol{v}_{n_k}(\tau)\|^2d\tau+\frac{3}{2}r^2c_\psi^2\int_0^t|\tilde{\boldsymbol{v}}(\tau)-\boldsymbol{v}_{n_k}(\tau)|_{\boldsymbol{H}}^2d\tau\\
&+\frac{1}{2}\int_0^t|\tilde{\boldsymbol{g}}(\tau)-\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2d\tau+\frac{1}{2}\int_0^t|\tilde{\boldsymbol{v}}(\tau)-\boldsymbol{v}_{n_k}(\tau)|_{\boldsymbol{H}}^2d\tau+\frac{1}{2}(1-r)\int_0^t|\tilde{\boldsymbol{g}}(\tau)|_{\boldsymbol{H}}^2d\tau\\
&+\frac{1}{2}(1-r)M_1t+\frac{1}{2}(1-r)\int_0^t|\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2d\tau+\frac{1}{2}(1-r)\int_0^t|\tilde{\boldsymbol{v}}(\tau)|_{\boldsymbol{H}}^2d\tau\quad\text{for all } k\geq N_r^*.
\end{aligned}
$$

Letting $k\to+\infty$ and $r\to1$, we have

$$
\begin{aligned}
&|\tilde{\boldsymbol{v}}(t)-\boldsymbol{v}(t)|_{\boldsymbol{H}}^2\\
&\quad\leq\ |\tilde{\boldsymbol{v}}_0-\boldsymbol{v}_0|_{\boldsymbol{H}}^2+\int_0^t|\tilde{\boldsymbol{g}}(\tau)-\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2d\tau+(3c_\psi^2+1)\int_0^t|\tilde{\boldsymbol{v}}(\tau)-\boldsymbol{v}(\tau)|_{\boldsymbol{H}}^2d\tau,
\end{aligned}
$$

for all $t \in [0,T]$. Hence, by Gronwall's inequality,

$$|\tilde{\boldsymbol{v}}(t)-\boldsymbol{v}(t)|_{\boldsymbol{H}}^2 \leq \left(|\tilde{\boldsymbol{v}}_0-\boldsymbol{v}_0|_{\boldsymbol{H}}^2+\int_0^T |\tilde{\boldsymbol{g}}(\tau)-\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2 d\tau\right)\exp((3c_\psi^2+1)T) \quad \text{for all } t \in [0,T].$$

If we take $\boldsymbol{g} = \tilde{\boldsymbol{g}}$ and $\boldsymbol{v}_0 = \tilde{\boldsymbol{v}}_0$, then we see that the solution obtained using Theorem 2.1 is unique. Thus the continuous dependence for the solutions holds regardless of their construction. □

4 Proofs of auxiliary proposition and lemmas

In this section we give proofs of the proposition and lemmas which were used in the previous section.

4.1 Proofs of auxiliary lemmas

Proof of Lemma 3.1. First, for each $n \in \mathbb{N}$, $t \in [0,T]$ and $\boldsymbol{u} \in \mathcal{V}(-\delta_0,T)$, it is easy to see that the functionals $\varphi_n^t(\boldsymbol{u};\cdot)$ and φ_0 are proper and convex on $\boldsymbol{H}$. Second, we show that $\varphi_n^t(\boldsymbol{u};\cdot)$ is lower semi-continuous on $\boldsymbol{H}$. Let $\boldsymbol{z}_m \to \boldsymbol{z}$ in $\boldsymbol{H}$ as $m \to +\infty$. If $\liminf_{m\to+\infty}\varphi_n^t(\boldsymbol{u};\boldsymbol{z}_m) = +\infty$, then it is evident that $\liminf_{m\to+\infty}\varphi_n^t(\boldsymbol{u};\boldsymbol{z}_m) \geq \varphi_n^t(\boldsymbol{u};\boldsymbol{z})$. So we only consider the case when $\alpha := \liminf_{m\to+\infty}\varphi_n^t(\boldsymbol{u};\boldsymbol{z}_m) < +\infty$. We can choose a subsequence $\{\boldsymbol{z}_{m_k}\}_{k\in\mathbb{N}}$ of $\{\boldsymbol{z}_m\}_{m\in\mathbb{N}}$ satisfying $\lim_{k\to+\infty}\varphi_n^t(\boldsymbol{u};\boldsymbol{z}_{m_k}) = \alpha$ and $\boldsymbol{z}_{m_k} \in K_n(t)$ for all $k \in \mathbb{N}$. Then $\{\boldsymbol{z}_{m_k}\}_{k\in\mathbb{N}}$ is bounded in $\boldsymbol{V}$, so we may assume that $\{\boldsymbol{z}_{m_k}\}_{k\in\mathbb{N}}$ such that $\boldsymbol{z}_{m_k} \to \boldsymbol{z}$ weakly in $\boldsymbol{V}$ as $k \to +\infty$. Hence $\boldsymbol{z} \in K_n(t)$, because $K_n(t)$ is closed and convex in $\boldsymbol{V}$, namely $K_n(t)$ is weakly closed in $\boldsymbol{V}$. Since $\boldsymbol{u}(t) \in K$, we see that

$$\begin{aligned}
\liminf_{m\to+\infty}\varphi_n^t(\boldsymbol{u};\boldsymbol{z}_m) &= \alpha \\
&= \lim_{k\to+\infty}\varphi_n^t(\boldsymbol{u};\boldsymbol{z}_{m_k}) \\
&= \liminf_{k\to+\infty}\varphi_n^t(\boldsymbol{u};\boldsymbol{z}_{m_k}) \\
&\geq \varphi_n^t(\boldsymbol{u};\boldsymbol{z}).
\end{aligned}$$

We also find that the functional φ_0 is lower semi-continuous on $\boldsymbol{H}$. Third, inequality (10) is obtained from definition (7) of the subdifferential. In fact, for each $n \in \mathbb{N}$, $t \in [0,T]$ and $\boldsymbol{u} \in \mathcal{V}(-\delta_0,t)$, definition (7) means that $\boldsymbol{z}^* \in \partial\varphi_n^t(\boldsymbol{u};\boldsymbol{z})$ if and only if $\boldsymbol{z} \in K_n(t)$, $\boldsymbol{z}^* \in \boldsymbol{H}$ and

$$(\boldsymbol{z}^*,\bar{\boldsymbol{z}}-\boldsymbol{z})_{\boldsymbol{H}} \leq \frac{1}{2}\|\bar{\boldsymbol{z}}\|^2 - \frac{1}{2}\|\boldsymbol{z}\|^2 - b(\boldsymbol{u}(t),\boldsymbol{u}(t),\bar{\boldsymbol{z}}-\boldsymbol{z}) \quad \text{for all } \bar{\boldsymbol{z}} \in K_n(t).$$

Now, for each $\tilde{\boldsymbol{z}} \in K_n(t)$ and $\delta \in (0,1)$, if we choose the vector function $\bar{\boldsymbol{z}} := \delta\tilde{\boldsymbol{z}}+(1-\delta)\boldsymbol{z} \in K_n(t)$, divide by δ and let $\delta \to 0$, then we obtain (10). Finally, inequality (9) comes from the definition of $\varphi_n^t(\boldsymbol{u};\cdot)$. In fact, $\boldsymbol{u}(t) \in K$ for all $t \in [0,T]$, and

$$\begin{aligned}
\varphi_n^t(\boldsymbol{u};\boldsymbol{z}) &= \frac{1}{2}\|\boldsymbol{z}\|^2 - b(\boldsymbol{u}(t),\boldsymbol{z},\boldsymbol{u}(t)) + \frac{1}{2}c_\varphi^2 \\
&\geq \frac{1}{2}\|\boldsymbol{z}\|^2 - c_\psi^2 \sum_{i,j=1}^{3} \int_\Omega \left|\frac{\partial z_i}{\partial x_j}(x)\right| dx + \frac{1}{2}c_\varphi^2 \\
&\geq \frac{1}{2}\|\boldsymbol{z}\|^2 - c_\psi^2 \int_\Omega \left(\sum_{i,j=1}^{3} \left|\frac{\partial z_i}{\partial x_j}(x)\right|^2\right)^{\frac{1}{2}} 9^{\frac{1}{2}} dx + \frac{1}{2}c_\varphi^2 \\
&\geq \frac{1}{2}\|\boldsymbol{z}\|^2 - 3c_\psi^2 \left\{\int_\Omega \sum_{i,j=1}^{N} \left|\frac{\partial z_i}{\partial x_j}(x)\right|^2 dx\right\}^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} + \frac{1}{2}c_\varphi^2 \\
&= \varphi_0(\boldsymbol{z}) \\
&\geq 0.
\end{aligned}$$

Thus inequality (9) holds. □

Proof of Lemma 3.2. For each $n \in \mathbb{N}$, recall that $T_0^{(n)} \in (0,T]$ and

$$\frac{1}{c_0}|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})} < 1 \quad \text{for all } s,t \in [0,T] \text{ with } |s-t| \leq T_0^{(n)}.$$

For each $\boldsymbol{z} \in K_n(s)$, put

$$\tilde{\boldsymbol{z}} := \left(1 - \frac{1}{c_0}|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})}\right)\boldsymbol{z}.$$

Using $\psi_n(s,x)/c_0 \geq 1$, we have

$$\begin{aligned}
|\tilde{\boldsymbol{z}}(x)| &\leq \left(1 - \frac{1}{c_0}|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})}\right)\psi_n(s,x) \\
&\leq \psi_n(s,x) - \psi_n(t,x) + \psi_n(t,x) - |\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})} \\
&\leq \psi_n(t,x) \quad \text{for a.a. } x \in \Omega.
\end{aligned}$$

Since $\mathrm{div}\tilde{\boldsymbol{z}} = 0$, $\tilde{\boldsymbol{z}} \in K_n(t)$ and (11) hold. Next, using the inequality

$$\begin{aligned}
\|\boldsymbol{z}\| &\leq \|\boldsymbol{z}\| + b(\boldsymbol{u}(s),\boldsymbol{u}(s),\boldsymbol{z}) + c_\varphi\|\boldsymbol{z}\| \\
&\leq \frac{1}{2}(1+c_\varphi)^2 + \frac{1}{2}\|\boldsymbol{z}\|^2 + b(\boldsymbol{u}(s),\boldsymbol{u}(s),\boldsymbol{z}) \\
&\leq 1 + \frac{c_\varphi^2}{2} + \varphi_n^s(\boldsymbol{u};\boldsymbol{z}) \\
&\leq \left(1 + \frac{c_\varphi^2}{2}\right)(1 + \varphi_n^s(\boldsymbol{u};\boldsymbol{z})),
\end{aligned}$$

we obtain

$$
\begin{aligned}
&\varphi_n^t(\boldsymbol{u};\tilde{\boldsymbol{z}}) - \varphi_n^s(\boldsymbol{u};\boldsymbol{z}) \\
\leq\ & \left(1 - \frac{1}{c_0}|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})}\right)^2 \frac{1}{2}\|\boldsymbol{z}\|^2 - \frac{1}{2}\|\boldsymbol{z}\|^2 \\
& + \left(\frac{1}{c_0}|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})} - 1\right) b(\boldsymbol{u}(t),\boldsymbol{z},\boldsymbol{u}(t)) + b(\boldsymbol{u}(s),\boldsymbol{z},\boldsymbol{u}(s)) \\
\leq\ & \frac{1}{c_0}|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})} b(\boldsymbol{u}(t),\boldsymbol{z},\boldsymbol{u}(t)) - b(\boldsymbol{u}(t)-\boldsymbol{u}(s),\boldsymbol{z},\boldsymbol{u}(s)) \\
& - b(\boldsymbol{u}(s),\boldsymbol{z},\boldsymbol{u}(t)-\boldsymbol{u}(s)) \\
\leq\ & \frac{1}{c_0}|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})} 3c_\psi^2|\Omega|^{\frac{1}{2}}\|\boldsymbol{z}\| + 2\cdot\sqrt{3}c_\psi|\boldsymbol{u}(t)-\boldsymbol{u}(s)|_{\boldsymbol{H}}\|\boldsymbol{z}\| \\
\leq\ & c_1\left\{|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})} + |\boldsymbol{u}(t)-\boldsymbol{u}(s)|_{\boldsymbol{H}}\right\}(1+\varphi_n^s(\boldsymbol{u};\boldsymbol{z})),
\end{aligned}
$$

where $c_1 := (1 + c_\varphi^2/2)(c_\varphi/c_0 + 2\sqrt{3}c_\psi)$. Thus (12) holds. □

4.2 Proof of Proposition 3.1.

The essential idea in the proof of Proposition 3.1 comes from Theorems 2.1 and 2.2 in [14]. It is enough to ensure that the time-dependent of $\varphi_n^t(\boldsymbol{u};\cdot)$ depends on $n \in \mathbb{N}$ (see Lemma 3.2), but positive constants M_1 and M_2 in (18) and (19) are independent of $n \in \mathbb{N}$. To solve the Cauchy problem in (16) and (17), the fixed point theorem is applied. We first prepare a vector-valued function space in which the solution is constructed. For each $R > 0$,

$$
\mathcal{V}_R(\boldsymbol{v}_{o,n};-\delta_0,T) := \left\{ \boldsymbol{u} \in \mathcal{V}(-\delta_0,T);\ \begin{array}{c} \boldsymbol{u}(t) = \boldsymbol{v}_{o,n}(t) \quad \text{for all } t \in [-\delta_0,0], \\ \displaystyle\sup_{s\in[0,T]} \left\{ \varphi_0(\boldsymbol{u}(s)) + \frac{1}{2}\int_0^s |\boldsymbol{u}'(\tau)|_{\boldsymbol{H}}^2 d\tau \right\} \leq R \end{array} \right\}.
$$

We now fix $n \in \mathbb{N}$, and take a positive constant $R > 0$ so that $R > \varphi_n^0(\boldsymbol{v}_{o,n};\boldsymbol{v}_{o,n}(0))$. Then we see that $\mathcal{V}_R(\boldsymbol{v}_{o,n};-\delta_0,T)$ is non-empty, convex and compact in $C([-\delta_0,T];\boldsymbol{H})$. Now, we recall the basic concepts of the resolvent and the Yosida approximation for convex functionals and their subdifferentials. For any $\lambda > 0$, $t \in [0,T]$ and $\boldsymbol{u} \in \mathcal{V}(-\delta_0,t)$, the resolvent $J_{n,\lambda}^t(\boldsymbol{u};\cdot) := (I + \lambda\partial\varphi_n^t(\boldsymbol{u};\cdot))^{-1} : \boldsymbol{H} \to \boldsymbol{H}$ and the Yosida approximation $(\partial\varphi_n^t(\boldsymbol{u};\cdot))^\lambda$ of $\partial\varphi_n^t(\boldsymbol{u};\cdot)$ are as follows:

$$
(\partial\varphi_n^t(\boldsymbol{u};\cdot))^\lambda(\boldsymbol{z}) := \frac{1}{\lambda}(\boldsymbol{z} - J_{n,\lambda}^t(\boldsymbol{u};\boldsymbol{z})) = \partial\varphi_{n,\lambda}^t(\boldsymbol{u};\boldsymbol{z}) \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H},
$$

where $\varphi_{n,\lambda}^t(\boldsymbol{u};\cdot)$ is the Moreau-Yosida regularization of $\varphi_n^t(\boldsymbol{u};\cdot)$ defined by

$$
\begin{aligned}
\varphi_{n,\lambda}^t(\boldsymbol{u};\boldsymbol{z}) &:= \inf_{\boldsymbol{y}\in\boldsymbol{H}} \left\{ \frac{1}{2\lambda}|\boldsymbol{z}-\boldsymbol{y}|_{\boldsymbol{H}}^2 + \varphi_n^t(\boldsymbol{u};\boldsymbol{y}) \right\} \\
&= \frac{1}{2\lambda}|\boldsymbol{z} - J_{n,\lambda}^t(\boldsymbol{u};\boldsymbol{z})|_{\boldsymbol{H}}^2 + \varphi_n^t(\boldsymbol{u};J_{n,\lambda}^t(\boldsymbol{u};\boldsymbol{z})) \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H}.
\end{aligned}
$$

For further fundamental properties of convex functionals, refer to [4, 9].

We need the following auxiliary lemma for the proof of Proposition 3.1:

Lemma 4.1. [[18], Lemma 1.2.1, Lemma 1.5.4 and [21], Remark 1.3] *Let $\boldsymbol{u}$ be any vector function in $\mathcal{V}_R(\boldsymbol{v}_{o,n}; -\delta_0, T)$. Then,*

$$|J^t_{n,\lambda}(\boldsymbol{u};\boldsymbol{z})|_{\boldsymbol{H}} \le c_\varphi + |\boldsymbol{z}|_{\boldsymbol{H}} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H},\ t \in [0, T_0^{(n)}] \text{ and } \lambda \in (0,1]. \tag{24}$$

Moreover, $s \mapsto \varphi^s_{n,\lambda}(\boldsymbol{u};\boldsymbol{z})$ is differentiable for a.a. $s \in [0, T_0^{(n)}]$ and its derivative is integrable on $[0, T_0^{(n)}]$, such that

$$\begin{aligned} \frac{d}{ds}\varphi^s_{n,\lambda}(\boldsymbol{u};\boldsymbol{z}) \le\ & \frac{1}{c_0}|\psi_n'(s)|_{C(\overline{\Omega})}(c_\varphi + |\boldsymbol{z}|_{\boldsymbol{H}})|\partial\varphi^s_{n,\lambda}(\boldsymbol{u};\boldsymbol{z})|_{\boldsymbol{H}} \\ & + c_1\Big\{|\psi_n'(s)|_{C(\overline{\Omega})} + |\boldsymbol{u}'(s)|_{\boldsymbol{H}}\Big\}(1 + \varphi^s_{n,\lambda}(\boldsymbol{u};\boldsymbol{z})), \end{aligned} \tag{25}$$

for all $\boldsymbol{z} \in \boldsymbol{H}$ and for a.a. $s \in [0, T_0^{(n)}]$.

Proof. Put $\boldsymbol{w} := J^t_{n,\lambda}(\boldsymbol{u};\boldsymbol{z})$. Then $\boldsymbol{w} \in K_n(t) \subset K$ and $\boldsymbol{w} + \lambda\partial\varphi^t_n(\boldsymbol{u};\boldsymbol{w}) \ni \boldsymbol{z}$ in $\boldsymbol{H}$. Since $\boldsymbol{u} \in K$, from definition (7) of subdifferential we see that

$$|\boldsymbol{w}|^2_{\boldsymbol{H}} + \lambda\varphi^t_n(\boldsymbol{u};\boldsymbol{w}) \le \frac{\lambda}{2}c_\varphi^2 + \frac{1}{2}|\boldsymbol{z}|^2_{\boldsymbol{H}} + \frac{1}{2}|\boldsymbol{w}|^2_{\boldsymbol{H}}.$$

Therefore, $\lambda \le 1$ implies estimate (24). Next, for each $\boldsymbol{z} \in \boldsymbol{H}$, by Lemmas 3.2 and the fact that $\tilde{\boldsymbol{w}} := J^s_{n,\lambda}(\boldsymbol{u};\boldsymbol{z}) \in K_n(s)$, there exists $\tilde{\boldsymbol{z}} \in K_n(t)$ such that two estimates similar to (11) and (12) hold, namely

$$\begin{aligned} & \varphi^t_{n,\lambda}(\boldsymbol{u};\boldsymbol{z}) - \varphi^s_{n,\lambda}(\boldsymbol{u};\boldsymbol{z}) \\ \le\ & \frac{1}{2\lambda}|\boldsymbol{z} - \tilde{\boldsymbol{z}}|^2_{\boldsymbol{H}} + \varphi^t_n(\boldsymbol{u};\tilde{\boldsymbol{z}}) - \frac{1}{2\lambda}|\boldsymbol{z} - \tilde{\boldsymbol{w}}|^2_{\boldsymbol{H}} - \varphi^s_n(\boldsymbol{u};\tilde{\boldsymbol{w}}) \\ \le\ & |\tilde{\boldsymbol{z}} - \tilde{\boldsymbol{w}}|_{\boldsymbol{H}}\left|\frac{\boldsymbol{z} - \tilde{\boldsymbol{w}}}{\lambda}\right|_{\boldsymbol{H}} + \frac{1}{2\lambda}|\tilde{\boldsymbol{z}} - \tilde{\boldsymbol{w}}|^2_{\boldsymbol{H}} + \varphi^t_n(\boldsymbol{u};\tilde{\boldsymbol{z}}) - \varphi^s_n(\boldsymbol{u};\tilde{\boldsymbol{w}}) \\ \le\ & \frac{1}{c_0}|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})}|\tilde{\boldsymbol{w}}|_{\boldsymbol{H}}|\partial\varphi^s_{n,\lambda}(\boldsymbol{u};\boldsymbol{z})|_{\boldsymbol{H}} + \frac{1}{2\lambda}\frac{1}{c_0^2}|\psi_n(t) - \psi_n(s)|^2_{C(\overline{\Omega})}|\tilde{\boldsymbol{w}}|^2_{\boldsymbol{H}} \\ & + c_1\Big\{|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})} + |\boldsymbol{u}(t) - \boldsymbol{u}(s)|_{\boldsymbol{H}}\Big\}\Big(1 + \varphi^s_n(\boldsymbol{u};\tilde{\boldsymbol{w}})\Big) \\ \le\ & \frac{1}{c_0}|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})}(c_\psi + |\boldsymbol{z}|_{\boldsymbol{H}})|\partial\varphi^s_{n,\lambda}(\boldsymbol{u};\boldsymbol{z})|_{\boldsymbol{H}} \\ & + \frac{1}{2\lambda}\frac{1}{c_0^2}|\psi_n(t) - \psi_n(s)|^2_{C(\overline{\Omega})}(c_\psi + |\boldsymbol{z}|_{\boldsymbol{H}})^2 \\ & + c_1\Big\{|\psi_n(t) - \psi_n(s)|_{C(\overline{\Omega})} + |\boldsymbol{u}(t) - \boldsymbol{u}(s)|_{\boldsymbol{H}}\Big\}(1 + \varphi^s_{n,\lambda}(\boldsymbol{u};\boldsymbol{z})). \end{aligned}$$

Thus (25) holds. □

Proof of Proposition 3.1. The proof of Proposition 3.1 consists of three steps. In Step 1, the auxiliary problem of the Yosida approximation for the convex functional is considered, the component $\boldsymbol{u} \in \mathcal{V}(\boldsymbol{v}_{o,n}; -\delta_0, T)$ of $\varphi_n^t(\boldsymbol{u}; \cdot)$ being fixed. In Step 2, we seek for the local solution $\boldsymbol{v}_n$ of (16) and (17) using Schauder's fixed point theorem. For each $n \in \mathbb{N}$, these local solutions satisfy the evolution equation on the time interval $[0, T_0^{(n)}]$. In Step 3, we consider the prolongation of this solution in time, and construct solutions $\boldsymbol{v}_n$ on all time interval $[0, T]$. Moreover, we show the uniqueness of these solutions.

Step 1. For each fixed $n \in \mathbb{N}$, let $\boldsymbol{u}$ be any vector function in $\mathcal{V}(\boldsymbol{v}_{o,n}; -\delta_0, T)$. By Lemmas 3.1, 3.2 and 4.1, the abstract theory (see [9, 17, 18, 39]) shows that for each $\lambda \in (0, 1]$ there exists a unique vector function $\boldsymbol{v}_{n,\lambda} \in W^{1,2}(-\delta_0, T_0^{(n)}; \boldsymbol{H})$ such that

$$\boldsymbol{v}_{n,\lambda}'(t) + \partial\varphi_{n,\lambda}^t(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(t)) = \boldsymbol{g}(t) \quad \text{in } \boldsymbol{H} \text{ for a.a. } t \in (0, T_0^{(n)}), \tag{26}$$

$$\boldsymbol{v}_{n,\lambda}(t) = \boldsymbol{v}_{o,n}(t) \quad \text{in } \boldsymbol{H} \text{ for all } t \in [-\delta_0, 0]. \tag{27}$$

Multiplying (26) at $t = \tau$ by $\boldsymbol{v}_{n,\lambda}(\tau)$ and using Young's inequality we have

$$\frac{1}{2}\frac{d}{d\tau}|\boldsymbol{v}_{n,\lambda}(\tau)|_{\boldsymbol{H}}^2 + \varphi_{n,\lambda}^\tau(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(\tau)) \leq \frac{1}{2}|\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2 + \frac{1}{2}|\boldsymbol{v}_{n,\lambda}(\tau)|_{\boldsymbol{H}}^2.$$

Now, from this inequality and Gronwall's inequality we deduce that

$$|\boldsymbol{v}_{n,\lambda}(t)|_{\boldsymbol{H}}^2 \leq \left(|\boldsymbol{v}_0|_{\boldsymbol{H}}^2 + c_\varphi^2 T + \int_0^T |\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2 d\tau\right) \exp T =: M_1^* \quad \text{for all } t \in [0, T_0^{(n)}].$$

Moreover,

$$\begin{aligned} \int_0^t \varphi_{n,\lambda}^\tau(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(\tau))d\tau &\leq \frac{1}{2}|\boldsymbol{v}_0|_{\boldsymbol{H}}^2 + \frac{1}{2}c_\varphi^2 T + \frac{1}{2}\int_0^t |\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2 d\tau + \frac{1}{2}\int_0^t |\boldsymbol{v}_{n,\lambda}(\tau)|_{\boldsymbol{H}}^2 d\tau \\ &\leq \frac{1}{2}M_1^* + \frac{1}{2}M_1^* T =: M_2 \quad \text{for all } t \in [0, T_0^{(n)}]. \end{aligned}$$

Next, multiplying (26) at $t = \tau$ by $\boldsymbol{v}_{n,\lambda}'(\tau)$ yields

$$|\boldsymbol{v}_{n,\lambda}'(\tau)|_{\boldsymbol{H}}^2 + \left(\partial\varphi_{n,\lambda}^\tau(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(\tau)), \boldsymbol{v}_{n,\lambda}'(\tau)\right)_{\boldsymbol{H}} = (\boldsymbol{g}(\tau), \boldsymbol{v}_{n,\lambda}'(\tau))_{\boldsymbol{H}} \quad \text{for all } \tau \in (0, T_0^{(n)}). \tag{28}$$

According to Lemma 1.2.5 of [18], we see that the function $t \mapsto \varphi_{n,\lambda}^t(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(t))$ is differentiable for a.a. $t \in [0, T_0^{(n)}]$, its derivative is integrable on $[0, T_0^{(n)}]$, and it satisfies

$$\varphi_{n,\lambda}^t(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(t)) - \varphi_{n,\lambda}^s(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(s)) \leq \int_s^t \frac{d}{d\tau}\varphi_{n,\lambda}^\tau(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(\tau))d\tau \quad \text{for all } s, t \in [0, T_0^{(n)}]; s \leq t.$$

Using Lemma 4.1, we obtain

$$\begin{aligned} &\frac{d}{dt}\varphi_{n,\lambda}^t(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(t)) - \left(\boldsymbol{v}_{n,\lambda}'(t), \partial\varphi_{n,\lambda}^t(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(t))\right)_{\boldsymbol{H}} \\ &\leq \frac{1}{c_0}|\psi_n'(t)|_{C(\overline{\Omega})}\left(c_\varphi + (M_1^*)^{\frac{1}{2}}\right)(|\boldsymbol{v}_{n,\lambda}'(t)|_{\boldsymbol{H}} + |\boldsymbol{g}(t)|_{\boldsymbol{H}}) \\ &\quad + c_1\left\{|\psi_n'(t)|_{C(\overline{\Omega})} + |\boldsymbol{u}(t)|_{\boldsymbol{H}}\right\}\left(1 + \varphi_{n,\lambda}^t(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(t))\right) \\ &\leq \frac{1}{4}|\boldsymbol{v}_{n,\lambda}'(t)|_{\boldsymbol{H}}^2 + \frac{1}{2}|\boldsymbol{g}(t)|_{\boldsymbol{H}}^2 + \left(1 + \frac{1}{2}\right)\frac{1}{c_0^2}|\psi_n'(t)|_{C(\overline{\Omega})}^2\left(c_\varphi + (M_1^*)^{\frac{1}{2}}\right)^2 \\ &\quad + c_1\left\{|\psi_n'(t)|_{C(\overline{\Omega})} + |\boldsymbol{u}(t)|_{\boldsymbol{H}}\right\}\left(1 + \varphi_{n,\lambda}^t(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(t))\right), \end{aligned}$$

for all $t \in (0, T_0^{(n)})$. Therefore, it follows from (28) that

$$
\begin{aligned}
& |\boldsymbol{v}'_{n,\lambda}(\tau)|_{\boldsymbol{H}}^2 + \frac{d}{d\tau}\varphi^{\tau}_{n,\lambda}(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(\tau)) \\
\leq\ & \left(\frac{1}{4}|\boldsymbol{v}'_{n,\lambda}(\tau)|_{\boldsymbol{H}}^2 + |\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2\right) + \frac{1}{4}|\boldsymbol{v}'_{n,\lambda}(\tau)|_{\boldsymbol{H}}^2 + \frac{1}{2}|\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2 + \frac{3}{c_0^2}|\psi'_n(\tau)|^2_{C(\overline{\Omega})}(c_\varphi^2 + M_1^*) \\
& + c_1\left\{|\psi'_n(\tau)|_{C(\overline{\Omega})} + |\boldsymbol{u}'(\tau)|_{\boldsymbol{H}}\right\} + c_1\left\{|\psi'_n(\tau)|_{C(\overline{\Omega})} + |\boldsymbol{u}'(\tau)|_{\boldsymbol{H}}\right\}\varphi^{\tau}_{n,\lambda}(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(\tau)),
\end{aligned}
$$

for all $\tau \in (0, T_0^{(n)})$. Using Gronwall's inequality again, we derive

$$
\begin{aligned}
& \varphi^{t}_{n,\lambda}(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda}(t)) \\
\leq\ & \left(\varphi^0_n(\boldsymbol{u}; \boldsymbol{v}_{o,n}(0)) + \frac{3}{2}\int_0^t |\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2 d\tau + \frac{3}{c_0^2}(c_\varphi^2 + M_1^*)\int_0^t |\psi'_n(\tau)|^2_{C(\overline{\Omega})} d\tau \right. \\
& \left. + c_1\int_0^t \left\{|\psi'_n(\tau)|_{C(\overline{\Omega})} + |\boldsymbol{u}'(\tau)|_{\boldsymbol{H}}\right\} d\tau\right) \exp\left(c_1\int_0^t \left\{|\psi'_n(\tau)|_{C(\overline{\Omega})} + |\boldsymbol{u}'(\tau)|_{\boldsymbol{H}}\right\} d\tau\right) \\
\leq\ & \left(\frac{1}{2}\|\boldsymbol{v}_0\|^2 + 9c_\psi^4|\Omega|\ \|\boldsymbol{v}_0\| + \frac{1}{2}c_\varphi^2 + \frac{3}{2}\int_0^T |\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2 d\tau + \frac{3}{c_0^2}(c_\varphi^2 + M_1^*)\int_0^T |\psi'_n(\tau)|^2_{C(\overline{\Omega})} d\tau \right. \\
& \left. + c_1\int_0^T |\psi'_n(\tau)|_{C(\overline{\Omega})} d\tau + c_1 R^{\frac{1}{2}} T^{\frac{1}{2}}\right) \exp\left(c_1\int_0^T |\psi'_n(\tau)|_{C(\overline{\Omega})} d\tau + c_1 R^{\frac{1}{2}} T^{\frac{1}{2}}\right) \\
=:\ & M_3^{(n,R)} \quad \text{for all } t \in [0, T_0^{(n)}],
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^t |\boldsymbol{v}'_{n,\lambda}(\tau)|_{\boldsymbol{H}}^2 d\tau \\
\leq\ & 2\varphi^0_n(\boldsymbol{u}; \boldsymbol{v}_{o,n}(0)) + 3\int_0^t |\boldsymbol{g}(\tau)|_{\boldsymbol{H}}^2 d\tau + \frac{6}{c_0^2}(c_\varphi^2 + M_1^*)\int_0^t |\psi'_n(\tau)|^2_{C(\overline{\Omega})} d\tau \\
& + 2c_1\int_0^t \left\{|\psi'_n(\tau)|^2_{C(\overline{\Omega})} + |\boldsymbol{u}'(\tau)|_{\boldsymbol{H}}\right\} d\tau + 2c_1 M_3^{(n,R)}\int_0^t \left\{|\psi'_n(\tau)|_{C(\overline{\Omega})} + |\boldsymbol{u}'(\tau)|_{\boldsymbol{H}}\right\} d\tau \\
\leq\ & 2M_3^{(n,R)} + 2c_1 M_3^{(n,R)}\left(\int_0^T |\psi'_n(\tau)|_{C(\overline{\Omega})} d\tau + R^{\frac{1}{2}} T^{\frac{1}{2}}\right) =: M_4^{(n,R)} \quad \text{for all } t \in [0, T_0^{(n)}].
\end{aligned}
$$

As a consequence of these uniform estimates for $\lambda \in (0,1]$, there exists a subsequence $\{\boldsymbol{v}_{n,\lambda_k}\}_{k\in\mathbb{N}}$ of $\{\boldsymbol{v}_{n,\lambda}\}_{\lambda\in(0,1]}$, a vector function $\boldsymbol{v}_n \in W^{1,2}(0, T_0^{(n)}; \boldsymbol{H})$ and $\boldsymbol{v}_n^* \in L^2(0, T_0^{(n)}; \boldsymbol{H})$, such that

$$
\boldsymbol{v}_{n,\lambda_k} \to \boldsymbol{v}_n \quad \text{weakly-}* \text{ in } L^\infty(0, T_0^{(n)}; \boldsymbol{H}), \quad \boldsymbol{v}'_{n,\lambda_k} \to \boldsymbol{v}'_n \quad \text{weakly in } L^2(0, T_0^{(n)}; \boldsymbol{H}),
$$

$$
\partial\varphi^{(\cdot)}_{n,\lambda_k}(\boldsymbol{u}; \boldsymbol{v}_{n,\lambda_k}(\cdot)) \to \boldsymbol{v}_n^* \quad \text{weakly in } L^2(0, T_0^{(n)}; \boldsymbol{H}) \quad \text{as } k \to +\infty.
$$

Moreover, by the standard argument in the theory of nonlinear evolution equations (cf. Lemma 2.4 in [12] or Lemma 1.4.1 in [18]), we have the following strong convergences:

$$
\boldsymbol{v}_{n,\lambda_k} \to \boldsymbol{v}_n \quad \text{in } C([0, T_0^{(n)}]; \boldsymbol{H}),
$$

$$\partial\varphi^{(\cdot)}_{n,\lambda_k}(\boldsymbol{u};\boldsymbol{v}_{n,\lambda_k}(\cdot)) \to \boldsymbol{v}^*_n \quad \text{in } L^2(0,T_0^{(n)};\boldsymbol{H}) \quad \text{as } k\to+\infty.$$

By the demi-closedness of the subdifferentials we have

$$\boldsymbol{v}^*_n(t) \in \partial\varphi^t_n(\boldsymbol{u};\boldsymbol{v}_n(t)) \quad \text{in } \boldsymbol{H} \text{ for a.a. } t\in[0,T_0^{(n)}].$$

Thus $\boldsymbol{v}_n$ satisfies

$$\boldsymbol{v}'_n(t) + \partial\varphi^t_n(\boldsymbol{u};\boldsymbol{v}_n(t)) \ni \boldsymbol{g}(t) \quad \text{in } \boldsymbol{H} \text{ for a.a. } t\in(0,T_0^{(n)}),$$

$$\boldsymbol{v}_n(t) = \boldsymbol{v}_{o,n}(t) \quad \text{in } \boldsymbol{H} \text{ for all } t\in[-\delta_0,0].$$

Moreover,

$$\begin{aligned}\int_0^t \varphi^\tau_n(\boldsymbol{u};\boldsymbol{v}_n(\tau))d\tau &\le \int_0^t \liminf_{k\to+\infty}\varphi^\tau_{n,\lambda_k}(\boldsymbol{u};\boldsymbol{v}_{n,\lambda_k}(\tau))d\tau \\ &\le \liminf_{k\to+\infty}\int_0^t \varphi^\tau_{n,\lambda_k}(\boldsymbol{u};\boldsymbol{v}_{n,\lambda_k}(\tau))d\tau \\ &\le M_2, \end{aligned} \tag{29}$$

$$\varphi^t_n(\boldsymbol{u};\boldsymbol{v}_n(t)) \le M_3^{(n,R)}, \tag{30}$$

$$\begin{aligned}\int_0^t |\boldsymbol{v}'_n(\tau)|^2_{\boldsymbol{H}}d\tau &\le \liminf_{k\to+\infty}\int_0^t |\boldsymbol{v}'_{n,\lambda_k}(\tau)|^2_{\boldsymbol{H}}d\tau \\ &\le M_4^{(n,R)} \quad \text{for all } t\in[0,T_0^{(n)}].\end{aligned}$$

The estimate (30) means that $\boldsymbol{v}_n(t)\in K_n(t)\subset K$ for all $t\in[0,T_0^{(n)}]$, and thus

$$|\boldsymbol{v}_n(t)|^2_{\boldsymbol{H}} \le c_\psi^2|\Omega|^{\frac{1}{2}} =: M_1 \quad \text{for all } t\in[0,T_0^{(n)}]. \tag{31}$$

Step 2. The parameter $n\in\mathbb{N}$ is still fixed. We remind the reader that the solution $\boldsymbol{v}_n$ of (16) and (17) is the same as a solution of (26) and (27) with $\boldsymbol{u}=\boldsymbol{v}_n$. We shall seek for a solution of (26) and (27) with the help of Schauder's fixed point theorem. Let $\tau_0>0$ such that

$$\varphi^0_n(\boldsymbol{v}_{o,n};\boldsymbol{v}_{o,n}(0)) + \tau_0 < R.$$

Then there exists a positive number $T_1^{(n)}\in(0,T_0^{(n)}]$ such that

$$\begin{aligned}&\varphi^t_n(\boldsymbol{u};\boldsymbol{v}_n(t)) + \int_0^t |\boldsymbol{v}'_n(\tau)|^2_{\boldsymbol{H}}d\tau \\ &\le \varphi^0_n(\boldsymbol{v}_{o,n};\boldsymbol{v}_{o,n}(0)) + \tau_0 \quad \text{for all } t\in[0,T_1^{(n)}] \text{ and } \boldsymbol{u}\in\mathcal{V}_R(\boldsymbol{v}_{o,n};-\delta_0,T_0^{(n)}),\end{aligned} \tag{32}$$

where $\boldsymbol{v}_n$ is the solution of (26) and (27) obtained by Step 1. In fact, by equation (28) and using Lemma 4.1, we have

$$\begin{aligned}
&\varphi^t_{n,\lambda}(\boldsymbol{u};\boldsymbol{v}_{n,\lambda}(t)) + \int_0^t |\boldsymbol{v}'_{n,\lambda}(\tau)|^2_{\boldsymbol{H}} d\tau \\
&\le \varphi^0_{n,\lambda}(\boldsymbol{u};\boldsymbol{v}_{o,n}(0)) + \int_0^t |\boldsymbol{g}(\tau)|^2_{\boldsymbol{H}} d\tau + \frac{1}{4}\int_0^t |\boldsymbol{v}'_{n,\lambda}(\tau)|^2_{\boldsymbol{H}} d\tau \\
&\quad + \frac{1}{c_0}\Big(c_\varphi + (M_1^*)^{\frac{1}{2}}\Big)\int_0^t |\psi'_n(\tau)|_{C(\overline{\Omega})}\Big(|\boldsymbol{v}'_{n,\lambda}(\tau)|_{\boldsymbol{H}} + |\boldsymbol{g}(\tau)|_{\boldsymbol{H}}\Big) d\tau \\
&\quad + c_1\int_0^t \Big\{|\psi'_n(\tau)|_{C(\overline{\Omega})} + |\boldsymbol{u}'(\tau)|_{\boldsymbol{H}}\Big\}\Big(1 + \varphi^\tau_{n,\lambda}(\boldsymbol{u};\boldsymbol{v}_{n,\lambda}(\tau))\Big) d\tau,
\end{aligned}$$

namely

$$\begin{aligned}
&\varphi^t_{n,\lambda}(\boldsymbol{u};\boldsymbol{v}_{n,\lambda}(t)) + \frac{1}{2}\int_0^t |\boldsymbol{v}'_{n,\lambda}(\tau)|^2_{\boldsymbol{H}} d\tau \\
&\le \varphi^0_n(\boldsymbol{v}_{o,n};\boldsymbol{v}_{o,n}(0)) + \int_0^t |\boldsymbol{g}(\tau)|^2_{\boldsymbol{H}} d\tau + \frac{1}{c_0^2}\Big(c_\varphi + (M_1^*)^{\frac{1}{2}}\Big)^2 \int_0^t |\psi'_n(\tau)|^2_{C(\overline{\Omega})} d\tau \\
&\quad + \frac{1}{c_0}\Big(c_\varphi + (M_1^*)^{\frac{1}{2}}\Big)\left\{\int_0^t |\psi'_n(\tau)|^2_{C(\overline{\Omega})} d\tau\right\}^{\frac{1}{2}}\left\{\int_0^t |\boldsymbol{g}(\tau)|^2_{\boldsymbol{H}} d\tau\right\}^{\frac{1}{2}} \\
&\quad + c_1 M_3^{(n,R)}\int_0^t |\psi'_n(\tau)|_{C(\overline{\Omega})} d\tau + c_1 M_3^{(n,R)} R^{\frac{1}{2}} t^{\frac{1}{2}},
\end{aligned}$$

for all $t \in [0, T_0^{(n)}]$. Since the right-hand side of the above inequality is independent of $\lambda \in (0,1]$ and $\boldsymbol{u} \in \mathcal{V}_R(\boldsymbol{v}_{o,n}; -\delta_0, T)$, there exists a small positive number $T_1^{(n)} \in (0, T_0^{(n)}]$ such that the condition (32) holds. Now, we define the mapping $S : \mathcal{V}_R(\boldsymbol{v}_o; -\delta_0, T) \to C([-\delta_0, T]; \boldsymbol{H})$ as follows:

$$S\boldsymbol{u}(t) := \begin{cases} \boldsymbol{v}_{o,n}(t) & \text{if } t \in [-\delta_0, 0], \\ \boldsymbol{v}_n(t) & \text{if } t \in (0, T_1^{(n)}], \\ \boldsymbol{v}_n(T_1^{(n)}) & \text{if } t \in (T_1^{(n)}, T], \end{cases}$$

where $\boldsymbol{v}_n$ is the solution obtained in Step 1 associated with $\boldsymbol{u} \in \mathcal{V}_R(\boldsymbol{v}_{o,n}; -\delta_0, T)$. Then, we conclude from (32) that the inclusion $S(\mathcal{V}_R(\boldsymbol{v}_{o,n}; -\delta_0, T)) \subset \mathcal{V}_R(\boldsymbol{v}_{o,n}; -\delta_0, T)$ holds. Moreover, let $\{\boldsymbol{w}_k\}_{k\in\mathbb{N}} \subset \mathcal{V}_R(\boldsymbol{v}_{o,n}; -\delta_0, T)$ and $\boldsymbol{w} \in \mathcal{V}_R(\boldsymbol{v}_{o,n}; -\delta_0, T)$ such that

$$\boldsymbol{w}_k \to \boldsymbol{w} \quad \text{in } C([-\delta_0, T]; \boldsymbol{H}) \quad \text{as } k \to +\infty.$$

Then $\varphi^t_n(\boldsymbol{w}_k; \cdot) \to \varphi^t_n(\boldsymbol{w}; \cdot)$ on $\boldsymbol{H}$ in the sense of Mosco as $k \to +\infty$ (see [14, 18, 28]). Thus the mapping S is continuous in $\mathcal{V}_R(\boldsymbol{v}_{o,n}; -\delta_0, T)$ with respect to the topology of $C([-\delta_0, T]; \boldsymbol{H})$. Schauder's fixed point theorem can be now applied to the solution operator S in $\mathcal{V}_R(\boldsymbol{v}_{o,n}; -\delta_0, T)$ to find a fixed point $S\boldsymbol{v}_n = \boldsymbol{v}_n$, that is, to find $\boldsymbol{v}_n$ satisfying (16) on $(0, T_1^{(n)})$ and (17).

Step 3. Next, we consider the prolongation of this solution onto the entire time interval $[0, T]$. Let T^* be the supremum of all finite $T_1^{(n)} > 0$ such that the problem has a solution $\boldsymbol{v}_n$ on $[0, T_1^{(n)}]$. Assume that $T^* < T$. For a certain positive constant M_3, depending only on $|\boldsymbol{v}_0|_{\boldsymbol{H}}$, c_ψ, T and $|\boldsymbol{g}|_{L^2(0,T;\boldsymbol{H})}$, independent of $n \in \mathbb{N}$,we have

$$\int_0^{T^*} \varphi_n^\tau(\boldsymbol{v}_n; \boldsymbol{v}_n(\tau))d\tau \leq M_3 \quad \text{for all } n \in \mathbb{N},$$

namely $\boldsymbol{v}_n \in L^2(0, T^*; \boldsymbol{V})$. Moreover $|\boldsymbol{v}_n(t,x)| \leq \psi(t,x)$ for a.a. $x \in \Omega$ for all $t \in [0, T^*)$ and $n \in \mathbb{N}$. So there exists $B_n \in L^2(0, T^*; \boldsymbol{H})$ such that

$$(B_n(t), \boldsymbol{z})_{\boldsymbol{H}} = b(\boldsymbol{v}_n(t), \boldsymbol{v}_n(t), \boldsymbol{z}) \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H} \quad \text{and for a.a. } t \in [0, T^*),$$

and $\boldsymbol{v}_n$ is a unique solution of

$$\boldsymbol{v}_n'(t) + \partial\tilde{\varphi}_n^t(\boldsymbol{v}_n(t)) \ni \boldsymbol{g}(t) - B_n(t) \quad \text{in } \boldsymbol{H} \text{ for a.a. } t \in (0, T^*),$$

$$\boldsymbol{v}_n(0) = \boldsymbol{v}_{0,n} \quad \text{in } \boldsymbol{H},$$

where

$$\tilde{\varphi}_n^t(\boldsymbol{z}) := \begin{cases} \frac{1}{2}\|\boldsymbol{z}\|^2 & \text{if } \boldsymbol{z} \in K_n(t), \\ +\infty & \text{if } \boldsymbol{z} \in \boldsymbol{H} \setminus K_n(t), \end{cases}$$

On account of the general theory ([17, 18]), the above problem has a unique solution in $W^{1,2}(0, T^*; \boldsymbol{H}) \cap L^\infty(0, T^*; \boldsymbol{V})$, which implies that $\boldsymbol{v}_n \in C([0, T^*]; \boldsymbol{H})$ and $\varphi_n^{T^*}(\boldsymbol{v}_n; \boldsymbol{v}_n(T^*)) < +\infty$, namely $\boldsymbol{v}_n(T^*) \in K_n(T^*)$. Hence, by taking T^* as the initial time and $\boldsymbol{v}_n(T^*)$ as the initial condition, and by repeating the same arguments as above, the solution can be extended beyond T^*. This is a contradiction. Thus there must exist a solution of (16) and (17) on $[0, T]$ for all $n \in \mathbb{N}$. The uniform estimates (18) and (19) come from (29) and (31), and are independent of $n \in \mathbb{N}$. Uniqueness also holds because of the uniform estimate $|\boldsymbol{v}_n(t,x)| \leq c_\psi$ for a.a. $x \in \Omega$ and for all $t \in [0, T]$. □

References

[1] A. Azevedo and L. Santos, A diffusion problem with gradient constraint depending on the temperature, Adv. Math. Sci. Appl., **20**(2010), 153–168.

[2] T. Aiki and E. Minchev, A prey-predator model with hysteresis effect, SIAM J. Math. Anal., **36**(2005), 2020–2032.

[3] C. Baiocchi and A. Capelo, *Vriational and quasivariational inequalities, Applications to free-boundary problems*, John Wiley and Sons, 1984

[4] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden, 1976.

[5] V. Barbu and S. S. Sritharan, Flow invariance preserving feedback controllers for the Navier-Stokes equation, J. Math. Anal. Appl., **255**(2001), 281–307.

[6] M. Biroli, Sur l'inéquation d'évolution de Navier-Stokes. I,II,III, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (8), **52**(1972), 457–460, 591–598, 811-820.

[7] M. Biroli, Sur la solution faible des inéquations d'évolution du type de Navier-Stokes avec convexe dépendant du temps, Boll. Unione Mat. Ital. (4), **11**(1975), 309–321.

[8] H. Brézis, Problème unilatéraux, J. Math. Pures Appl. (9), **51**(1972), 1–168.

[9] H. Brézis, Inéquations variationnelles relatives á l'opérateur de Navier-Stokes, J. Math. Anal. Appl., **39**(1972), 159–165.

[10] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam-London-New York, 1973.

[11] P. Colli, N. Kenmochi and M. Kubo, A phase-field model with temperature dependent constraint, J. Math. Anal. Appl., **256**(2001), 668–685.

[12] M. G. Crandall and A. Pazy, Semi-groups of nonlinear contractions and dissipative sets, J. Funct. Anal., **3**(1969), 376–418.

[13] T. Fukao, Variational inequality for the Stokes equations with constraint, pp.437–446 in Discrete Contin. Dyn. Syst. Supplement 2011, 2011.

[14] T. Fukao and N. Kenmochi, Variational inequality for the Navier-Stokes equations with time-dependent constraint, to appear in GAKUTO Internat. Ser. Math. Sci. Appl.

[15] T. Fukao and N. Kenmochi, A thermohydraulics model with temperature dependent constraint on velocity fields, in preparation.

[16] R. Kano, N. Kenmochi and Y. Murase, Nonlinear evolution equations generated by subdifferentials with nonlocal constraints, pp. 175–194 in Nonlocal and Abstract Parabolic Equations and their Applications, **Vol.86**, Banach Center Publ., 2009.

[17] N. Kenmochi, Some nonlinear parabolic variational inequalities, Israel Journal of Mathematics, **22**(1975), 304–331.

[18] N. Kenmochi, Solvability of nonlinear evolution equations with time-dependent constraints and applications, Bull. Fac. Edu., Chiba Univ., **30**(1981), 1–87.

[19] N. Kenmochi, Parabolic quasi-variational diffusion problems with gradient constraints, to appear in Discrete Contin. Dyn. Syst. Ser. S.

[20] N. Kenmochi and U. Stefanelli, Existence for a class of nonlinear quasivariational evolution problems, pp.253–264 in *Nonlinear phenomena with energy dissipation*, **Vol.29**, GAKUTO Internat. Ser. Math. Sci. Appl., 2008.

[21] M. Kubo, Characterization of a class of evolution operators generated by time-dependent subdifferentials, Funkcial. Ekvac., **32**(1989), 301–321.

[22] M. Kubo and N. Yamazaki, Quasilinear parabolic variational inequalities with time-dependent constraints, Adv. Math. Sci. Appl., **15**(2005), 335–354.

[23] M. Kunze and J. F. Rodrigues, An elliptic quasi-variational inequality with gradient constraints and some of its applications, Math. Meth. Appl. Sci., **23**(2000), 897–908.

[24] A. I. Lefter, Navier-Stokes equations with potentials, Abstr. Appl. Anal., **2007**(2007), ID 79406, 1–30.

[25] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Études Mathématiques, Dunod Gauthier-Villas, Paris, 1968.

[26] F. Mignot and J. P. Puel, Inéquations d'evolution paraboliques avec convexes dépendant du temps. Applications aux inéquations quasi-variationnelles d'evolution, Arch. Rational Mech. Anal., **64**(1977), 59–91.

[27] E. Minchev, On a system of nonlinear PDE's for phase transitions with vector order parameter, Adv. Math. Sci. Appl., **14**(2004), 187–209.

[28] U. Mosco, Convergence of convex set and of solutions of variational inequalities, Adv. Math., **3**(1969), 510–585.

[29] J. Naumann, On evolution inequalities of Navier-Stokes type in three dimensions, Ann. Mat. Pure Appl.(4), **124**(1980), 107–125.

[30] M. Ôtani, Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators, Cauchy problems, J. Differential Equations, **46**(1982), 268–299.

[31] G. Prouse, On an inequality related to the motion, in any dimension, of viscous, incompressible fluids, Nota I,II, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. (8), **67**(1979), 191–196, 282–288.

[32] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, Amsterdam-New York, North-Holland, 1977.

[33] J. F. Rodrigues, On the evolution Boussinesq-Stefan problem for non-Newtonian fluids, pp.390–397 in *Free Boundary Problems:Theory and Applications*, **Vol.14**, GAKUTO Internat. Ser. Math. Sci. Appl., 2000.

[34] J. F. Rodrigues and L. Santos, A parabolic quasi-variational inequality arising in a superconductivity model, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4), **29**(2000), 153–169.

[35] J. F. Rodrigues and J. M. Urbano, On the stationary Boussinesq-Stefan problem with constitutive power-laws, Int. J. Non-Linear Mechanics, **33**(1998), 555–566.

[36] J. F. Rodrigues and J. M. Urbano, On a three-dimensional convective Stefan problem for a non-Newtonian fluid, pp.457–468 in *Applied Nonlinear Analysis*, Kluwer Acad. Plenum Publ., 1999.

[37] R. Rossi and U. Stefanelli, An order approach to a class of quasivariational sweeping process, Adv. Differential Equations, **10**(2005), 527–552.

[38] U. Stefanelli, Nonlocal quasivariational evolution problems, J. Differential Equations, **229**(2006), 204–228.

[39] Y. Yamada, On evolution equations generated by subdifferential operators, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **23**(1976), 491–515.