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Weyl modules and principal series modules

奈良工業高等専門学校　吉井　豊 (Yutaka Yoshii)
Nara National College of Technology

1 Introduction

Let G be a connected, simply connected and semisimple algebraic group over an algebraically closed field k of characteristic $p>0$, which is defined and split over $\mathbb{F}_p$, and set $q=p^n$. We fix a maximal split torus T and a Borel subgroup B. We shall use the following standard notation:

$X=X(T)$: character group of T;
$\Phi=\Phi(G,T)$: root system relative to the pair (G,T);
$\alpha^{\vee}$: coroot corresponding to $\alpha\in\Phi$;
$W=N_G(T)/T$: Weyl group;
$\langle\cdot,\cdot\rangle$: W-invariant inner product on $\mathbb{E}=X\otimes\mathbb{R}$;
w_0 : the longest element of W;
B^+ : Borel subgroup opposite to B;
Φ^+ : set of positive roots corresponding to B^+;
Δ : set of simple roots;
h : Coxeter number;
$\rho=(\sum_{\alpha\in\Phi^+}\alpha)/2$;
$X^+=\{\lambda\in X|\langle\lambda,\alpha^{\vee}\rangle\geq 0,\ \forall\alpha\in\Delta\}$: set of dominant weights;
$X_n=\{\lambda\in X^+|\langle\lambda,\alpha^{\vee}\rangle<q,\ \forall\alpha\in\Delta\}$: set of q-restricted weights;
ω_{α} : fundamental weight for $\alpha\in\Delta$;
$F:G\rightarrow G$: Frobenius map relative to $\mathbb{F}_p$.

The simple (rational) G-modules are parametrized by the elements of X^+, and they are denoted by $L(\lambda)$ for $\lambda\in X^+$. For $\lambda\in X^+$, let k_{λ} be the one-dimensional T-(B- or B^+-)module with weight λ and we set $V(\lambda)=(\mathrm{Ind}_B^G k_{-w_0\lambda})^*$ and call it the Weyl module with highest weight λ. The Weyl module $V(\lambda)$ is generated by an element of weight λ, which is unique up to scalar multiple and called the highest weight vector.

Let $G(n) = G^{F^n}$ be the finite Chevalley group corresponding to G, and set $B^+(n) = B^{+F^n}$. For $\lambda \in X_n$, the simple G-module is also simple as a $kG(n)$-module and any simple $kG(n)$-module can be obtained in this way. For $\lambda \in X_n$, we set $M_n(\lambda) = \mathrm{Ind}_{B^+(n)}^{G(n)} k_\lambda$ and call it a principal series module.

Pillen has given a kind of relation between Weyl modules and principal series modules:

Theorem 1.1 ([3, Theorem 1.2]) *Suppose that $q > 2h - 1$. Let $\lambda \in X_n$ and let v be the highest weight vector of $V(\lambda + (q-1)\rho)$. Then the $kG(n)$-submodule generated by v is isomorphic to $M_n(\lambda)$ if and only if $\langle \lambda, \alpha^\vee \rangle > 0$ for any $\alpha \in \Delta$.*

In this article, we report that this theorem can be extended to the case $\langle \lambda, \alpha^\vee \rangle = 0$ for some $\alpha \in \Delta$.

2 Main result

Without loss of generality, we assume that G is simple for the rest of this article.

We shall introduce some further notation to describe the main result. For a subset $I \subseteq \Delta$, let I^c be the complement of I in Δ and set $\rho_I = \sum_{\alpha \in I} \omega_\alpha$. For $\lambda \in X_n$, set

$$I_0(\lambda) = \{\alpha \in \Delta | \langle \lambda, \alpha^\vee \rangle = 0\}$$

and

$$I_{q-1}(\lambda) = \{\alpha \in \Delta | \langle \lambda, \alpha^\vee \rangle = q - 1\}.$$

It is known that $M_n(\lambda)$ can be decomposed as

$$M_n(\lambda) \cong \bigoplus_{J \subseteq I_0(\lambda)} \bigoplus_{J' \subseteq I_{q-1}(\lambda)} Y(\lambda + (q-1)\rho_J - (q-1)\rho_{J'}),$$

where each $Y(\mu)$ has a simple $G(n)$-head which is isomorphic to $L(\mu)$ (see [2, 4.6 (1)] and [4, §3]).

Now we can state the main result:

Theorem 2.1 ([5, Theorem 2.1]) *Suppose that $q > h + 1$. Let $\lambda \in X_n$ and let v be the highest weight vector of $V(\lambda + (q-1)\rho)$. Then the $kG(n)$-submodule generated by v is isomorphic to*

$$\bigoplus_{J \subseteq I_{q-1}(\lambda)} Y(\lambda + (q-1)\rho_{I_0(\lambda)} - (q-1)\rho_J).$$

Remarks (1) Actually Pillen's original proof in [3] contains an error and the assumption $q > 2h - 1$ is not appropriate. However, after modifying it, this generalized theorem holds under the weaker assumption $q > h + 1$.
(2) If $I_0(\lambda)$ is empty, then the resulting direct sum is isomorphic to $M_n(\lambda)$ and so this theorem is certainly a generalization of Theorem 1.1.

Example Consider the case $G = \mathrm{SL}_5(k)$ and $q = 7$. Set $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where α_i are in the standard numbering of type A_4 as in [1, 11.4]. Let ω_i be the fundamental weight corresponding to α_i. Then any dominant weight is of the form $\sum_{i=1}^{4} c_i\omega_i$ with $c_i \in \mathbb{Z}_{\geq 0}$, which is usually abbreviated (c_1, c_2, c_3, c_4). Now we take $\lambda = (0,0,2,6)$. Then $I_0(\lambda) = \{\alpha_1, \alpha_2\}$ and $I_{q-1}(\lambda) = \{\alpha_4\}$. The principal series module $M_n(0,0,2,6)$ is decomposed as

$$Y(0,0,2,0) \oplus Y(0,0,2,6) \oplus Y(0,6,2,0) \oplus Y(0,6,2,6)$$

$$\oplus Y(6,0,2,0) \oplus Y(6,0,2,6) \oplus Y(6,6,2,0) \oplus Y(6,6,2,6)$$

(the entries of λ whose values are 0 or $q-1$ 'split' into 0 and $q-1$), and the highest weight vector v of the Weyl module $V(\lambda + (q-1)\rho) = V(6,6,8,12)$ generates a $kG(n)$-submodule which is isomorphic to

$$Y(0,0,2,6) \oplus Y(0,6,2,6) \oplus Y(6,0,2,6) \oplus Y(6,6,2,6)$$

(the entries of λ whose values are 0 'split' into 0 and $q-1$).

3 Strategy of the proof

The method of proof of the main theorem is essentially similar to Pillen's original proof. But we need to use the following two generalized lemmas instead of Lemmas 1.5 and 1.6 in [3]:

Lemma 3.1 **[5, Lemma 2.4]** *Let $I \subseteq \Delta$. Suppose that $q > \langle \rho_I, \alpha_0^\vee \rangle + 2$ and that $\mu \in X_n$ satisfies $\mu \geq (q-1)\rho + w_0\rho_I$. Then the multiplicity of $L(\mu)$ in the composition factors of the $kG(n)$-module $M_n(\rho_I)$ is one if $\mu = (q-1)\rho + w_0\rho_I$ and zero otherwise.*

Lemma 3.2 **[5, Lemma 2.5]** *Let $I \subseteq \Delta$, and suppose that $q > \langle \rho_I, \alpha_0^\vee \rangle + 2$. Then the $kG(n)$-submodule generated by the highest weight vector of $V(\rho_I + (q-1)\rho)$ is isomorphic to $Y(\rho_I + (q-1)\rho_{I^c})$.*

Lemma 3.1 is used to prove Lemma 3.2. Now we outline the proof of Theorem 1.1.

Let m_0 and m_1 be the generators of the $kG(n)$-modules $M_n(\lambda)$ and $M_n(\rho_{I_0(\lambda)^c})$ respectively, and let v_1 and v_2 be the highest weight vectors of the Weyl modules $V((q-1)\rho+\rho_{I_0(\lambda)^c})$ and $V(\lambda-\rho_{I_0(\lambda)^c})$ respectively. To begin with, consider the composite map of two $kG(n)$-module homomorphisms:

$$f\otimes id: M_n(\rho_{I_0(\lambda)^c})\otimes V(\lambda-\rho_{I_0(\lambda)^c})\to V((q-1)\rho+\rho_{I_0(\lambda)^c})\otimes V(\lambda-\rho_{I_0(\lambda)^c}),$$

$$\varphi: M_n(\lambda)\to M_n(\rho_{I_0(\lambda)^c})\otimes V(\lambda-\rho_{I_0(\lambda)^c}),$$

where $f\otimes id$ is defined by $m_1\otimes v_2\mapsto v_1\otimes v_2$ and φ is defined by $m_0\mapsto m_1\otimes v_2$ (and is injective). It is enough to show that the image of the composite map $(f\otimes id)\circ\varphi$ is isomorphic to the desired $kG(n)$-module since $v_1\otimes v_2$ generates $V(\lambda+(q-1)\rho)$ as a G-module.

For a subset $I\subseteq\Delta$, let G_I be the Levi subgroup relative to I and let $G_I(n)$ be the corresponding finite group. An analogous notation will be used for G_I, for example, $L_I(\lambda)$, $V_I(\lambda)$, $M_{n,I}(\lambda)$ and $Y_I(\mu)$. Now consider the $kG_{I_0(\lambda)}(n)$-module embedding

$$\varphi_{I_0(\lambda)}: M_{n,I_0(\lambda)}(\lambda)\to M_{n,I_0(\lambda)}(\rho_{I_0(\lambda)^c})\otimes V_{I_0(\lambda)}(\lambda-\rho_{I_0(\lambda)^c})$$

which is analogous to φ. Since the $V_{I_0(\lambda)}(\lambda-\rho_{I_0(\lambda)^c})$ is one-dimensional $(=k_{\lambda-\rho_{I_0(\lambda)^c}})$, $\varphi_{I_0(\lambda)}$ is bijective and maps the summand $Y_{I_0(\lambda)}(\lambda+(q-1)\rho_J)$ onto $Y_{I_0(\lambda)}(\rho_{I_0(\lambda)^c}+(q-1)\rho_J)\otimes V_{I_0(\lambda)}(\lambda-\rho_{I_0(\lambda)^c})$ for any $J\subseteq\Delta$. We shall denote this restriction map by $\varphi_{I_0(\lambda),J}$. Taking Harish-Chandra induction $\mathrm{HCInd}_{G_{I_0(\lambda)}(n)}^{G(n)}$ we have

$$\begin{aligned}\varphi &= \mathrm{HCInd}(\varphi_{I_0(\lambda)})=\mathrm{HCInd}\Big(\bigoplus_{J\subseteq I_0(\lambda)}\varphi_{I_0(\lambda),J}\Big)\\ &= \bigoplus_{J\subseteq I_0(\lambda)}\mathrm{HCInd}(\varphi_{I_0(\lambda),J}).\end{aligned}$$

Moreover, one can prove that

$$\mathrm{HCInd}_{G_{I_0(\lambda)}(n)}^{G(n)}Y_{I_0(\lambda)}(\lambda+(q-1)\rho_J)=\bigoplus_{J'\subseteq I_{q-1}(\lambda)}Y(\lambda+(q-1)\rho_J-(q-1)\rho_{J'}),$$

and

$$\mathrm{HCInd}_{G_{I_0(\lambda)}(n)}^{G(n)}Y_{I_0(\lambda)}(\rho_{I_0(\lambda)^c}+(q-1)\rho_J)=Y(\rho_{I_0(\lambda)^c}+(q-1)\rho_J)$$

by using Frobenius reciprocity. These three formulas imply that φ maps the right-hand side of the second formula to $Y(\rho_{I_0(\lambda)^c}+(q-1)\rho_J)\otimes V(\lambda-\rho_{I_0(\lambda)^c})$ injectively. Moreover, Lemma 3.2 implies that the restriction of $f\otimes id$ on the tensor product is injective for $J=I_0(\lambda)$, and zero otherwise. Therefore, the theorem follows.

References

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