数理解析研究所講究録
第 1795 巻 2012 年 135-143

A BASIS FOR THE MODULE OF DIFFERENTIAL OPERATORS OF ORDER 2 ON THE BRAID HYPERPLANE ARRANGEMENT

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ABSTRACT. The braid arrangement is one of most important arrangement. The study of the braid arrangement was developed from several ways. In this article, we prove that the module of differential operators on the braid arrangement is free by constructing a basis. In addition, we discuss the action of the symmetric group on the elements of the basis.

1. INTRODUCTION

The theory of hyperplane arrangements has been developed by many researchers. The hyperplane arrangement defined by the direct product is so called the braid arrangement. The braid arrangement is the Coxeter arrangement of type A_{n-1}. It was proved by Saito [6] that Coxeter arrangements are free. An excellent reference on arrangements is the book by Orlik and Terao [5].

Let K be a field, and $S = K[x_1, \dots, x_n]$ be the polynomial ring of n variables. Let $D^{(m)}(S) := \bigoplus_{|\alpha|=m} S\partial^{\alpha}$ be the module of differential operators (of order m) of S, where $\alpha \in \mathbb{N}^{\ell}$ is a multi-index. For a central arrangement $\mathscr{A}$, we fix the defining polynomial $Q(\mathscr{A}) = \prod_{H\in\mathscr{A}} p_H$ where $\ker(p_H) = H$. We define the module $D^{(m)}(\mathscr{A})$ of $\mathscr{A}$-differential operators of order m as follows:

$$D^{(m)}(\mathscr{A}) := \left\{\theta \in D^{(m)}(S) \mid \theta(Q(\mathscr{A})S) \subseteq Q(\mathscr{A})S\right\}.$$

In the case $m = 1$, $D^{(1)}(\mathscr{A})$ is the module of $\mathscr{A}$-derivations.

Holm began to study $D^{(m)}(\mathscr{A})$ in his PhD thesis. Holm proved the idealizer of the ideal generated by the defining polynomial of a central arrangement is the direct sum of the module of $\mathscr{A}$-differential operators. We can describe the ring of differential operators of the coordinate ring of a central arrangement.

Holm proved that $D^{(m)}(\mathscr{A})$ are free for all $m \geq 1$ when $\mathscr{A}$ is a 2-dimensional central arrangement. Let $\mathscr{A}$ be a generic arrangement. It was already known that $D^{(m)}(\mathscr{A})$ is not free if $n > 3, |\mathscr{A}| > n, m < |\mathscr{A}| - n + 1$, and is free if $n > 3, |\mathscr{A}| > n, m = |\mathscr{A}| - n + 1$ [2].In addition, the author, Okuyama and Saito [4] proved that $D^{(m)}(\mathscr{A})$ is free if $n > 3, |\mathscr{A}| > n, m > |\mathscr{A}| - n + 1$.

Let $\mathcal{A}$ be the braid arrangement. In this article, we prove that $D^{(2)}(\mathcal{A})$ is free by constructing a basis, and calculate an action of the symmetric group on $D^{(2)}(\mathcal{A})$.

2. Determinants of matrices

Let $n \geq m > 0$. Let e_ℓ denote the ℓ-th elementary symmetric polynomial in m variables. For $n > m$ and $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, we define row vectors $\boldsymbol{v}_{i_1,\dots,i_m}$ by

(2.1)

$$\boldsymbol{v}_{i_1,\dots,i_m} = \left(\dots, e_{\ell_1}(x_{i_1}, \dots, x_{i_m}) \cdots e_{\ell_{n-m}}(x_{i_1}, \dots, x_{i_m}), \dots\right)_{0 \leq \ell_1 \leq \cdots \leq \ell_{n-m} \leq m},$$

and an $\binom{n}{m}$-th square matrix $E_m(x_1, \dots, x_n)$ as the matrix whose rows are $\boldsymbol{v}_{i_1,\dots,i_m}$. Namely

$$E_m(x_1, \dots, x_n) = \begin{pmatrix} \vdots \\ \boldsymbol{v}_{i_1,\dots,i_m} \\ \vdots \end{pmatrix} \qquad (\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}).$$

We agree $E_n(x_1, \dots, x_n) = (1)$.

Let

$$\Delta = \Delta(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

be the difference product.

Theorem 2.1.

$$\det E_m(x_1, \dots, x_n) = c\Delta^{\binom{n-2}{m-1}},$$

where $c = \pm 1$.

To prove the theorem above, we first consider the degree of this determinant. Let $a_{n,m}$ be the total sum of degrees of polynomials in the set

$$\left\{e_{\ell_1} \cdots e_{\ell_{n-m}} \mid 0 \leq \ell_1 \leq \cdots \leq \ell_{n-m} \leq m\right\}. \tag{2.2}$$

Namely

$$a_{n,m} = \sum_{0 \leq \ell_1 \leq \cdots \leq \ell_{n-m} \leq m} \deg e_{\ell_1} \cdots e_{\ell_{n-m}}.$$

Proposition 2.2.

$$a_{n,m} = \binom{n}{2}\binom{n-2}{m-1}.$$

Proof. It is clear that $\deg(e_{\ell_1}\cdots e_{\ell_k}) = \ell_1+\cdots+\ell_k$. Since the set (2.2) is equal to the set

$$\bigcup_{1\le k\le n-m} \{e_{\ell_1}\cdots e_{\ell_k} \mid 1\le \ell_1\le\cdots\le\ell_k\le m\}, \tag{2.3}$$

it follows that

$$\begin{aligned} a_{n,m} &= \sum_{0\le\ell_1\le\cdots\le\ell_{n-m}\le m} (\ell_1+\cdots+\ell_{n-m}) \\ &= \sum_{1\le k\le n-m}\ \sum_{0\le\ell_1\le\cdots\le\ell_k\le m} (\ell_1+\cdots+\ell_{n-m}). \end{aligned}$$

Therefore

$$\begin{aligned} &a_{n,m} \\ =& \sum_{1\le k\le n-1-m}\ \sum_{0\le\ell_1\le\cdots\le\ell_k\le m} (\ell_1+\cdots+\ell_{n-m}) + \sum_{1\le\ell_1\le\cdots\le\ell_{n-m}\le m} (\ell_1+\cdots+\ell_{n-m}) \\ =& a_{n-1,m} + \sum_{1\le\ell_1\le\cdots\le\ell_{n-m}\le m} ((\ell_1-1)+\cdots+(\ell_{n-m}-1)) + (n-m)\binom{n-1}{m-1} \\ =& a_{n-1,m} + \sum_{0\le\ell_1\le\cdots\le\ell_{n-m}\le m-1} (\ell_1+\cdots+\ell_{n-m}) + (n-m)\binom{n-1}{m-1} \\ =& a_{n-1,m} + a_{n-1,m-1} + (n-m)\binom{n-1}{m-1}. \end{aligned}$$

The assertion is completed by induction. □

To prove Theorem 2.1, we use the following two lemmas. We agree $\det E_0 = 1$.

Lemma 2.3. *Assume that $n > m$. The coefficient of $x_1^{(n-1)\binom{n-2}{m-1}}$ in $\det E_m(x_1,\dots,x_n)$ is equal to*

$$\det E_{m-1}(x_2,\dots,x_n)\cdot\det E_m(x_2,\dots,x_n).$$

Proof. After fundamental operations, we may assume that the upper $\binom{n-1}{m-1}$ rows contain x_1 and the right $\binom{n-1}{m-1}$ colums are of indexes $1\le \ell_1\le\cdots\le\ell_{n-m}\le m$. Let A be the upper right $\binom{n-1}{m-1}\times\binom{n-1}{m-1}$ submatrix of $E_m(x_1,\dots,x_n)$, i.e.,

$$E_m(x_1,\dots,x_n) = \begin{bmatrix} * & A \\ E_m(x_2,\dots,x_n) & * \end{bmatrix}.$$

Since the x_1-degree $\deg_{x_1} e_{\ell_1}\cdots e_{\ell_{n-m}}(x_1,x_{i_2},\dots,x_{i_m})$ is equal to $n-m$ for $1\le\ell_1\le\cdots\le\ell_{n-m}\le m$ and the x_1-degree of every component which does not appear in A is less than $n-m$, the term containing $x_1^{(n-m)\binom{n-1}{m-1}} = x_1^{(n-1)\binom{n-2}{m-1}}$ appears only in $\det A \cdot \det E_m(x_2, \ldots, x_n)$.

It remains to prove that the coefficient of $x_1^{(n-1)\binom{n-2}{m-1}}$ in $\det A$ is equal to $\det E_{m-1}(x_2, \ldots, x_n)$. Since

$$e_\ell\left(x_1, x_{i_2}, \ldots, x_{i_m}\right) = e_{\ell-1}\left(x_{i_2}, \ldots, x_{i_m}\right) x_1 + e_\ell\left(x_{i_2}, \ldots, x_{i_m}\right),$$

we have that

$$e_{\ell_1} \cdots e_{\ell_{n-m}}\left(x_1, x_{i_2}, \ldots, x_{i_m}\right) = e_{\ell_1-1} \cdots e_{\ell_{n-m}-1}\left(x_{i_2}, \ldots, x_{i_m}\right) x_1^{n-m} + b$$

where $\deg_{x_1} b < n-m$. Therefore we conclude that

$$\det A = \det E_{m-1}(x_2, \ldots, x_n) x_1^{(n-m)\binom{n-1}{m-1}} + B$$

where $\deg_{x_1} B < (n-m)\binom{n-1}{m-1} = (n-1)\binom{n-2}{m-1}$. □

In general, the following holds.

Lemma 2.4. *Let R be a ring of characteristic zero. Let $f \in R[y]$ be a nonzero polynomial and $\alpha \in R$. Suppose $\deg f \geq m$. Then $(y-\alpha)^m$ divides f if and only if $f^{(\ell)}(\alpha) = 0$ for $\ell = 0, 1, \ldots, m-1$.*

Proof of Theorem 2.1. We will prove the theorem by induction on n. It is clear that the case $n = m$. Assume that $n > m$. By renumbering $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\binom{n}{m}}$ defined by (2.1), we may assume that x_1-degrees of all components of $\boldsymbol{v}_\ell$ are 0 for $\binom{n-1}{m-1} + 1 \leq \ell \leq \binom{n}{m}$. Thus we see that for $1 \leq k \leq \binom{n-2}{m-1} - 1$

$$\det E_m(x_1, \ldots, x_n)^{(k)} = \sum_{k_1 + \cdots + k_{\binom{n}{m}} = k} \det \begin{pmatrix} \vdots \\ \boldsymbol{v}_i^{(k_i)} \\ \vdots \end{pmatrix} \tag{2.4}$$

where $f^{(k)} = \frac{\partial^k}{\partial x_1^k}(f)$ for any $f \in S$ and $\boldsymbol{v}^{(k)} = \left(v_1^{(k)}, \ldots, v_\ell^{(k)}\right)$. Since $\frac{\partial}{\partial x_1}\boldsymbol{v}_j = 0$ for $\binom{n-1}{m-1} + 1 \leq j \leq \binom{n}{m}$, we may consider $k_{\binom{n-1}{m-1}+1} = 0, \ldots, k_{\binom{n}{m}} = 0$ on (2.4). For any $\{i_2, \ldots, i_m\} \subseteq \{2, \ldots, t-1, t+1, \ldots, n\}$, there exist $\binom{n-1}{m-1} + 1 \leq \ell \leq \binom{n}{m}$ such that

$$\boldsymbol{v}_{\{1, i_2, \ldots, i_m\}}|_{x_1 = x_t} = \boldsymbol{v}_\ell.$$

A cardinality of $\{\{i_2, \ldots, i_m\} \subseteq \{2, \ldots, t-1, t+1, \ldots, n\}\}$ equals $\binom{n-2}{m-1}$. We have that every term of RHS on (2.4) has $k_i = 0$ with $1 \leq i \leq \binom{n-1}{m-1}$. Thus we conclude that

$$\det E_m(x_1, \ldots, x_n)^{(k)}|_{x_1 = x_t} = 0$$

for $1 \leq k \leq \binom{n-2}{m-1} - 1$. By Lemma 2.4,

$$\det E_m(x_1, \ldots, x_n) \in \prod_{i \neq 1} (x_1 - x_i) S. \tag{2.5}$$

By the induction hypothesis and Lemma 2.3, we see that the coefficient of $x_1^{(n-1)\binom{n-2}{m-1}}$ in $\det E_m(x_1, \ldots, x_n)$ is

$$c\Delta\left(x_2, \ldots, x_n\right)^{\binom{n-3}{m-2}} \cdot \Delta\left(x_2, \ldots, x_n\right)^{\binom{n-3}{m-1}} = c\Delta\left(x_2, \ldots, x_n\right)^{\binom{n-2}{m-1}} \neq 0$$

where $c = \pm 1$. So it follows form (2.5) that

$$\det E_m(x_1, \ldots, x_n) \in \Delta(x_1, \ldots, x_n)^{\binom{n-2}{m-1}} S \setminus \{0\}.$$

By Proposition 2.2, the degree of $\det E_m(x_1, \ldots, x_n)$ equals $\binom{n}{2}\binom{n-2}{m-1}$. Comparing degrees, we see that $\det E_m(x_1, \ldots, x_n) = c\Delta^{\binom{n-2}{m-1}}$. Moreover $c = \pm 1$. □

3. Elements of the module of $\mathcal{A}$-differential operators on the braid arrangement

Throughout the remaining of this paper, let $\mathcal{A}$ be the n-th braid hyperplane arrangement, and let $D^{(m)}(\mathcal{A})$ be the module of $\mathcal{A}$-differential operators which preserve the ideal generated by $Q(\mathcal{A})$. We assume that the characteristic of K is zero. By [3, Proposition 2.3] and [3, Theorem 2.4], we have

$$D^{(m)}(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} D^{(m)}(p_H S), \tag{3.1}$$

where $D^{(m)}(p_H S) = \left\{\theta \in D^{(m)}(S) \mid \theta(p_H x^{\boldsymbol{\alpha}}) \in p_H S \text{ for any } |\boldsymbol{\alpha}| = m-1\right\}$ for $H \in \mathcal{A}$.

Put

$$h_t = (x_t - x_1) \cdots (x_t - x_{t-1})(x_t - x_{t+1}) \cdots (x_t - x_n).$$

Proposition 3.1. *Let k be a positive integer. The operators*

$$\eta_{t,k} = h_t \sum_{|\boldsymbol{\alpha}| = m; \alpha_t \geq k} \frac{1}{\boldsymbol{\alpha}!} \partial^{\boldsymbol{\alpha}} \quad (t = 1, \ldots, n) \tag{3.2}$$

belong to $D^{(m)}(\mathcal{A})$.

Proof. For any $1 \leq i < j \leq n$ such that $i \neq t, j \neq t$ and $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| = m - 1$,

$$\sum_{|\boldsymbol{\alpha}| = m; \alpha_t \geq k} \frac{1}{\boldsymbol{\alpha}!} \partial^{\boldsymbol{\alpha}} \left((x_i - x_j) x^{\boldsymbol{\beta}}\right) = \begin{cases} 0 & \text{if } \beta_t < k \\ 1 - 1 = 0 & \text{if } \beta_t \geq k. \end{cases}$$

This implies that $h_t \sum_{|\alpha|=m;\alpha_t \geq k} \frac{1}{\alpha!}\partial^{\alpha} \in D^{(m)}((x_i - x_j)S)$ for any $1 \leq i < j \leq n$. It follows from (3.1) that operators (3.2) belong to $D^{(m)}(\mathcal{A})$ as required. □

We consider other polynomials relating with elementary symmetric polynomials. Let $e^k_\ell = e_\ell(y_1, \ldots, y_k)$ be the ℓ-th elementary symmetric polynomial in k variables. For a multi -index $\boldsymbol{\alpha}$, we define

$$e^k_\ell(\boldsymbol{\alpha}) = e^k_\ell(x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_n, \ldots, x_n) \in S$$

where the number of x_t is α_t.

Proposition 3.2. *Let k be a positive integer. For any sequence of nonnegative integers $\ell_1, \ldots, \ell_k$, the operator*

$$\theta_{\ell_1,\ldots,\ell_k} = \sum_{|\boldsymbol{\alpha}|=m} e^m_{\ell_1}(\boldsymbol{\alpha}) \cdots e^m_{\ell_k}(\boldsymbol{\alpha}) \frac{1}{\boldsymbol{\alpha}!}\partial^{\boldsymbol{\alpha}}. \tag{3.3}$$

belongs to $D^{(m)}(\mathcal{A})$.

Proof. Since $e_\ell(y_1, \ldots, y_m) = e_\ell(y_1, \ldots, y_{m-1}) + e_{\ell-1}(y_1, \ldots, y_{m-1})y_m$ for $1 \leq i < j \leq n$ and $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| = m - 1$,

$$\begin{aligned}
&\theta_{\ell_1,\ldots,\ell_k}\left((x_i - x_j)x^{\boldsymbol{\beta}}\right) \\
=&e^m_{\ell_1}(\boldsymbol{\beta} + \boldsymbol{e}_i) \cdots e^m_{\ell_k}(\boldsymbol{\beta} + \boldsymbol{e}_i) - e^m_{\ell_1}(\boldsymbol{\beta} + \boldsymbol{e}_j) \cdots e^m_{\ell_k}(\boldsymbol{\beta} + \boldsymbol{e}_j) \\
=& \left(e^m_{\ell_1}(\boldsymbol{\beta}) + e^m_{\ell_1}(\boldsymbol{\beta})x_i\right) \cdots \left(e^m_{\ell_k}(\boldsymbol{\beta}) + e^m_{\ell_k}(\boldsymbol{\beta})x_i\right) \\
&\qquad - \left(e^m_{\ell_1}(\boldsymbol{\beta}) + e^m_{\ell_1}(\boldsymbol{\beta})x_j\right) \cdots \left(e^m_{\ell_k}(\boldsymbol{\beta}) + e^m_{\ell_k}(\boldsymbol{\beta})x_j\right).
\end{aligned}$$

Substitute x_j for x_i, then we get $\theta_{\ell_1,\ldots,\ell_k}\left((x_i - x_j)x^{\boldsymbol{\beta}}\right)|_{x_i = x_j} = 0$. This follows that a polynomial $x_i - x_j$ divides a polynomial $\theta_{\ell_1,\ldots,\ell_k}\left((x_i - x_j)x^{\boldsymbol{\beta}}\right)$, so $\theta_{\ell_1,\ldots,\ell_k} \in D^{(m)}((x_i - x_j)S)$. Therefore we conclude $\theta_{\ell_1,\ldots,\ell_k} \in D^{(m)}(\mathcal{A})$ by (3.1). □

4. A basis for $D^{(2)}(\mathcal{A})$ and its representation

In this section, we assume that the characteristic of K is zero and K is a algebraically closed. We find a basis for the module $D^{(2)}(\mathcal{A})$ relating with the Specht modules. For $f, g \in S$, it is convenient to write $f \doteq g$ if $f = cg$ for some $c \in K \setminus \{0\}$.

Theorem 4.1. *Let $\eta_t = \eta_{t,1}$. The set*

$$\{\eta_1, \ldots, \eta_n\} \cup \left\{\theta_{\ell_1,\ldots,\ell_{n-2}} \mid 0 \leq \ell_1 \leq \cdots \leq \ell_{n-2} \leq 2\right\} \tag{4.1}$$

forms a basis for $D^{(2)}(\mathcal{A})$.

Proof. We have already seen, by Proposition 3.1 and Proposition 3.2, $\eta_{t,1} \in D^{(2)}(\mathcal{A})$ and $\theta_{\ell_1,\ldots,\ell_k} \in D^{(2)}(\mathcal{A})$.

By Saito-Holm criterion [8, Theorem 4.10.], it is sufficient to show that

(4.2)
$$\det M_m\left(\eta_t, \theta_{\ell_1,\ldots,\ell_{n-2}} \mid t = 1,\ldots,n,\ 0 \le \ell_1 \le \cdots \le \ell_{n-2} \le 2\right) \doteq Q(\mathcal{A})^n$$

where $M_m\left(\eta_t, \theta_{\ell_1,\ldots,\ell_{n-2}} \mid t = 1,\ldots,n,\ 0 \le \ell_1 \le \cdots \le \ell_{n-2} \le 2\right)$ is the coefficient matrix of the operators in (4.1). By Theorem 2.1.

$$\det M_m \doteq Q^2 \begin{vmatrix} I_n & * \\ 0 & E_m(x_1,\ldots,x_n) \end{vmatrix} \doteq Q^2 \cdot Q^{n-2} = Q^n,$$

as required. □

Define the K-vector space

$$V = \sum_{t=1}^{n} K\eta_t + \sum_{0 \le \ell_1 \le \cdots \le \ell_{n-2} \le 2} K\theta_{\ell_1,\ldots,\ell_{n-2}}.$$

Now we retake a basis which is also a basis for the decomposition of V into Specht modules.

Let $\lambda = (\lambda_1,\ldots,\lambda_n)\,(\lambda_1 \ge \cdots \ge \lambda_n \ge 0)$ be a Young diagram of n cells. We say that λ is a partition of n and write $\lambda \vdash n$. Define $\mathrm{Tab}(\lambda)$ (resp. $\mathrm{STab}(\lambda)$) be the set of Young tableaux (resp. standard tableaux) of shape λ.

We say that $T(i,j)$ is a number in the (i,j)-box (i.e. the box of i-th row and j-th column) of a tableau $T \in \mathrm{Tab}(\lambda)$. Let

$$\Delta_T = \prod_{j=1}^{\lambda_1} \prod_{1 \le i_1 < i_2 \le \lambda'_j} \left(x_{T(i_1,j)} - x_{T(i_2,j)}\right) \in S$$

be the Specht polynomial, for $T \in \mathrm{Tab}(\lambda)$. For each partition λ of n, an $K[S_n]$-module

$$V_\lambda = \sum_{T \in \mathrm{Tab}(\lambda)} K\Delta_T$$

is called the Specht module. The following proposition is well known (cf. [1]).

Proposition 4.2. *Let* $\lambda \vdash n$.

(1) *The set* $\{\Delta_T \mid T \in \mathrm{STab}(\lambda)\}$ *forms a* K*-basis for* V_λ.
(2) *The representation* V_λ *is irreducible. Every irreducible representation is isomorphic to one of* V_λ.

The symmetric group S_n acts on the Weyl algebra by

$$\sigma \cdot x_i = x_{\sigma^{-1}(i)}, \qquad \sigma \cdot \partial_i = \partial_{\sigma^{-1}(i)}$$

for $\sigma \in S_n$. Then for any homogeneous differential operator $\theta \in D^{(m)}(S)$ and $\sigma \in S_n$,

$$(\sigma\theta)(f) = \sigma\left(\theta(\sigma^{-1}f)\right) \qquad (f \in S). \tag{4.3}$$

We show that the symmetric group also acts on $D^{(m)}(\mathcal{A})$;

Proposition 4.3.

$$S_n.D^{(m)}(\mathcal{A}) \subseteq D^{(m)}(\mathcal{A}).$$

Proof. Let $\theta \in D^{(m)}(\mathcal{A})$ and $\sigma \in S_n$. For $f \in S$, there exist $g \in S$ such that

$$\theta\left(\mathrm{sgn}(\sigma^{-1})Q(\sigma^{-1}f)\right) = Qg.$$

We see that

$$\begin{aligned}(\sigma\theta)(Qf) &= \sigma\left(\theta(\sigma^{-1}(Qf))\right) = \sigma\left(\theta(\mathrm{sgn}(\sigma^{-1})Q\sigma^{-1}(f))\right)\\ &= \sigma(Qg) = \mathrm{sgn}(\sigma)Q \cdot \sigma g \in QS\end{aligned}$$

by (4.3). Therefore $\sigma\theta \in D^{(m)}(\mathcal{A})$. □

Let $\theta \in \left\{\theta_{\ell_1,\ldots,\ell_{n-2}} \mid 0 \le \ell_1 \le \cdots \le \ell_{n-2} \le 2\right\}$. It is clear that $\sigma\theta = \theta$ for $\sigma \in S_n$. So we see that $K\theta$ is isomorphic to $V_{(n)} = K$.

It only remains to decompose $W = \sum_{t=1}^{n} K\eta_t$ into Specht modules. For $i < j$, a transposition (i,j) acts on η's as follows:

$$(i,j)\eta_i = \eta_j, (i,j)\eta_j = \eta_i, (i,j)\eta_k = \eta_k \quad (k \ne i,j).$$

We have that W is isomorphic to the $K[S_n]$-module of homogeneous polynomials of degree 1. So it is well-known a $K[S_n]$-module decomposition

$$W = K(\eta_1 + \cdots + \eta_n) \oplus \sum_{t=2}^{n} K(\eta_1 - \eta_t) \simeq V_{(n)} \oplus V_{(n-1,1)}.$$

We retake a basis for for $D^{(2)}(\mathcal{A})$.

Corollary 4.4. *The set*

$$\{\eta_1 + \cdots + \eta_n, \eta_1 - \eta_2, \ldots, \eta_1 - \eta_n\} \cup \left\{\theta_{\ell_1,\ldots,\ell_{n-2}} \mid 0 \le \ell_1 \le \cdots \le \ell_{n-2} \le 2\right\}$$

forms a basis for $D^{(2)}(\mathcal{A})$.

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