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# KHOVANOV-LAUDA-ROUQUIER ALGEBRAS AND CRYSTAL BASES FOR FINITE CLASSICAL TYPE

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ABSTRACT. We present a crystal basis theoretic construction of irreducible modules over Khovanov-Lauda-Rouquier algebras and their cyclotomic quotients of finite classical type.

## INTRODUCTION

The *Khovanov-Lauda-Rouquier algebras* (or *Hecke quiver algebras*) were introduced independently Khovanov-Lauda [11, 12] and Rouquier [16] for providing a categorification of quantum groups associated with symmetrizable Cartan data. Let $U_q(\mathfrak{g})$ be a quantum group and let $R(\alpha)$ be the corresponding Khovanov-Lauda-Rouquier algebra of weight $\alpha \in \mathsf{Q}^+$. For a dominant integral weight $\lambda \in \mathsf{P}^+$, the algebra $R(\alpha)$ has a special quotient $R^\lambda(\alpha)$, which gives a categorification of the irreducible highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ with highest weight $\lambda$ [4, 18]. The crystal structures of $U_q^-(\mathfrak{g})$ and $V(\lambda)$ also were interpreted in [14] in terms of irreducible modules over $R(\alpha)$ and $R^\lambda(\alpha)$. The Khovanov-Lauda-Rouquier algebras were generalized to the quantum generalized Kac-Moody algebras [5, 7] and, when the Cartan datum is symmetric, geometric realizations of the Khovanov-Lauda-Rouquier algebras were given in [6, 17] via quiver varieties.

In this paper, we announce the main result of our previous work [1, 8], which is an explicit construction of irreducible modules over $R(\alpha)$ and $R^\lambda(\alpha)$ of finite classical type. This construction differs from the one given in [2, 13] and is based on the theory of crystal bases. Though this paper is rely on [1], the description of irreducible modules in this paper is different and more combinatorial than the description given in [1].

Let us explain more precisely. Let $\mathfrak{B}(\infty)$ (resp. $\mathfrak{B}(\lambda)$) be the set of all isomorphism classes of irreducible graded $R(\alpha)$-modules (resp. $R^\lambda(\alpha)$-modules) for $\alpha \in \mathsf{Q}^+$. It is shown in [14] that there exists an crystal isomorphism $\mathfrak{B}(\infty) \simeq B(\infty)$ (resp. $\mathfrak{B}(\lambda) \simeq B(\lambda)$). We first define *segments* $\mathfrak{s}$ to be unordered pairs of comparable elements in the basic crystals $\mathbb{B}_X$ $(X = A, B, C, D)$ given in Section 3 and give a partial order $\preceq$ to them. Then we set *multisegments* $\underline{\mathfrak{m}}$ of $\mathbb{B}_X$ to be multisets of segments satisfying the conditions (3.1). Proposition

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3.1 says that the set $\mathcal{M}_X$ of multisegements is in a 1-1 correspondence $\Upsilon$ to the crystal $B(\infty)$. For each segment $\mathfrak{s}$, we define 1 or 2-dimensional module $\nabla(\mathfrak{s})$ out of the crystal $\mathbb{B}_X$. Then a multisegment $\underline{\mathfrak{m}} = \{\mathfrak{s}_1 \preceq \mathfrak{s}_2 \preceq \mathfrak{s}_3 \preceq \cdots\}$ of $\mathbb{B}_X$ gives the outer tensor product

$$\nabla(\underline{\mathfrak{m}}) = \nabla(\mathfrak{s}_1) \boxtimes \nabla(\mathfrak{s}_2) \boxtimes \nabla(\mathfrak{s}_3) \boxtimes \cdots .$$

It follows from [1] that

$$\mathrm{hdInd}\nabla(\underline{\mathfrak{m}}) = \tilde{f}_{\nu_X}^{\mathbf{e}_1(\mathfrak{m})} \cdots \tilde{f}_{\nu_X}^{\mathbf{e}_n(\mathfrak{m})} \mathbf{1}$$

for $\underline{\mathfrak{m}} \in \mathcal{M}_X$ and the map $\Psi : \mathcal{M}_X \longrightarrow \mathfrak{B}(\infty)$ defined by

$$\Psi(\underline{\mathfrak{m}}) = \mathrm{hdInd}\nabla(\underline{\mathfrak{m}}) \quad \text{for } \underline{\mathfrak{m}} \in \mathcal{M}_X$$

is bijective (Theorem 3.2). Hence it can be deduced from Proposition 2.3 that the composition $\Psi \circ \Upsilon^{-1} : B(\infty) \longrightarrow \mathfrak{B}(\infty)$ is a crystal isomorphism. In the cyclotomic cases, using the crystal embedding $B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C$, we obtain the same results; i.e., the composition $\Psi^\lambda \circ \Upsilon_\lambda^{-1} : B(\lambda) \longrightarrow \mathfrak{B}(\lambda)$ is a crystal isomorphism (Corollary 3.4).

This paper is organized as follows. Section 1 contains a brief review of the crystal basis theory for quantum generalized Kac-Moody algebras. In Section 2, we give the definition of Khovanov-Lauda-Rouquier algebras which is the most general version given in [6] associated with a Borcherds-Cartan datum (or a quiver positively with loops), and introduce some results of Khovanov-Lauda-Rouquier algebras on crystal bases. In Section 3, we restrict to the case of finite classical types and present a crystal basis theoretic construction of irreducible modules over Khovanov-Lauda-Rouquier algebras and their cyclotomic quotients in terms of segments.

## 1. Quantum generalized Kac-Moody algebras

Let $I$ be an index set. A square matrix $\mathsf{A} = (a_{ij})_{i,j \in I}$ is called a *symmetrizable Borcherds-Cartan matrix* if it satisfies (i) $a_{ii} = 2$ or $a_{ii} \in 2\mathbb{Z}_{\le 0}$ for $i \in I$, (ii) $a_{ij} \in \mathbb{Z}_{\le 0}$ for $i \neq j$, (iii) $a_{ij} = 0$ if $a_{ji} = 0$ for $i, j \in I$, (vi) there is a diagonal matrix $\mathsf{D} = \mathrm{diag}(\mathsf{d}_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that $\mathsf{DA}$ is symmetric. Let $I^{\mathrm{re}} = \{i \in I \mid a_{ii} = 2\}$ and $I^{\mathrm{im}} = I \setminus I^{\mathrm{re}}$.

A *Borcherds-Cartan datum* $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$ consists of

(1) a symmetrizable Borcherds-Cartan matrix $\mathsf{A}$,
(2) a free abelian group $\mathsf{P}$, called the *weight lattice*,
(3) the set $\Pi = \{\alpha_i \mid i \in I\} \subset \mathsf{P}$ of *simple roots*,
(4) the set $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathsf{P}^\vee := \mathrm{Hom}(\mathsf{P}, \mathbb{Z})$ of *simple coroots*,

which satisfy the following properties:

(i) $\langle h_i, \alpha_j \rangle := \alpha_j(h_i) = a_{ij}$ for all $i, j \in I$,
(ii) $\Pi \subset \mathfrak{h}^*$ is linearly independent, where $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} \mathsf{P}^\vee$,

(iii) for each $i \in I$, there exists $\Lambda_i \in \mathsf{P}$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for all $j \in I$.

We denote by $\mathsf{P}^+ = \{\lambda \in \mathsf{P} \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0},\ i \in I\}$ the set of *dominant integral weights*. The free abelian group $\mathsf{Q} = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ is the *root lattice*, and $\mathsf{Q}^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ is the *positive root lattice*. For $\alpha = \sum_{i\in I} k_i\alpha_i \in \mathsf{Q}^+$, the *height* of $\alpha$ is $|\alpha| := \sum_{i\in I} k_i$. There is a symmetric bilinear form $( \mid )$ on $\mathfrak{h}^*$ such that

$$(\alpha_i|\alpha_j) = \mathsf{d}_i a_{ij} \text{ for } i,j \in I, \qquad \langle h_i, \lambda\rangle = \frac{2(\alpha_i|\lambda)}{(\alpha_i|\alpha_i)} \text{ for } \lambda \in \mathfrak{h}^* \text{ and } i \in I.$$

Let $q$ be an indeterminate and $m, n \in \mathbb{Z}_{\geq 0}$. For $i \in I^{\mathrm{re}}$, let $q_i = q^{\mathsf{d}_i}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i![n]_i!}.$$

**Definition 1.1.** The *quantum generalized Kac-Moody algebra* $U_q(\mathfrak{g})$ associated with a Borcherds-Cartan datum $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$ is the associative algebra over $\mathbb{Q}(q)$ with 1 generated by $e_i, f_i$ $(i \in I)$ and $q^h$ $(h \in \mathsf{P}^\vee)$ satisfying following relations:

(1) $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in \mathsf{P}^\vee$,

(2) $q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i,\ q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i$ for $h \in \mathsf{P}^\vee, i \in I$,

(3) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$, where $K_i = q_i^{h_i}$,

(4) $\displaystyle\sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0$ if $i \in I^{\mathrm{re}}$ and $i \neq j$,

(5) $\displaystyle\sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0$ if $i \in I^{\mathrm{re}}$ and $i \neq j$,

(6) $e_i e_j - e_j e_i = 0,\ f_i f_j - f_j f_i = 0$ if $a_{ij} = 0$.

Note that, if all diagonal entries of $\mathsf{A}$ are 2, then $\mathsf{A}$ is a generalized Cartan matrix and $U_q(\mathfrak{g})$ is the usual quantum group associated with $\mathsf{A}$.

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements $e_i$ (resp. $f_i$) for $i \in I$. For $n \in \mathbb{Z}_{>0}$, set

$$e_i^{(n)} = \begin{cases} \dfrac{e_i^n}{[n]_i!} & \text{if } i \in I^{\mathrm{re}}, \\ e_i^n & \text{if } i \in I^{\mathrm{im}}, \end{cases} \qquad f_i^{(n)} = \begin{cases} \dfrac{f_i^n}{[n]_i!} & \text{if } i \in I^{\mathrm{re}}, \\ f_i^n & \text{if } i \in I^{\mathrm{im}}. \end{cases}$$

For an element $u \in U_q^-(\mathfrak{g})$, $u$ can be expressed uniquely as

$$u = \sum_{k\geq 0} f_i^{(k)} u_k, \tag{1.1}$$

where $u_k \in \ker e_i'$ and $u_k = 0$ for $k \gg 0$. Here, $e_i'$ is the endomorphism $U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$ given in [3, Section 6] and [9, Section 3.3]. The *Kashiwara operators* $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ of $U_q^-(\mathfrak{g})$ are defined by

$$\tilde{e}_i u = \sum_{k\geq 1} f_i^{(k-1)} u_k, \qquad \tilde{f}_i u = \sum_{k\geq 0} f_i^{(k+1)} u_k.$$

Let $\mathbb{A}_0 = \{f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q], g(0) \neq 0\}$.

**Definition 1.2.** A *crystal basis* of $U_q^-(\mathfrak{g})$ is a pair $(L, B)$ satisfying the following conditions:

(1) $L$ is a free $\mathbb{A}_0$-module of $U_q^-(\mathfrak{g})$ such that $U_q^-(\mathfrak{g}) = \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L$ and $L = \bigoplus_{\alpha\in Q^+} L_{-\alpha}$, where $L_{-\alpha} := L \cap U_q^-(\mathfrak{g})_{-\alpha}$,
(2) $B$ is a $\mathbb{Q}$-basis of $L/qL$ such that $B = \bigsqcup_{\alpha\in Q^+} B_{-\alpha}$, where $B_{-\alpha} := B \cap (L_{-\alpha}/qL_{-\alpha})$,
(3) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \subset B$ for all $i \in I$,
(4) For $b, b' \in B$ and $i \in I$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$.

Let $L(\infty)$ be the free $\mathbb{A}_0$-module of $U_q^-(\mathfrak{g})$ generated by $\{\tilde{f}_{i_1}\cdots\tilde{f}_{i_r}1 \mid r \geq 0, i_k \in I\}$ and let

$$B(\infty) = \{\tilde{f}_{i_1}\cdots\tilde{f}_{i_r}1 + qL(\infty) \mid r \geq 0, i_k \in I\} \setminus \{0\}.$$

Then, it is proved in [3, 9] that the pair $(L(\infty), B(\infty))$ is a unique crystal basis of $U_q^-(\mathfrak{g})$.

Let $M$ be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{int}$ defined in [3, Definition 3.1]. For any $i \in I$, any element $u \in M_\mu$ can be expressed uniquely as

$$u = \sum_{k\geq 0} f_i^{(k)} u_k,$$

where $u_k \in M_{\mu+k\alpha_i} \cap \ker e_i$. The *Kashiwara operators* $\tilde{e}_i, \tilde{f}_i$ $(i \in I)$ are defined by

$$\tilde{e}_i u = \sum_{k\geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k\geq 0} f_i^{(k+1)} u_k.$$

**Definition 1.3.** A *crystal basis* of $U_q(\mathfrak{g})$-module $M$ is a pair $(L, B)$ satisfying the following conditions:

(1) $L$ is a free $\mathbb{A}_0$-module of $M$ such that $M = \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L$ and $L = \bigoplus_{\lambda\in P} L_\lambda$, where $L_\lambda := L \cap M_\lambda$,
(2) $B$ is $\mathbb{Q}$-basis of $L/qL$ such that $B = \bigsqcup_{\lambda\in P} B_\lambda$, where $B_\lambda := B \cap L_\lambda/qL_\lambda$,
(3) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \subset B \sqcup \{0\}$ for all $i \in I$,
(4) For $b, b' \in B$ and $i \in I$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$.

For a dominant integral weight $\lambda \in \mathsf{P}^+$, we denote by $V(\lambda)$ the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda$. Note that $V(\lambda)$ is contained in $\mathcal{O}_{int}$. Let $L(\lambda)$ be the free $\mathbb{A}_0$-module of $V(\lambda)$ generated by $\{\tilde{f}_{i_1}\cdots\tilde{f}_{i_r}v_\lambda \mid r \geq 0, i_k \in I\}$ and let

$$B(\lambda) = \{\tilde{f}_{i_1}\cdots\tilde{f}_{i_r}v_\lambda + qL(\lambda) \mid r \geq 0, i_k \in I\} \setminus \{0\}.$$

Then the pair $(L(\lambda), B(\lambda))$ is a unique crystal basis of $V(\lambda)$ [3, 9].

## 2. Khovanov-Lauda-Rouquier algebras

Let $\mathbf{k} = \bigoplus_{n\in\mathbb{Z}} \mathbf{k}_n$ be a commutative graded ring such that $\mathbf{k}_0$ is a field and $\mathbf{k}_n = 0$ for $n < 0$. For $\alpha \in \mathsf{Q}^+$ with $|\alpha| = m$, let

$$I^\alpha = \{\nu = (\nu_1, \dots, \nu_m) \in I^m \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_m} = \alpha\}.$$

Note that the symmetric group $\mathrm{S}_m = \langle s_k \mid k = 1, \dots m-1\rangle$ acts naturally on $I^\alpha$. For $t = 1, \dots, m-1$, we define the operator $\partial_t$ on $\mathbf{k}[x_1, \dots, x_m]$ by

$$\partial_t(f) = \frac{s_t f - f}{x_t - x_{t+1}},$$

where $wf(x_1, \dots, x_m) = f(x_{w(1)}, \dots, x_{w(m)})$ for $w \in \mathrm{S}_m$ and $f(x_1, \dots, x_m) \in \mathbf{k}[x_1, \dots, x_m]$.

For each $i \in I$, we choose a polynomial $\mathcal{P}_i(u, v) \in \mathbf{k}[u. v]$ of the form

$$\mathcal{P}_i(u, v) = \sum_{k,l\geq 0} p_{i;k,l} u^k v^l,$$

where $p_{i;k,l} \in \mathbf{k}_{\mathsf{d}_i(2-a_{ii})-2\mathsf{d}_i k-2\mathsf{d}_i l}$ and $p_{i;1-\frac{a_{ii}}{2},0}, p_{i;0,1-\frac{a_{ii}}{2}} \in \mathbf{k}_0^\times$. We also take a matrix $(\mathcal{Q}_{ij}(u,v))_{i,j\in I}$ in $\mathbf{k}[u, v]$ such that $\mathcal{Q}_{ij}(u, v) = \mathcal{Q}_{ji}(v, u)$ and $\mathcal{Q}_{ij}(u, v)$ has the form

$$\mathcal{Q}_{ij}(u, v) = \begin{cases} 0 & \text{if } i = j, \\ \sum_{k,l\geq 0} q_{i,j;k,l} u^k v^l & \text{if } i \neq j, \end{cases}$$

where $q_{i,j;k,l} \in \mathbf{k}_{-2(\alpha_i|\alpha_j)-2\mathsf{d}_i k-2\mathsf{d}_j l}$ and $q_{i,j;-a_{ij},0} \in \mathbf{k}_0^\times$.

**Definition 2.1.** Let $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$ be a Borcherds-Cartan datum and $\alpha \in \mathsf{Q}^+$ with height $m$. The *Khovanov-Lauda-Rouquier algebra* $R(\alpha)$ of weight $\alpha$ associated with the data $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$, $(\mathcal{P}_i)_{i\in I}$ and $(\mathcal{Q}_{ij})_{i,j\in I}$ is the associative graded $\mathbf{k}$-algebra generated by $e(\nu)$ ($\nu = (\nu_1, \dots, \nu_m) \in I^\alpha$), $x_k$ ($1 \leq k \leq m$), $\tau_t$ ($1 \leq t \leq m-1$) satisfying the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'} e(\nu), \ \sum_{\nu\in I^\alpha} e(\nu) = 1, \ x_k e(\nu) = e(\nu) x_k, \ x_k x_l = x_l x_k,$$

$$\tau_t e(\nu) = e(s_t(\nu))\tau_t, \ \tau_t\tau_s = \tau_s\tau_t \text{ if } |t-s| > 1,$$

$$\tau_t^2 e(\nu) = \begin{cases} \partial_t \mathcal{P}_{\nu_t}(x_t, x_{t+1})\tau_t e(\nu) & \text{if } \nu_t = \nu_{t+1}, \\ \mathcal{Q}_{\nu_t\nu_{t+1}}(x_t, x_{t+1}) e(\nu) & \text{if } \nu_t \neq \nu_{t+1}, \end{cases}$$

$$(\tau_t x_k - x_{s_t(k)}\tau_t) e(\nu) = \begin{cases} -\mathcal{P}_{\nu_t}(x_t, x_{t+1}) e(\nu) & \text{if } k = t \text{ and } \nu_t = \nu_{t+1}, \\ \mathcal{P}_{\nu_t}(x_t, x_{t+1}) e(\nu) & \text{if } k = t+1 \text{ and } \nu_t = \nu_{t+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$
\begin{aligned}
&(\tau_{t+1}\tau_t\tau_{t+1} - \tau_t\tau_{t+1}\tau_t)e(\nu) \\
&\quad = \begin{cases} \mathcal{P}_{\nu_t}(x_t, x_{t+2})\overline{\mathcal{Q}}_{\nu_t,\nu_{t+1}}(x_t, x_{t+1}, x_{t+2})e(\nu) & \text{if } \nu_t = \nu_{t+2} \neq \nu_{t+1}, \\ \overline{\mathcal{P}}'_{\nu_t}(x_t, x_{t+1}, x_{t+2})\tau_t e(\nu) + \overline{\mathcal{P}}''_{\nu_t}(x_t, x_{t+1}, x_{t+2})\tau_{t+1}e(\nu) & \text{if } \nu_t = \nu_{t+1} = \nu_{t+2}, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{\mathcal{P}}'_i(u. v, w) &:= \frac{\mathcal{P}_i(v, u)\mathcal{P}_i(u, w)}{(u - v)(u - w)} + \frac{\mathcal{P}_i(u, w)\mathcal{P}_i(v, w)}{(u - w)(v - w)} - \frac{\mathcal{P}_i(u, v)\mathcal{P}_i(v, w)}{(u - v)(v - w)}, \\
\overline{\mathcal{P}}''_i(u, v, w) &:= -\frac{\mathcal{P}_i(u, v)\mathcal{P}_i(u, w)}{(u - v)(u - w)} - \frac{\mathcal{P}_i(u. w)\mathcal{P}_i(w, v)}{(u - w)(v - w)} + \frac{\mathcal{P}_i(u, v)\mathcal{P}_i(v, w)}{(u - v)(v - w)}, \\
\overline{\mathcal{Q}}_{i,j}(u, v, w) &:= \frac{\mathcal{Q}_{i,j}(u, v) - \mathcal{Q}_{i,j}(w, v)}{u - w}.
\end{aligned}
$$

The algebra $R(\alpha)$ has the $\mathbb{Z}$-grading given by

$$
\deg(e(\nu)) = 0, \quad \deg(x_k(\nu)) = 2\mathsf{d}_{\nu_k}, \quad \deg(\tau_t(\nu)) = -(\alpha_{\nu_t}|\alpha_{\nu_{t+1}}),
$$

where $x_k(\nu) = x_k e(\nu)$ and $\tau_t(\nu) = \tau_t e(\nu)$ for $\nu \in I^\alpha$. A diagrammatic presentation of $R(\alpha)$ using planar diagrams with dots and strands is given in [7, 11, 12].

For $\lambda \in \mathsf{P}^+$ and $i \in I$, let us fix a polynomial $a_i^\lambda(u)$ of the form

$$
a_i^\lambda(u) = \sum_{k=0}^{\lambda(h_i)} c_{i;k}^\lambda u^{\lambda(h_i)-k},
$$

where $c_{i;k}^\lambda \in \mathbf{k}_{2\mathsf{d}_i k}$ and $c_{i;0}^\lambda = 1$. Set $a^\lambda(x) = \sum_{\nu \in I} a_{\nu_1}^\lambda(x)e(\nu)$. Then the *cyclotomic Khovanov-Lauda-Rouquier algebra* $R^\lambda(\alpha)$ is defined to be the quotient algebra

$$
R^\lambda(\alpha) = R(\alpha)/R(\alpha)a^\lambda(x)R(\alpha).
$$

Let $R(\alpha)$-mod (resp. $R^\lambda(\alpha)$-mod) be the category of finite-dimensional graded left $R(\alpha)$-modules (resp. finite-dimensional graded left $R^\lambda(\alpha)$-modules). For a $\mathbb{Z}$-graded module $M = \bigoplus_{k\in\mathbb{Z}} M_k$ and $t \in \mathbb{Z}$, let $M\langle t\rangle = \bigoplus_{k\in\mathbb{Z}} M\langle t\rangle_k$ be the $\mathbb{Z}$-graded module obtained from $M$ by setting $M\langle t\rangle_k := M_{t+k}$. The *q-character* $\mathsf{ch}_q(M)$ and *character* $\mathsf{ch}(M)$ of $M$ are defined by

$$
\mathsf{ch}_q(M) = \sum_{\nu\in I^\alpha} \dim_q(e(\nu)M)\ \nu, \qquad \mathsf{ch}(M) = \sum_{\nu\in I^\alpha} \dim(e(\nu)M)\ \nu,
$$

where $\dim_q(N) := \sum_{i\in\mathbb{Z}}(\dim N_i)q^i$ for any graded module $N = \bigoplus_{i\in\mathbb{Z}} N_i$.

For $M, N \in R(\alpha)$-mod, let $\mathsf{Hom}(M, N)$ be the set of homogeneous homomorphisms of degree 0, and let $\mathsf{HOM}(M, N) = \bigoplus_{k\in\mathbb{Z}} \mathsf{Hom}(M, N\langle k\rangle)$. For $\beta_1, \dots, \beta_m \in \mathsf{Q}^+$, we define

$$
e(\beta_1, \dots, \beta_m) = \sum_{\nu_j \in I^{\beta_j}} e(\nu_1 * \cdots * \nu_m),
$$

where $\nu_1 * \cdots * \nu_m$ is the concatenation of $\nu_k$'s. The natural embedding $R(\beta_1)\otimes\cdots\otimes R(\beta_m) \hookrightarrow R(\beta_1+\cdots+\beta_m)$ gives the functors

$$\mathrm{Ind}_{\beta_1,\ldots,\beta_m} : R(\beta_1)\otimes\cdots\otimes R(\beta_m)\text{-mod} \to R(\beta_1+\cdots+\beta_m)\text{-mod},$$
$$\mathrm{Res}_{\beta_1,\ldots,\beta_m} : R(\beta_1+\cdots+\beta_m)\text{-mod} \to R(\beta_1)\otimes\cdots\otimes R(\beta_m)\text{-mod}$$

defined by $\mathrm{Ind}_{\beta_1,\ldots,\beta_m} L := R(\beta_1+\cdots+\beta_m)\otimes_{R(\beta_1)\otimes\cdots\otimes R(\beta_m)} L$ and $\mathrm{Res}_{\beta_1,\ldots,\beta_m} M := e(\beta_1,\ldots,\beta_m)M$ for $L \in R(\beta_1)\otimes\cdots\otimes R(\beta_m)$-mod and $M \in R(\beta_1+\cdots+\beta_m)$-mod.

Let $\mathfrak{B}(\infty)$ (resp. $\mathfrak{B}(\lambda)$) be the set of all isomorphism classes of irreducible graded $R(\beta)$-modules (resp. $R^\lambda(\beta)$-modules) for all $\beta \in \mathsf{Q}^+$. Set $\mathbf{1}$ to be the 1-dimensional trivial $R(0)$-module. For $M \in R(\beta)$-mod, we define

$$\tilde{e}_i(M) = \mathrm{soc}(\mathrm{Res}_{\alpha_i,\beta-\alpha_i} M) \in R(\beta-\alpha_i)\text{-mod}$$
$$\tilde{f}_i(M) = \mathrm{hd}\,\mathrm{Ind}_{\alpha_i,\beta}(L(i)\boxtimes M) \in R(\beta+\alpha_i)\text{-mod},$$
$$\mathrm{wt}(M) = \begin{cases} -\beta & \text{if } M \in R(\beta)\text{-mod}, \\ \lambda-\beta & \text{if } M \in R^\lambda(\beta)\text{-mod}, \end{cases}$$
$$\varepsilon_i(M) = \max\{k \geq 0 \mid \tilde{e}_i^k M \neq 0\}, \quad \varphi_i(M) = \varepsilon_i(M) + \mathrm{wt}(M)(h_i),$$

where $L(i)$ is the 1-dimensional $R(\alpha_i)$-module with $\dim_q L(i) = 1$. Here, for an $R(\alpha)$-module $N$, $\mathrm{soc}(N)$ (resp. hd(N)) is the maximal completely reducible submodule (resp. the maximal completely reducible quotient) of $N$. Then, when $a_{ii} \neq 0$ for all $i \in I$, the sets $\mathfrak{B}(\infty)$ and $\mathfrak{B}(\lambda)$ with the above maps have crystal structures [7, 14].

**Theorem 2.2** ([7, 14]). *If $a_{ii} \neq 0$ for all $i \in I$, then the crystal $(\mathfrak{B}(\infty), \mathrm{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i)$ (resp. $(\mathfrak{B}(\lambda), \mathrm{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i)$) is isomorphic to the crystal $B(\infty)$ (resp. $B(\lambda)$).*

We now assume that $a_{ii} \neq 0$ for all $i \in I$. Let $n = |I|$ be the rank of $U_q(\mathfrak{g})$, and let $I_{(k)} \subset I$ $(k = 1, \ldots, n)$ be subsets of $I = I_{(n+1)}$ such that $I_{(k)} \subset I_{(k+1)}$ and $|I_{(k)}| = k$ for all $k$. Let $U_k$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i$, $f_i$ $(i \in I_{(k)})$ and $q^h$ $(h \in \mathsf{P}^\vee)$ and let $\mathcal{B}_k$ be the crystal obtained from $B(\infty)$ by forgetting the $i$-arrows for $i \notin I_{(k)}$. Then $\mathcal{B}_k$ can be understood as a $U_k$-crystal and every connected component of $\mathcal{B}_k$ has a unique highest weight vector [1, Lemma 1.9].

Take an element $v \in B(\infty)$. Let $u_0 = v$ and let $u_k$ be the highest weight vector of the connected component $\mathcal{C}_k$ of $\mathcal{B}_k$ containing $v$ for $k = 1, \ldots, n$. By construction, there is a chain of injective maps

$$\mathcal{C}_1 \hookrightarrow \mathcal{C}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_{n-1} \hookrightarrow B(\infty).$$

For $k = 1, \ldots, n$, let

$$\mathcal{N}_k(v) = \tilde{f}_{\nu_{k,1}} \cdots \tilde{f}_{\nu_{k,t_k}} \mathbf{1} \in \mathfrak{B}(\infty), \tag{2.1}$$

where $\nu_k = (\nu_{k,1}, \dots, \nu_{k,t_k})$ is a sequence of $I$ such that $u_{k-1} = \tilde{f}_{\nu_{k,1}} \cdots \tilde{f}_{\nu_{k,t_k}} u_k$.

Hence, for each $v \in B(\infty)$, we obtain the corresponding $n$-tuple $(\mathcal{N}_1(v), \mathcal{N}_2(v), \dots, \mathcal{N}_n(v))$ of irreducible modules in $\mathfrak{B}(\infty)$. Then, using the same argument as in [1, Proposition 1.10], one can obtain the following proposition.

**Proposition 2.3.** [1, Proposition 1.10]

(1) *For* $v \in B(\infty)$, $\operatorname{hd}\operatorname{Ind}(\boxtimes_{k=1}^{n} \mathcal{N}_k(v))$ *is irreducible.*
(2) *The map* $\Phi : B(\infty) \longrightarrow \mathfrak{B}(\infty)$ *defined by*

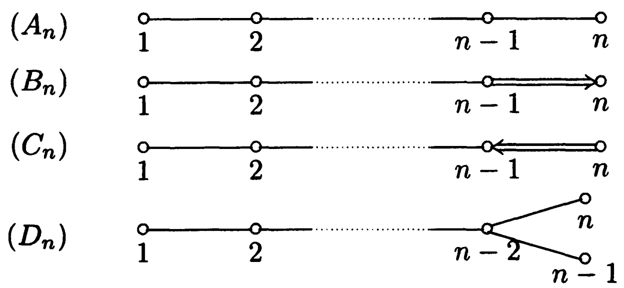
$$\Phi(v) = \operatorname{hd}\operatorname{Ind}(\boxtimes_{k=1}^{n} \mathcal{N}_k(v)) \quad \textit{for } v \in B(\infty)$$

*is a crystal isomorphism.*

## 3. Crystal bases and irreducible representations

In this section, we give a crystal basis theoretic construction of irreducible modules over Khovanov-Lauda-Rouquier algebras and their cyclotomic quotients for finite classical types. Though this section is based on our previous work [1], the description of irreducible modules in this section is different and more combinatorial than the description given in [1]. Throughout this section, we assume that $\mathbf{k} = \mathbb{C}$ and $\mathsf{A}$ is a generalized Cartan matrix of finite classical type $A_n$, $B_n$, $C_n$ and $D_n$.

Let $I = \{1, 2, \dots, n\}$ and let $\Gamma$ be the following Dynkin diagram:




We set $\mathbb{B}_X$ $(X = A, B, C, D)$ to be the crystal defined by

$(A_n)$ $$\boxed{1} \xleftarrow{1} \boxed{2} \xleftarrow{2} \cdots \xleftarrow{n-1} \boxed{n} \xleftarrow{n} \boxed{n+1}$$

$(B_n)$ $$\boxed{\overline{1}} \xleftarrow{1} \cdots \xleftarrow{n-1} \boxed{\overline{n}} \xleftarrow{n} \boxed{0} \xleftarrow{n} \boxed{n} \xleftarrow{n-1} \cdots \xleftarrow{1} \boxed{1}$$

$(C_n)$ $$\boxed{\overline{1}} \xleftarrow{1} \cdots \xleftarrow{n-2} \boxed{\overline{n-1}} \xleftarrow{n-1} \boxed{\overline{n}} \xleftarrow{n} \boxed{n} \xleftarrow{n-1} \boxed{n-1} \xleftarrow{n-2} \cdots \xleftarrow{1} \boxed{1}$$

$$(D_n)\qquad \begin{tikzcd}
 & & & \boxed{n} \arrow[dl, "n"'] & & & \\
\boxed{\bar{1}} & \cdots \arrow[l, "1"'] & \boxed{\overline{n-1}} \arrow[l, "n-2"'] & & \boxed{n-1} \arrow[ul, "n-1"'] \arrow[dl, "n"] & \cdots \arrow[l, "n-2"'] & \boxed{1} \arrow[l, "1"'] \\
 & & & \boxed{\bar{n}} \arrow[ul, "n-1"] & & &
\end{tikzcd}$$

with the entries ordered by

$$\begin{array}{ll}
(A_n) & \bar{1} \succ \bar{2} \succ \cdots \succ \overline{n+1}, \\
(B_n) & \bar{1} \succ \bar{2} \succ \cdots \succ \bar{n} \succ 0 \succ n \succ \cdots \succ 2 \succ 1, \\
(C_n) & \bar{1} \succ \bar{2} \succ \cdots \succ \overline{n-1} \succ \bar{n} \succ n \succ n-1 \succ \cdots \succ 1, \\
(D_n) & \bar{1} \succ \bar{2} \succ \cdots \succ \overline{n-1} \succ \bar{n}, n \succ n-1 \succ \cdots \succ 1.
\end{array}$$

For $\nu = (\nu_1, \dots, \nu_m) \in I^m$ and $\mathbf{k} = (k_1, \dots, k_m) \in (\mathbb{Z}_{\geq 0})^m$, let $\tilde{f}_\nu^{\mathbf{k}} = \tilde{f}_{\nu_1}^{k_1} \cdots \tilde{f}_{\nu_m}^{k_m}$ (resp. $\tilde{e}_\nu^{\mathbf{k}} = \tilde{e}_{\nu_1}^{k_1} \cdots \tilde{e}_{\nu_m}^{k_m}$). If $\mathbf{k} = (1, \dots, 1)$, then we write $\tilde{f}_\nu$ (resp. $\tilde{e}_\nu$) for $\tilde{f}_\nu^{\mathbf{k}}$ (resp. $\tilde{e}_\nu^{\mathbf{k}}$).

A *segment* $\mathfrak{s}$ of the crystal $\mathbb{B}_X$ is a subset $\mathfrak{s} = \{a, b\} \subset \mathbb{B}_X$ such that the two elements $a$ and $b$ of $\mathfrak{s}$ are distinct and comparable. Note that any distinct two elements $\{a, b\}$ of $\mathbb{B}_X$ except the case $\{a, b\} = \{n, \bar{n}\}$ $(X = D)$ can be viewed as a segment of $\mathbb{B}_X$. For a segment $\mathfrak{s} = \{a, b\}$ with $a \succ b$, set $\mathsf{h}(\mathfrak{s}) = a$ and $\mathsf{t}(\mathfrak{s}) = b$, respectively. We define a partial order $\succeq$ on the set of segments of $\mathbb{B}_X$ as follows: for two segments $\mathfrak{s}$ and $\mathfrak{s}'$ of $\mathbb{B}_X$,

$$\mathfrak{s} \succeq \mathfrak{s}' \quad \text{if and only if} \quad \big(\mathsf{h}(\mathfrak{s}) \succ \mathsf{h}(\mathfrak{s}')\big) \text{ or } \big(\mathsf{h}(\mathfrak{s}) = \mathsf{h}(\mathfrak{s}') \text{ and } (\mathsf{t}(\mathfrak{s}') \succ \mathsf{t}(\mathfrak{s}))\big)$$

For $b \in \mathbb{B}_X$ and $X = B, C, D$, we set $b^\vee$ to be a unique element in $\mathbb{B}_X$ such that $\mathrm{wt}(b^\vee) = -\mathrm{wt}(b)$. Let $b_X = \bar{n}$ $(X = A, B, C)$, $b_X = \overline{n-1}$ $(X = D)$. A *multisegment* $\underline{\mathfrak{m}}$ of $\mathbb{B}_X$ is a multiset of segments of the crystal $\mathbb{B}_X$ such that

$$\begin{array}{ll}
 & \text{(i) } \mathsf{h}(\mathfrak{s}) \succeq b_X \text{ for } \mathfrak{s} \in \underline{\mathfrak{m}}, \\
(3.1) & \text{(ii) } \mathsf{t}(\mathfrak{s}) \succeq \mathsf{h}(\mathfrak{s})^\vee \text{ if } X = B, C, D, \\
 & \text{(iii) any two segments in } \underline{\mathfrak{m}} \text{ are comparable.}
\end{array}$$

When no confusion can arise, we write $\underline{\mathfrak{m}} = \{\mathfrak{s}_1 \preceq \mathfrak{s}_2 \preceq \mathfrak{s}_3 \preceq \cdots\}$. For $i = 1, \dots, n$, we set

$$\underline{\mathfrak{m}}(i) = \{\mathfrak{s} \in \underline{\mathfrak{m}} \mid \tilde{e}_i(\mathsf{h}(\mathfrak{s})) \neq 0\}.$$

Note that $\underline{\mathfrak{m}} = \bigcup_{i \in I} \underline{\mathfrak{m}}(i)$.

Let $\mathcal{M}_X$ be the set of all multisegments of $\mathbb{B}_X$. We will show that $\mathcal{M}_X$ parameterizes the crystal $B(\infty)$. Let $\ell_A = n$, $\ell_B = \ell_C = 2n - 1$, $\ell_D = 2n - 2$ and let $\nu_X$ be the sequence of the colors over the arrows of $\mathbb{B}_X$, i.e.,

$$\nu_A = (1, 2, \dots, n-1, n),$$
$$\nu_B = \nu_C = (1, 2, \dots, n-1, n, n-1, \dots, 2, 1),$$
$$\nu_D = (1, 2, \dots, n-1, n, n-2, \dots, 2, 1).$$

For a segment $\mathfrak{s}$, let $\mathsf{e}(\mathfrak{s}) = (\mathsf{e}_1, \dots, \mathsf{e}_{\ell_X}) \in \mathbb{Z}^{\ell_X}$ be the $\ell_X$-tuple of $\mathbb{Z}$ defined by

$$\mathsf{h}(\mathfrak{s}) = \tilde{f}_{\nu_X}^{\mathsf{e}(\mathfrak{s})}\mathsf{t}(\mathfrak{s}), \quad \mathsf{e}_j \neq 0 \text{ for some } 1 \le j \le n.$$

Here we consider $\mathbb{Z}^{\ell_X}$ as an abelian group. Let $\mathfrak{e}_i = (0, \dots, 0, \overset{i\text{th}}{\overset{\downarrow}{1}}, 0, \dots, 0)$ for $i = 1, \dots, \ell_X$.

**Proposition 3.1.** *Let $\Upsilon : \mathcal{M}_X \longrightarrow B(\infty)$ be the map defined by*

$$\Upsilon(\underline{\mathrm{m}}) = \tilde{f}_{\nu_X}^{\mathsf{e}_1(\underline{\mathrm{m}})} \cdots \tilde{f}_{\nu_X}^{\mathsf{e}_n(\underline{\mathrm{m}})} 1,$$

$$\textit{where } \mathsf{e}_i(\underline{\mathrm{m}}) = \begin{cases} \sum_{\mathfrak{s} \in \underline{\mathrm{m}}(n+1-i)} \mathsf{e}(\mathfrak{s}) & \textit{if either } X = A, B, C \textit{ or } i \neq 1, 2 \ (X = D), \\ \#\underline{\mathrm{m}}(n-1)\mathfrak{e}_{n-1} & \textit{if } i = 2 \ (X = D), \\ \#\underline{\mathrm{m}}(n)\mathfrak{e}_n & \textit{if } i = 1 \ (X = D). \end{cases}$$

*Then the map $\Upsilon$ is bijective.*

*Proof.* We focus on the case $X = D$ since the remaining cases can be proved in a similar manner. Let $\mathcal{S}$ be the set of $\mathbf{t} = (\mathbf{t}_1, \dots, \mathbf{t}_n) \in \mathbb{Z}_{\ge 0}^{n(2n-2)}$ such that

(a) $\mathbf{t}_j = (t_{j,1}, \dots, t_{j,2n-2}) \in \mathbb{Z}_{\ge 0}^{(2n-2)}$ for $j = 1, \dots, n$,
(b) $t_{j,n+1-j} \ge t_{j,n+2-j} \ge \cdots \ge t_{j,n-1}, t_{j,n} \ge t_{j,n+1} \ge \cdots \ge t_{j,n-2+j}$ for $j = 3, \dots, n$,
(c) if $j = 3, \dots, n$, then $t_{j,k} = 0$ for either $k < n+1-j$ or $k > n-2+j$,
(d) $t_{2,k} = 0$ for $k \neq n-1$ and $t_{1,k} = 0$ for $k \neq n$.

It follows from [15, Section 7] that the map $\mathcal{S} \to B(\infty)$ mapping $\mathbf{t} = (\mathbf{t}_1, \dots, \mathbf{t}_n) \in \mathcal{S}$ to $\tilde{f}_{\nu_D}^{\mathbf{t}_1} \cdots \tilde{f}_{\nu_D}^{\mathbf{t}_n} 1 \in B(\infty)$ is bijective.

Let $\underline{\mathrm{m}} \in \mathcal{M}_X$ and write $\mathsf{e}_j(\underline{\mathrm{m}}) = (\mathsf{e}_{j,1}, \dots, \mathsf{e}_{j,2n-2})$ for $j = 1, \dots, n$. By (ii) and (iii) of (3.1), the sequences $\mathsf{e}_j(\underline{\mathrm{m}})$ satisfy the above condition (b), (c) and (d). Hence the map $\phi : \mathcal{M}_X \to \mathcal{S}$ given by

$$\phi(\underline{\mathrm{m}}) = (\mathsf{e}_1(\underline{\mathrm{m}}), \dots, \mathsf{e}_n(\underline{\mathrm{m}})) \quad (\underline{\mathrm{m}} \in \mathcal{M}_X)$$

is well-defined.

On the other hand, let

$$\widehat{i} = \begin{cases} \bar{i} & \text{if } 1 \le i \le n, \\ 2n+1-i & \text{if } n+1 \le i \le 2n. \end{cases}$$

and, for a segment $\mathfrak{s}$ and a multisegment $\underline{\mathfrak{m}} \in \mathcal{M}_X$, let us denote by $\sigma(\underline{\mathfrak{m}}, \mathfrak{s})$ the multiplicity of $\mathfrak{s}$ in $\underline{\mathfrak{m}}$. Let $\psi : \mathcal{S} \to \mathcal{M}_X$ be the map mapping $\mathbf{t} \in \mathcal{S}$ to the multisegment $\psi(\mathbf{t}) \in \mathcal{M}_X$ such that, if $i = 1, \ldots, n-2$, then

$$\sigma(\psi(\mathbf{t}), \{\widehat{i}, \widehat{j+1}\}) = \begin{cases} t_{n+1-i,j} - t_{n+1-i,j+1} & \text{if } j \le n-3, \\ t_{n+1-i,n-2} - \max\{t_{n+1-i,n-1}, t_{n+1-i,n}\} & \text{if } \ j = n-2, \\ \max\{0, t_{n+1-i,n-1} - t_{n+1-i,n}\} & \text{if } \ j = n-1, \\ \max\{0, t_{n+1-i,n} - t_{n+1-i,n-1}\} & \text{if } \ j = n, \\ \min\{t_{n+1-i,n-1}, t_{n+1-i,n}\} - t_{n+1-i,n+1} & \text{if } \ j = n+1, \\ t_{n+1-i,j-1} - t_{n+1-i,j} & \text{if } \ j \ge n+2, \end{cases}$$

and, if $i = n-1$, then

$$\sigma(\psi(\mathbf{t}), \{\widehat{n-1}, \widehat{j+1}\}) = \begin{cases} \max\{0, t_{2,n-1} - t_{1,n}\} & \text{if } \ j = n-1, \\ \max\{0, t_{1,n} - t_{2,n-1}\} & \text{if } \ j = n, \\ \min\{t_{1,n}, t_{2,n-1}\} & \text{if } j = n+1. \end{cases}$$

Then it is straightforward to verify that $\phi \circ \psi = \mathrm{id}_{\mathcal{S}}$ and $\psi \circ \phi = \mathrm{id}_{\mathcal{M}_X}$. □

We now return to the Khovanov-Lauda-Rouquier algebras. Let $\mathbf{1}$ be the trivial $R(0)$-module. For a segment $\mathfrak{s}$, we define

$$\nabla(\mathfrak{s}) = \tilde{f}_{\nu_X}^{\mathsf{e}(\mathfrak{s})}\mathbf{1} \in \mathfrak{B}(\infty).$$

We give a explicit description of the module structure of $\nabla(\mathfrak{s})$ as follows.

Let $l = |\mathrm{wt}(\mathsf{h}(\mathfrak{s})) - \mathrm{wt}(\mathsf{t}(\mathfrak{s}))|$. If one of the following holds: $\mathsf{h}(\mathfrak{s}) \succ \mathsf{t}(\mathfrak{s})$ $(A_n, C_n)$, either $\mathsf{t}(\mathfrak{s}) \succeq 0$ or $0 \succeq \mathsf{h}(\mathfrak{s})$ $(B_n)$, either $\mathsf{t}(\mathfrak{s}) \succ n-1$ or $\overline{n-1} \succ \mathsf{h}(\mathfrak{s})$ $(D_n)$, then the module $\nabla(\mathfrak{s})$ is the 1-dimensional module $\mathbb{C}v$ given by

$$x_i v = 0, \quad \tau_j v = 0, \quad e(\nu)v = \begin{cases} v & \text{if } \nu = \nu(\mathfrak{s}), \\ 0 & \text{otherwise}, \end{cases}$$

where $\nu(\mathfrak{s}) \in I^l$ such that $\mathsf{h}(\mathfrak{s}) = \tilde{f}_{\nu(\mathfrak{s})}\mathsf{t}(\mathfrak{s})$.

If $\mathsf{h}(\mathfrak{s}) \succ 0 \succ \mathsf{t}(\mathfrak{s})$ for type $B_n$, then $\nabla(\mathfrak{s})$ is the 2-dimensional module $\mathbb{C}u \oplus \mathbb{C}v$ with

$$x_i u = 0, \qquad \tau_j u = \begin{cases} v & \text{if } j = d, \\ 0 & \text{otherwise}, \end{cases} \qquad e(\nu)u = \begin{cases} u & \text{if } \nu = \nu(\mathfrak{s}), \\ 0 & \text{otherwise}, \end{cases}$$

$$x_i v = \begin{cases} u & \text{if } i = d, \\ -u & \text{if } i = d+1, \\ 0 & \text{otherwise} \end{cases} \qquad \tau_j v = 0, \qquad e(\nu)u = \begin{cases} u & \text{if } \nu = \nu(\mathfrak{s}), \\ 0 & \text{otherwise}, \end{cases}$$

where $\nu(\mathfrak{s}) \in I^l$ such that $\mathsf{h}(\mathfrak{s}) = \tilde{f}_{\nu(\mathfrak{s})}\mathsf{t}(\mathfrak{s})$, and $d$ is an integer such that $s_d(\nu(\mathfrak{s})) = \nu(\mathfrak{s})$.

If $\mathsf{h}(\mathfrak{s}) \succeq \overline{n-1}$ and $n-1 \succeq \mathsf{t}(\mathfrak{s})$ for type $D_n$, then the module $\nabla(\mathfrak{s})$ is the 2-dimensional module $\mathbb{C}u \oplus \mathbb{C}v$ defined by

$$x_i u = 0, \quad \tau_j u = \begin{cases} v & \text{if } j = d, \\ 0 & \text{otherwise}, \end{cases} \qquad e(\nu)u = \begin{cases} \mathcal{Q}_{n-1,n}(x_{n-1}, x_n)u & \text{if } \nu = \nu(\mathfrak{s})^+, \\ 0 & \text{otherwise}, \end{cases}$$

$$x_i v = 0, \quad \tau_j v = \begin{cases} u & \text{if } j = d, \\ 0 & \text{otherwise}, \end{cases} \qquad e(\nu)v = \begin{cases} v & \text{if } \nu = \nu(\mathfrak{s})^-, \\ 0 & \text{otherwise}, \end{cases}$$

where $\nu(\mathfrak{s})^+$, $\nu(\mathfrak{s})^- \in I^l$ such that $\nu(\mathfrak{s})^+ \neq \nu(\mathfrak{s})^-$ and $\mathsf{h}(\mathfrak{s}) = \tilde{f}_{\nu(\mathfrak{s})^+}\mathsf{t}(\mathfrak{s}) = \tilde{f}_{\nu(\mathfrak{s})^-}\mathsf{t}(\mathfrak{s})$, and $d$ is an integer such that $s_d(\nu(\mathfrak{s})^+) = \nu(\mathfrak{s})^-$. Note that $\mathcal{Q}_{n-1,n}(x_{n-1}, x_n) \in \mathbb{C}^*$. For the description above, the character $\mathsf{ch}\nabla(\mathfrak{s})$ is given as follows

$$\mathsf{ch}\nabla(\mathfrak{s}) = \begin{cases} \nu(\mathfrak{s})^+ + \nu(\mathfrak{s})^- & \text{if } \mathsf{h}(\mathfrak{s}) \succeq \overline{n-1},\ n-1 \succeq \mathsf{t}(\mathfrak{s}) \ \ (D_n), \\ 2\nu(\mathfrak{s}) & \text{if } \mathsf{h}(\mathfrak{s}) \succ 0 \succ \mathsf{t}(\mathfrak{s}) \ \ (B_n), \\ \nu(\mathfrak{s}) & \text{otherwise}. \end{cases} \tag{3.2}$$

For a multisegment $\underline{\mathfrak{m}} = \{\mathfrak{s}_1 \preceq \mathfrak{s}_2 \preceq \mathfrak{s}_3 \preceq \cdots\}$, we define

$$\nabla(\underline{\mathfrak{m}}) = \nabla(\mathfrak{s}_1) \boxtimes \nabla(\mathfrak{s}_2) \boxtimes \nabla(\mathfrak{s}_3) \boxtimes \cdots .$$

Then we have the following theorem.

**Theorem 3.2.** [1, Theorem 3.2]

(1) *For a multisegment* $\underline{\mathfrak{m}}$, *we have*

$$\mathrm{hdInd}\nabla(\underline{\mathfrak{m}}) = \tilde{f}_{\nu_X}^{\mathsf{e}_1(\underline{\mathfrak{m}})} \cdots \tilde{f}_{\nu_X}^{\mathsf{e}_n(\underline{\mathfrak{m}})}\mathbf{1},$$

*where* $\mathsf{e}_i(\underline{\mathfrak{m}}) = \begin{cases} \sum_{\mathfrak{s}\in\underline{\mathfrak{m}}(n+1-i)} \mathsf{e}(\mathfrak{s}) & \text{if either } X = A, B, C \text{ or } i \neq 1, 2\ (X = D), \\ \#\underline{\mathfrak{m}}(n-1)\mathfrak{e}_{n-1} & \text{if } i = 2\ (X = D), \\ \#\underline{\mathfrak{m}}(n)\mathfrak{e}_n & \text{if } i = 1\ (X = D). \end{cases}$

(2) *Let* $\Psi : \mathcal{M}_X \longrightarrow \mathfrak{B}(\infty)$ *be the map defined by*

$$\Psi(\underline{\mathfrak{m}}) = \mathrm{hdInd}\nabla(\underline{\mathfrak{m}}) \quad \text{for } \underline{\mathfrak{m}} \in \mathcal{M}_X.$$

*Then the map* $\Psi$ *is bijective.*

*Proof.* We give a sketch of the proof of [1, Theorem 3.2]. When $X = D$, without loss of generality, we may assume that

$$\sigma(\underline{\mathfrak{m}}, \{\overline{n-1}, \overline{n}\}) \geq \sigma(\underline{\mathfrak{m}}, \{\overline{n-1}, n\}),$$

where $\sigma(\underline{\mathfrak{m}}, \mathfrak{s})$ is the multiplicity of $\mathfrak{s}$ in $\underline{\mathfrak{m}}$. Note that $\sum_{\mathfrak{s}\in\underline{\mathfrak{m}}(n-1)} \mathsf{e}(\mathfrak{s}) = \#\underline{\mathfrak{m}}(n-1)\mathfrak{e}_{n-1} + \#\underline{\mathfrak{m}}(n)\mathfrak{e}_n$ if $X = D$. It follows from [1, Lemma 4.3] and the definition of $\underline{\mathfrak{m}}(k)$ that

(a) $\varepsilon_i(\mathrm{Ind}\nabla(\underline{\mathfrak{m}}(k))) = 0$ for $i = n+1-k, n+2-k, \ldots, n$,

$$\text{(b) } \mathrm{hdInd}\nabla(\underline{\mathrm{m}}(k)) = \begin{cases} \tilde{f}_{\nu_X}^{\mathrm{e}_{n+1-k}(\underline{\mathrm{m}})}\mathbf{1} & \text{if } k = 1,\dots,n \ (X = A,B,C), \\ & \quad k = 1,\dots,n-2 \ (X = D), \\ \tilde{f}_{\nu_X}^{\mathrm{e}_1(\underline{\mathrm{m}})}\tilde{f}_{\nu_X}^{\mathrm{e}_2(\underline{\mathrm{m}})}\mathbf{1} & \text{if } k = n-1 \ (X = D), \end{cases}$$

Let $I_{(k)} = \{n+1-k,\dots,n\}$ and let $\mathcal{N}_k$ be the module given in (2.1). Then, by the crystal description given in [15], we have

$$\mathrm{hdInd}\nabla(\underline{\mathrm{m}}(k)) \simeq \begin{cases} \mathcal{N}_{n+1-k} & \text{if } k = 1,\dots,n \ (X = A,B,C), \\ & \quad k = 1,\dots,n-2 \ (X = D), \\ \mathrm{Ind}(\mathcal{N}_1 \boxtimes \mathcal{N}_2) & \text{if } k = n-1 \ (X = D). \end{cases}$$

Combining Proposition 2.3 and [1, Lemma 1.8] with the above conditions (a) and (b), we obtain

$$\begin{aligned} \mathrm{hd}\,\mathrm{Ind}\,(\boxtimes_{k=1}^{n}\mathcal{N}_k(v)) &\simeq \mathrm{hd}\,\mathrm{Ind}\,\big(\boxtimes_{k=n'}^{1}\mathrm{Ind}\nabla(\underline{\mathrm{m}}(k))\big) \\ &\simeq \mathrm{hd}\,\mathrm{Ind}\,\big(\boxtimes_{k=n'}^{1}\nabla(\underline{\mathrm{m}}(k))\big) \\ &\simeq \mathrm{hdInd}\nabla(\underline{\mathrm{m}}), \end{aligned}$$

where $n' = n$ $(X = A,B,C)$ and $n' = n-1$ $(X = D)$. Therefore, the assertion follows from Proposition 2.3. □

From Proposition 3.1 and Theorem 3.2, we have the following corollary.

**Corollary 3.3.** *The composition $\Psi \circ \Upsilon^{-1} : B(\infty) \longrightarrow \mathfrak{B}(\infty)$ is a crystal isomorphism.*

Let $\lambda \in \mathsf{P}^+$ be the dominant integral weight and let $B(\lambda)$ be the crystal of the irreducible highest weight module $V(\lambda)$. It was shown in [10] that there is a unique strict crystal embedding

$$B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C, \quad v_\lambda \mapsto 1 \otimes t_\lambda \otimes c$$

where $v_\lambda$ is the highest weight vector of $B(\lambda)$. Here, $T_\lambda = \{t_\lambda\}$ (resp. $C = \{c\}$) is a crystal with $\mathrm{wt}(t_\lambda) = \lambda$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = 0$, $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = -\infty$ (resp. $\mathrm{wt}(t_\lambda) = 0$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = 0$, $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0$). We denote by $\imath_\lambda$ the composition of the strict embedding and the natural projection:

$$B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C \to B(\infty).$$

Let $\mathcal{M}_X(\lambda) = \Upsilon^{-1} \circ \imath_\lambda(B(\lambda))$. By Proposition 3.1, the set $B(\lambda)$ is in 1-1 correspondence to $\mathcal{M}_X(\lambda)$ via $\Upsilon^{-1} \circ \imath_\lambda$. Then the map $\Upsilon_\lambda := (\Upsilon^{-1} \circ \imath_\lambda)^{-1} : \mathcal{M}_X(\lambda) \to B(\lambda)$ is give by

$$\Upsilon_\lambda(\underline{\mathrm{m}}) = \tilde{f}_{\nu_X}^{\mathrm{e}_1(\underline{\mathrm{m}})} \cdots \tilde{f}_{\nu_X}^{\mathrm{e}_n(\underline{\mathrm{m}})} v_\lambda \quad \text{for } \underline{\mathrm{m}} \in \mathcal{M}_X(\lambda),$$

$$\text{where } \mathrm{e}_i(\underline{\mathrm{m}}) = \begin{cases} \sum_{\mathfrak{s} \in \underline{\mathrm{m}}(n+1-i)} \mathrm{e}(\mathfrak{s}) & \text{if either } X = A, B, C \text{ or } i \neq 1,2 \ (X = D), \\ \#\underline{\mathrm{m}}(n-1)\mathfrak{e}_{n-1} & \text{if } i = 2 \ (X = D), \\ \#\underline{\mathrm{m}}(n)\mathfrak{e}_n & \text{if } i = 1 \ (X = D). \end{cases}$$

We remark that the set $\mathcal{M}_X(\lambda)$ can be described explicitly from the string parametrization of $B(\lambda)$ given in [15]. By Theorem 3.2 and Corollary 3.3, we have the following Corollary.

**Corollary 3.4.** *Let $\lambda \in \mathsf{P}^+$ be a dominant integral weight.*

(1) *Let $\Psi^\lambda : \mathcal{M}_X(\lambda) \longrightarrow \mathfrak{B}(\lambda)$ be the map defined by*

$$\Psi^\lambda(\underline{\mathrm{m}}) = \mathrm{hdInd}\nabla(\underline{\mathrm{m}}) \quad \text{for } \underline{\mathrm{m}} \in \mathcal{M}_X(\lambda).$$

*Then the map $\Psi^\lambda$ is bijective.*

(2) *The composition $\Psi^\lambda \circ \Upsilon_\lambda^{-1} : B(\lambda) \longrightarrow \mathfrak{B}(\lambda)$ is a crystal isomorphism.*

**Example 3.5.** Let $U_q(\mathfrak{g})$ be of type $D_5$. Then the crystal $\mathbb{B}_D$ is given as follows:

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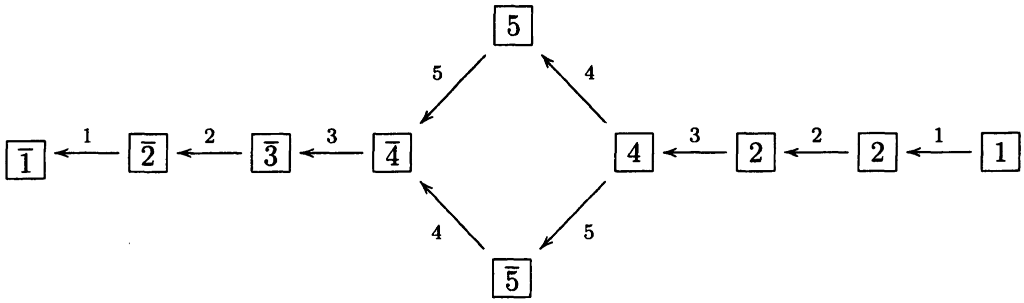


Note that $\nu_D = (1,2,3,4,5,3,2,1)$. We choose the following segments $\mathfrak{s}_k$ $(k = 1, \dots, 6)$ of $\mathbb{B}_X$:

$$\mathfrak{s}_1 := \{\bar{4}, \bar{5}\}, \qquad \mathfrak{s}_2 := \{\bar{3}, 5\}, \qquad \mathfrak{s}_3 := \{\bar{2}, \bar{4}\},$$
$$\mathfrak{s}_4 := \{\bar{2}, 4\}, \qquad \mathfrak{s}_5 := \{\bar{1}, \bar{4}\}, \qquad \mathfrak{s}_6 := \{\bar{1}, 4\}$$

and let $\underline{\mathrm{m}} = \{\mathfrak{s}_1 \preceq \mathfrak{s}_2 \preceq \mathfrak{s}_3 \preceq \mathfrak{s}_4 \preceq \mathfrak{s}_5 \preceq \mathfrak{s}_6\}$ be the multisegment consisting of $\mathfrak{s}_k$ $(k = 1, \dots, 6)$. Note that

$$\begin{aligned} &\mathsf{ch}\nabla(\mathfrak{s}_1) = (4), && \mathsf{ch}\nabla(\mathfrak{s}_2) = (3,5), \\ &\mathsf{ch}\nabla(\mathfrak{s}_3) = (2,3), && \mathsf{ch}\nabla(\mathfrak{s}_4) = (2,3,4,5) + (2,3,5,4), \\ &\mathsf{ch}\nabla(\mathfrak{s}_5) = (1,2,3), && \mathsf{ch}\nabla(\mathfrak{s}_6) = (1,2,3,4,5) + (1,2,3,5,4). \end{aligned}$$

Since $\underline{\mathrm{m}}(1) = \{\mathfrak{s}_5, \mathfrak{s}_6\}$, $\underline{\mathrm{m}}(2) = \{\mathfrak{s}_3, \mathfrak{s}_4\}$, $\underline{\mathrm{m}}(3) = \{\mathfrak{s}_2\}$, $\underline{\mathrm{m}}(4) = \{\mathfrak{s}_1\}$ and $\underline{\mathrm{m}}(5) = \emptyset$, we have

$$\begin{aligned} &\mathsf{e}_1(\underline{\mathrm{m}}) = (2,2,2,1,1,0,0,0), && \mathsf{e}_2(\underline{\mathrm{m}}) = (0,2,2,1,1,0,0,0), \\ &\mathsf{e}_3(\underline{\mathrm{m}}) = (0,0,1,0,1,0,0,0), && \mathsf{e}_4(\underline{\mathrm{m}}) = (0,0,0,1,0,0,0,0), \\ &\mathsf{e}_5(\underline{\mathrm{m}}) = (0,0,0,0,0,0,0,0). \end{aligned}$$

It follows from Theorem 3.2 that

$$\begin{aligned}\mathrm{hdInd}\nabla(\underline{\mathfrak{m}}) &\simeq \mathrm{hdInd}\,(\nabla(\mathfrak{s}_1)\boxtimes\nabla(\mathfrak{s}_2)\boxtimes\nabla(\mathfrak{s}_3)\boxtimes\nabla(\mathfrak{s}_4)\boxtimes\nabla(\mathfrak{s}_5)\boxtimes\nabla(\mathfrak{s}_6))\\ &\simeq \tilde{f}_4\tilde{f}_3\tilde{f}_5\tilde{f}_2^2\tilde{f}_3^2\tilde{f}_4\tilde{f}_5\tilde{f}_1^2\tilde{f}_2^2\tilde{f}_3^2\tilde{f}_4\tilde{f}_5\mathbf{1}.\end{aligned}$$

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