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O-MINIMAL TOPOLOGY

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Let R be a real closed field and assume an o-minimal structure over R. Between the last century, topology of real manifolds ($\mathbf{R}$-manifolds) was vastly and profoundly investigated. We consider manifolds over R with the o-minimal structure (called definable manifolds). I will explain whether known important results on $\mathbf{R}$-manifolds hold and what properties definable manifolds have but $\mathbf{R}$-manifolds do not necessarily have.

There are three kinds of $\mathbf{R}$-manifolds: C^0 $\mathbf{R}$-manifolds, PL $\mathbf{R}$-manifolds and C^r $\mathbf{R}$-manifolds $(r = 1, \ldots, \omega)$. A $*$ $\mathbf{R}$-manifold ($* = C^0$, PL or C^r) is defined by a local $*$ coordinate system, and a PL homeomorphism between polyhedra means a homeomorphism which is linear on each simplex of some simplicial decomposition of the domain of definition. It is easy to imbed a PL $\mathbf{R}$-manifold into a Euclidean space by a PL imbedding so that the image is a polyhedron. Hence we regard a PL $\mathbf{R}$-manifold as a polyhedron in some Euclidean space. Since there is not a large difference between topology of C^1 $\mathbf{R}$-manifolds and topology of C^r $\mathbf{R}$-manifolds $(r = 2, \ldots)$, we consider C^1 $\mathbf{R}$-manifolds only.

A non-compact C^0 PL or C^1 $\mathbf{R}$-manifold is called *compactifiable* if it is homeomorphic, PL homeomorphic or C^1 diffeomorphic, respectively, to the interior of a compact C^0, PL or C^1, respectively, $\mathbf{R}$-manifold with boundary. The simplest example of a non-compactifiable connected $\mathbf{R}$-manifold is $\mathbf{R}^2 - \mathbf{Z} \times \{0\}$. Note only that there are non-compactifiable contractible C^0, PL or C^1 $\mathbf{R}$-manifolds and, even if compactifiable, compactifications are not necessarily unique up to homeomorphisms, PL homeomorphisms or C^1 diffeomorphisms, respectively.

1. C^1 R-MANIFOLDS AND PL R-MANIFOLDS

In the following sense we can regard a C^1 $\mathbf{R}$-manifold as PL $\mathbf{R}$-manifold.

Cairns-Whitehead Theorem (see [5]). *Given a C^1 $\mathbf{R}$-manifold M, there exist a PL $\mathbf{R}$-manifold M^{PL}, a simplicial decomposition K of M^{PL} and a homeomorphism $\pi : M^{PL} \to M$ such that $\pi|_\sigma : \sigma \to \pi(\sigma)$ is a C^1 diffeomorphism for each $\sigma \in K$. Such M^{PL} is unique, i.e. if there is another M_1^{PL} of the same properties then M^{PL} and M_1^{PL} are PL homeomorphic.*

Hence the correspondence $M \to M^{PL}$ induces a well-defined map Π from the C^1 diffeomorphism classes of C^1 $\mathbf{R}$-manifolds to the PL homeomorphism classes of PL $\mathbf{R}$-manifolds. We call M^{PL} a C^1 triangulation of M. Then a natural questions is whether Π is bijective, which was posed by Thom. Namely, is any PL $\mathbf{R}$-manifold a C^1 triangulation of some C^1 $\mathbf{R}$-manifold? If C^1 triangulations of two C^1 $\mathbf{R}$-manifolds are PL homeomorphic, are the C^1 $\mathbf{R}$-manifolds C^1 diffeomorphic? Milnor, Munkres,

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Hirsch et al. studied the problems and obtained many results. For example, a PL $\mathbf{R}$-manifold of dimension ≤ 7 and a contractible PL $\mathbf{R}$-manifold are C^1 triangulations of some C^1 $\mathbf{R}$-manifolds; there are multi C^1 $\mathbf{R}$-manifolds whose C^1 triangulations are PL homeomorphic to a PL $\mathbf{R}$-sphere (the boundary of a simplex) of dimension ≥ 7 (we call such C^1 $\mathbf{R}$-manifolds exotic spheres); the restriction of Π to manifolds of dimension ≤ 3 is bijective (we say that a PL $\mathbf{R}$-manifold of dimension ≤ 3 admits a unique C^1 $\mathbf{R}$-manifold structure).

2. PL $\mathbf{R}$-MANIFOLDS AND C^0 $\mathbf{R}$-MANIFOLDS

A C^0 $\mathbf{R}$-manifold of dimension ≤ 3 admits a unique PL $\mathbf{R}$-manifold structure (i.e. a C^0 $\mathbf{R}$-manifold of dimension ≤ 3 is homeomorphic to some PL $\mathbf{R}$-manifold and such PL $\mathbf{R}$-manifolds are PL homeomorphic each other). However the relation of C^0 $\mathbf{R}$-manifolds and PL $\mathbf{R}$-manifolds of dimension 4 is very complicated. On the other hand, the relation of manifolds of dimension ≥ 5 is clarified by Kirby and Siebenmann.

Kirby-Siebenmann Theorem (see [3]). *Given a compact C^0 $\mathbf{R}$-manifold M of dimension ≥ 5, there is a well-defined obstruction $\tau(M)$ in $H^4(M, \mathbf{Z}_2)$ such that M admits a PL $\mathbf{R}$-manifold structure if and only if $\tau(M) = 0$. Given a compact PL $\mathbf{R}$-manifolds M of dimension ≥ 5, there is one-to-one correspondence from $H^3(M_1, \mathbf{Z}_2)$ to isotopy classes of PL $\mathbf{R}$-manifold structures on C^0 $\mathbf{R}$-manifold M.*

Consequently, there exists a compact C^0 $\mathbf{R}$-manifold M of dimension ≥ 5 which does not admit a PL $\mathbf{R}$-manifold structure; there exist compact PL $\mathbf{R}$-manifolds M_1 and M_2 of dimension ≥ 5 which are homeomorphic but not PL homeomorphic; a compact C^0 $\mathbf{R}$-manifold M of dimension ≥ 5 admits a PL $\mathbf{R}$-manifold structure if $H^4(M, \mathbf{Z}_2) = 0$; compact PL $\mathbf{R}$-manifolds M_1 and M_2 of dimension ≥ 5 are PL homeomorphic if they are homeomorphic and $H^3(M_1, \mathbf{Z}_2) = 0$.

3. O-MINIMAL STRUCTURES OVER A REAL CLOSED FIELD

An *o-minimal structure* over R is a sequence $\{S_n : n \in \mathbf{N}\}$ such that for each $n \in \mathbf{N}$,
(i) S_n is a boolean algebra of subsets of R^n,
(ii) if $X \in S_n$, then $R \times X$ and $X \times R$ are elements of S_{n+1},
(iii) every algebraic set in R^n is an element of S_n,
(iv) if $X \in S_{n+1}$, then the image of X under the projection of R^{n+1} onto the first n coordinates is an element of S_n, and
(v) an element of S_1 is a finite union of points and open intervals $(a, b) = \{x \in R : a < x < b\}$ $(a, b \in R \cup \{\pm\infty\})$.

An element of S_n is called *definable*, and a map between definable sets is called *definable* if its graph is definable. We call a definable set in R^n *compact* if it is closed and bounded in R^n. There are two fundaments on topology of definable sets.

Triangulation Theorem. *A definable set is definably homeomorphic to a finite union of open simplexes. A compact definable set is definably homeomorphic to a finite union of simplexes.*

Uniqueness Theorem. *Two definable polyhedra are definably PL homeomorphic if they are definably homeomorphic.*

We naturally define a definable C^0, PL or C^1 manifold. Then an immediate corollary is

Corollary. *A compact definable C^0 manifold is definably homeomorphic to a compact definable PL manifold, and a non-compact definable C^0 manifold is definably homeomorphic to the interior of a compact definable PL manifold with boundary, i.e., a non-compact definable C^0 or PL manifold is compactifiable. Moreover these compact definable PL manifolds possibly with boundary are unique up to PL homeomorphisms.*

Thus there is no difference between definable C^0 manifolds, definable PL manifolds and their compactifications. This is the difference between $\mathbf{R}$-manifolds and definable manifolds.

The triangulation theorem is proved in the same way as triangulations of semianalytic sets by Łojasiewicz [4] (see [7] and [9]). However the proof of the uniqueness theorem is very complicated [8]. It requires knowledge of PL topology, stratification theory and model theory.

4. COMPACT DEFINABLE PL MANIFOLDS POSSIBLY WITH BOUNDARY

It is not easy to study definable C^0 manifolds directly. However, by the above corollary it suffices to consider compact definable PL manifolds possibly with boundary, which are easy to treat as follows.

We naturally define a $\mathbf{Q}$-polyhedron and a PL $\mathbf{Q}$-manifold in $\mathbf{Q}^n$. Note that a compact $\mathbf{Q}$-polyhedron is a finite union of $\mathbf{Q}$-simplexes but an R-polyhedron closed and bounded in R^n is not necessarily a finite union of R-simplexes if R is non-Archimedean. For a $\mathbf{Q}$-simplex σ in $\mathbf{Q}^n$, let σ_R denote the simplex in R^n spanned by the vertices of σ. For a compact $\mathbf{Q}$-polyhedron X in $\mathbf{Q}^n$, we define X_R to be $\cup_{\sigma\in K}\sigma_R$ for a simplicial decomposition K of X. It is easy to see that if M is a compact PL $\mathbf{Q}$-manifold possibly with boundary then M_R is a compact definable PL manifold possibly with boundary.

Theorem. *The correspondence $M \longrightarrow M_R$ is a bijection from the PL homeomorphism classes of compact PL $\mathbf{Q}$-manifolds possibly with boundary to the definably PL homeomorphism classes of compact definable PL manifolds possibly with boundary.*

Thus we regard a compact definable PL manifolds possibly with boundary as a compact PL ($\mathbf{Q}$ or) $\mathbf{R}$-manifold possibly with boundary. The proof of the theorem is short but requires some elementary knowledge of PL topology and model theory [8].

5. DEFINABLE C^1 MANIFOLDS

We will reduce problems on definable C^1 manifolds to the $\mathbf{R}$-case. There are many o-minimal structures over fixed R. A PL manifold is definable if and only if it is a finite union of open simplexes. Hence definability of a PL manifold does not depend on the choice of an o-minimal structure. However this is not the case for a C^1 manifold. Indeed there is a C^1 manifold which is definable in an o-minimal structure but not so in another. Hence we need to introduce an o-minimal structure such that a C^1 manifold definable in this special structure is definable in any o-minimal structure. That is the *semialgebraic* structure. A *semialgebraic* set is a subset of R^n of the form $\cup\cap\{x \in R^n : f_i(x) *_i 0\}$, where f_i are a finite number of polynomial functions on R^n and $*_i$ means $=$ or $>$. An equivalent definition is that a *semialgebraic* set is a subset of R^n definable in any o-minimal structure. A *semialgebraic* map between semialgebraic sets is a map with semialgebraic graph. It follows that a semialgebraic C^1 manifold and a semialgebraic C^1 map between semialgebraic C^1 manifolds are definable in any o-minimal structure.

By the following theorem we can reduce problems on definable C^1 manifolds to the semialgebraic C^1 case.

Theorem [2]. *A definable C^1 manifold is definably C^1 diffeomorphic to a semialgebraic C^1 manifold.*

Next we will reduce to the $\mathbf{R}$-case. Let $\mathbf{R}_{\mathrm{alg}}$ denote the real algebraic numbers, which is the smallest real closed field. For a polynomial function f on $\mathbf{R}^n_{\mathrm{alg}}$, let f_R denote the polynomial function on R^n naturally extended from f. For a semialgebraic set $X = \cup\cap\{x \in \mathbf{R}^n_{\mathrm{alg}} : f_i(x) *_i 0\}$, define X_R to be $\cup\cap\{x \in R^n : f_{iR}(x) *_i 0\}$. Then we easily see that X is a semialgebraic C^1 $\mathbf{Q}$-manifold if and only if X_R is a semialgebraic C^1 manifold.

Theorem [1]. *The correspondence $M \to M_R$ is a bijection from the semialgebraic C^∞ diffeomorphism classes of semialgebraic C^∞ $\mathbf{R}_{alg}$-manifolds to the semialgebraic C^1 diffeomorphism classes of semialgebraic C^1 R-manifolds.*

Thus it suffices to consider semialgebraic C^∞ $\mathbf{R}$- (or $\mathbf{R}_{\mathrm{alg}}$-)manifolds. Compactification of a semialgebraic C^∞ $\mathbf{R}$-manifold is always possible as follows.

Theorem [6]. *A non-compact semialgebraic C^∞ $\mathbf{R}$-manifold is semialgebraically C^∞ diffeomorphic to the interior of a compact semialgebraic $\mathbf{R}$-manifold with boundary. Such a compact semialgebraic $\mathbf{R}$-manifold with boundary is unique up to semialgebraically C^∞ diffeomorphisms.*

The proof of the first theorem is the same as the proof of the second. The proof of the second is based on the Morse theory over R. The proof of the third uses the Artin-Mazur theorem and the Hironaka desingularization theorem.

Thus the facts shown in section 1 hold for definable C^1 manifolds and definable PL manifolds. However, the original proofs of the Cairns-Whitehead theorem and the facts that a PL $\mathbf{R}$-manifold of dimension ≤ 7 and a compact PL $\mathbf{R}$-manifold admit C^1 $\mathbf{R}$-manifold structures are false for non-Archimedean R. We can prove them using the above theorems. A typical example of a theorem which holds for $\mathbf{R}$ but not for non-Archimedean R is the simplicial approximation theorem. The simplicial approximation theorem states that any C^0 function on an $\mathbf{R}$-polyhedron is approximated by a PL function in the C^0 topology. However, if R is non-Archimedean, then the function $R \supset [0, 1] \ni x \to x^2 \in R$ cannot be approximated by a PL function (see [8]).

References

[1] M. Coste and M. Shiota, *Nash triviality in families of Nash manifolds*, Invent. Math., 108(1992), 349-368.

[2] J. Escribano, *Approximation theorems in o-minimal structures*, Illinois J. Math., 46(2002),111-128.

[3] R. C. Kirby and L.C. Siebenmann, Foundational essays on topological manifolds, smoothings, and triangulations, Princeton University Press., 1977.

[4] S. Łojasiewicz, *Triangulations of semianalytic sets*, Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat., 18(1964), 449-474.

[5] J. R. Munkres, Elementary differential topology, Princeton University Press, 1966.

[6] M. Shiota, Nash manifolds, Lecture Notes in Math., 1269, Springer-Verlag, 1987.

[7] M. Shiota, Geometry of subanalytic and semialgebraic sets, Progress in Math., 150, Birkhäuser, 1997.

[8] M. Shiota, *O-minimal Hauptvermutung for polyhedra*, to appear.

[9] L. van den Dries, Tame topology and o-minimal structures, Cambridge University Press, London Mathematical Society Lecture Notes, 248, 1998.

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