KHOVANOV-LAUDA-ROUQUIER ALGEBRAS AND CRYSTAL BASES FOR FINITE CLASSICAL TYPE

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ABSTRACT. We present a crystal basis theoretic construction of irreducible modules over Khovanov-Lauda-Rouquier algebras and their cyclotomic quotients of finite classical type.

INTRODUCTION

The Khovanov-Lauda-Rouquier algebras (or Hecke quiver algebras) were introduced independently Khovanov-Lauda [11, 12] and Rouquier [16] for providing a categorification of quantum groups associated with symmetrizable Cartan data. Let $U_q(g)$ be a quantum group and let $R(\alpha)$ be the corresponding Khovanov-Lauda-Rouquier algebra of weight $\alpha \in \mathbb{Q}^+$. For a dominant integral weight $\lambda \in \mathbb{P}^+$, the algebra $R(\alpha)$ has a special quotient $R^\lambda(\alpha)$, which gives a categorification of the irreducible highest weight $U_q(g)$-module $V(\lambda)$ with highest weight $\lambda$ [4, 18]. The crystal structures of $U_q^-(g)$ and $V(\lambda)$ also were interpreted in [14] in terms of irreducible modules over $R(\alpha)$ and $R^\lambda(\alpha)$. The Khovanov-Lauda-Rouquier algebras were generalized to the quantum generalized Kac-Moody algebras [5, 7] and, when the Cartan datum is symmetric, geometric realizations of the Khovanov-Lauda-Rouquier algebras were given in [6, 17] via quiver varieties.

In this paper, we announce the main result of our previous work [1, 8], which is an explicit construction of irreducible modules over $R(\alpha)$ and $R^\lambda(\alpha)$ of finite classical type. This construction differs from the one given in [2, 13] and is based on the theory of crystal bases. Though this paper is rely on [1], the description of irreducible modules in this paper is different and more combinatorial than the description given in [1].

Let us explain more precisely. Let $\mathcal{B}(\infty)$ (resp. $\mathcal{B}(\lambda)$) be the set of all isomorphism classes of irreducible graded $R(\alpha)$-modules (resp. $R^\lambda(\alpha)$-modules) for $\alpha \in \mathbb{Q}^+$. It is shown in [14] that there exists a crystal isomorphism $\mathcal{B}(\infty) \simeq B(\infty)$ (resp. $\mathcal{B}(\lambda) \simeq B(\lambda)$). We first define segments $s$ to be unordered pairs of comparable elements in the basic crystals $\mathbb{B}_X (X = A, B, C, D)$ given in Section 3 and give a partial order $\preceq$ to them. Then we set multisegments $m$ of $\mathbb{B}_X$ to be multisets of segments satisfying the conditions (3.1). Proposition

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3.1 says that the set $\mathcal{M}_X$ of multisegments is in a 1-1 correspondence $\Upsilon$ to the crystal $B(\infty)$. For each segment $s$, we define 1 or 2-dimensional module $\nabla(s)$ out of the crystal $B_X$. Then a multisegment $m = \{s_1 \leq s_2 \leq s_3 \leq \cdots\}$ of $B_X$ gives the outer tensor product

$$\nabla(m) = \nabla(s_1) \boxtimes \nabla(s_2) \boxtimes \nabla(s_3) \boxtimes \cdots.$$ 

It follows from [1] that

$$\text{hdInd}\nabla(m) = \tilde{f}_{\nu_1}(m) \cdots \tilde{f}_{\nu_n}(m) 1$$

for $m \in \mathcal{M}_X$ and the map $\Psi : \mathcal{M}_X \rightarrow B(\infty)$ defined by

$$\Psi(m) = \text{hdInd}\nabla(m) \quad \text{for} \quad m \in \mathcal{M}_X$$

is bijective (Theorem 3.2). Hence it can be deduced from Proposition 2.3 that the composition $\Psi \circ \Upsilon^{-1} : B(\infty) \rightarrow B(\infty)$ is a crystal isomorphism. In the cyclotomic cases, using the crystal embedding $B(\lambda) \hookrightarrow B(\infty) \otimes T_{\lambda} \otimes C$, we obtain the same results; i.e., the composition $\Psi^\lambda \circ \Upsilon^{-1}_\lambda : B(\lambda) \rightarrow B(\lambda)$ is a crystal isomorphism (Corollary 3.4).

This paper is organized as follows. Section 1 contains a brief review of the crystal basis theory for quantum generalized Kac-Moody algebras. In Section 2, we give the definition of Khovanov-Lauda-Rouquier algebras which is the most general version given in [6] associated with a Borcherds-Cartan datum (or a quiver positively with loops), and introduce some results of Khovanov-Lauda-Rouquier algebras on crystal bases. In Section 3, we restrict to the case of finite classical types and present a crystal basis theoretic construction of irreducible modules over Khovanov-Lauda-Rouquier algebras and their cyclotomic quotients in terms of segments.

1. Quantum generalized Kac-Moody algebras

Let $I$ be an index set. A square matrix $A = (a_{ij})_{i,j \in I}$ is called a symmetrizable Borcherds-Cartan matrix if it satisfies (i) $a_{ii} = 2$ or $a_{ii} \in 2\mathbb{Z}_{\leq 0}$ for $i \in I$, (ii) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$, (iii) $a_{ij} = 0$ if $a_{ji} = 0$ for $i, j \in I$, (vi) there is a diagonal matrix $D = \text{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that $DA$ is symmetric. Let $I^{re} = \{i \in I \mid a_{ii} = 2\}$ and $I^{im} = I \setminus I^{re}$.

A Borcherds-Cartan datum $(A, P, \Pi, \Pi^\vee)$ consists of

1. a symmetrizable Borcherds-Cartan matrix $A$,
2. a free abelian group $P$, called the weight lattice,
3. the set $\Pi = \{\alpha_i \mid i \in I\} \subset P$ of simple roots,
4. the set $\Pi^\vee = \{\alpha^*_i \mid i \in I\} \subset P^\vee := \text{Hom}(P, \mathbb{Z})$ of simple coroots,

which satisfy the following properties:

1. $\langle \alpha_i, \alpha_j \rangle := \alpha_j(\alpha_i) = a_{ij}$ for all $i, j \in I$,
2. $\Pi \subset \mathfrak{h}^*$ is linearly independent, where $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$, 

(iii) for each \( i \in I \), there exists \( \Lambda_i \in P \) such that \( \langle h_j, \Lambda_i \rangle = \delta_{ij} \) for all \( j \in I \).

We denote by \( P^+ = \{ \lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}, \ i \in I \} \) the set of dominant integral weights. The free abelian group \( Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \) is the root lattice, and \( Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \) is the positive root lattice. For \( \alpha = \sum_{i \in I} k_i \alpha_i \in Q^+ \), the height of \( \alpha \) is \( |\alpha| = \sum_{i \in I} k_i \).

We denote by \( p^+ = \{ \lambda \in P | \lambda(h_i) \in \mathbb{Z}_{\geq 0}, \ i \in I \} \) the set of dominant integral weights. The free abelian group \( Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \) is the root lattice, and \( Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \) is the positive root lattice. For \( \alpha = \sum_{i \in I} k_i \alpha_i \in Q^+ \), the height of \( \alpha \) is \( |\alpha| = \sum_{i \in I} k_i \).

Let \( q \) be an indeterminate and \( m, n \in \mathbb{Z}_{\geq 0} \). For \( i \in I^r \), let \( q_i = q^{a_i} \) and

\[
[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{|m|_i!}{|m-n|_i!|n|_i!}.
\]

**Definition 1.1.** The quantum generalized Kac-Moody algebra \( U_q(g) \) associated with a Borcherds-Cartan datum \((A, P, \Pi, \Pi^\vee)\) is the associative algebra over \( \mathbb{Q}(q) \) with 1 generated by \( e_i, f_i \) \((i \in I)\) and \( q^h \) \((h \in P^\vee)\) satisfying following relations:

1. \( q^0 = 1, q^h q^{h'} = q^{h+h'} \) for \( h, h' \in P^\vee \),
2. \( q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, \quad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i \) for \( h \in P^\vee, i \in I \),
3. \( e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \) where \( K_i = q_i^{h_i} \),
4. \( \sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0 \) if \( i \in I^r \) and \( i \neq j \),
5. \( \sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0 \) if \( i \in I^r \) and \( i \neq j \),
6. \( e_i e_j - e_j e_i = 0, \quad f_i f_j - f_j f_i = 0 \) if \( a_{ij} = 0 \).

Note that, if all diagonal entries of \( A \) are 2, then \( A \) is a generalized Cartan matrix and \( U_q(g) \) is the usual quantum group associated with \( A \).

Let \( U_q^+(g) \) (resp. \( U_q^-(g) \)) be the subalgebra of \( U_q(g) \) generated by the elements \( e_i \) (resp. \( f_i \)) for \( i \in I \). For \( n \in \mathbb{Z}_{\geq 0} \), set

\[
e_i^{(n)} = \begin{cases} e_i^n & \text{if } i \in I^r, \\ \frac{e_i^n}{[n]_i!} & \text{if } i \in I^m, \end{cases} \quad f_i^{(n)} = \begin{cases} f_i^n & \text{if } i \in I^r, \\ \frac{f_i^n}{[n]_i!} & \text{if } i \in I^m. \end{cases}
\]

For an element \( u \in U_q^-(g) \), \( u \) can be expressed uniquely as

\[
u = \sum_{k \geq 0} f_i^{(k)} u_k,
\]
where \( u_k \in \ker e'_i \) and \( u_k = 0 \) for \( k \gg 0 \). Here, \( e'_i \) is the endomorphism \( U_q^{-}(\mathfrak{g}) \to U_q^{-}(\mathfrak{g}) \) given in [3, Section 6] and [9, Section 3.3]. The Kashiwara operators \( \tilde{e}_i, \tilde{f}_i \) \((i \in I)\) of \( U_q^{-}(\mathfrak{g}) \) are defined by
\[
\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.
\]

Let \( A_0 = \{ f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q], g(0) \neq 0 \} \).

**Definition 1.2.** A crystal basis of \( U_q^{-}(\mathfrak{g}) \) is a pair \((L, B)\) satisfying the following conditions:

1. \( L \) is a free \( A_0 \)-module of \( U_q^{-}(\mathfrak{g}) \) such that \( U_q^{-}(\mathfrak{g}) = \mathbb{Q}(q) \otimes_{A_0} L \) and \( L = \bigoplus_{\alpha \in Q^+} L_{-\alpha} \), where \( L_{-\alpha} := L \cap U_q^{-}(\mathfrak{g})_{-\alpha} \).
2. \( B \) is a \( \mathbb{Q} \)-basis of \( L/qL \) such that \( B = \bigcup_{\alpha \in Q^+} B_{-\alpha} \), where \( B_{-\alpha} := B \cap (L_{-\alpha}/qL_{-\alpha}) \).
3. \( \tilde{e}_i B \subset B \cup \{ 0 \}, \tilde{f}_i B \subset B \) for all \( i \in I \).
4. For \( b, b' \in B \) and \( i \in I \), \( b' = \tilde{f}_i b \) if and only if \( b = \tilde{e}_i b' \).

Let \( L(\infty) \) be the free \( A_0 \)-module of \( U_q^{-}(\mathfrak{g}) \) generated by \( \{ \tilde{f}_i, \tilde{f}_i 1 \mid r \geq 0, i \in I \} \) and let
\[
B(\infty) = \{ \tilde{f}_i, \tilde{f}_i 1 + qL(\infty) \mid r \geq 0, i \in I \} \setminus \{ 0 \}.
\]

Then, it is proved in [3, 9] that the pair \((L(\infty), B(\infty))\) is a unique crystal basis of \( U_q^{-}(\mathfrak{g}) \).

Let \( M \) be a \( U_q(\mathfrak{g}) \)-module in the category \( \mathcal{O}_{int} \) defined in [3, Definition 3.1]. For any \( i \in I \), any element \( u \in M_{\mu} \) can be expressed uniquely as
\[
u = \sum_{k \geq 0} f_i^{(k)} u_k,
\]
where \( u_k \in M_{\mu+ka_i} \cap \ker e_i \). The Kashiwara operators \( \tilde{e}_i, \tilde{f}_i \) \((i \in I)\) are defined by
\[
\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.
\]

**Definition 1.3.** A crystal basis of \( U_q(\mathfrak{g}) \)-module \( M \) is a pair \((L, B)\) satisfying the following conditions:

1. \( L \) is a free \( A_0 \)-module of \( M \) such that \( M = \mathbb{Q}(q) \otimes_{A_0} L \) and \( L = \bigoplus_{\lambda \in P} L_{\lambda} \), where \( L_{\lambda} := L \cap M_{\lambda} \).
2. \( B \) is \( \mathbb{Q} \)-basis of \( L/qL \) such that \( B = \bigcup_{\lambda \in P} B_{\lambda} \), where \( B_{\lambda} := B \cap L_{\lambda}/qL_{\lambda} \).
3. \( \tilde{e}_i B \subset B \cup \{ 0 \}, \tilde{f}_i B \subset B \) for all \( i \in I \).
4. For \( b, b' \in B \) and \( i \in I \), \( b' = \tilde{f}_i b \) if and only if \( b = \tilde{e}_i b' \).

For a dominant integral weight \( \lambda \in P^+ \), we denote by \( V(\lambda) \) the irreducible highest weight \( U_q(\mathfrak{g}) \)-module with highest weight \( \lambda \). Note that \( V(\lambda) \) is contained in \( \mathcal{O}_{int} \). Let \( L(\lambda) \) be the free \( A_0 \)-module of \( V(\lambda) \) generated by \( \{ \tilde{f}_i, \tilde{f}_i v_\lambda \mid r \geq 0, i \in I \} \) and let
\[
B(\lambda) = \{ \tilde{f}_i, \tilde{f}_i v_\lambda + qL(\lambda) \mid r \geq 0, i \in I \} \setminus \{ 0 \}.
\]
Then the pair \((L(\lambda), B(\lambda))\) is a unique crystal basis of \(V(\lambda)\) \([3, 9]\).

2. KHOVANOV-LAUDA-ROUQUIER ALGEBRAS

Let \(k = \bigoplus_{n \in \mathbb{Z}} k_n\) be a commutative graded ring such that \(k_0\) is a field and \(k_n = 0\) for \(n < 0\). For \(\alpha \in Q^+\) with \(|\alpha| = m\), let

\[I^\alpha = \{ \nu = (\nu_1, \ldots, \nu_m) \in I^m \mid |\alpha| = m\} = \{ \nu = (\nu_1, \ldots, \nu_m) \in I^m \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_m} = \alpha \}.\]

Note that the symmetric group \(S_m = \langle s_k \mid k = 1, \ldots, m-1 \rangle\) acts naturally on \(I^\alpha\).

For each \(i \in I\), we choose a polynomial \(P_i(u, v) \in k[u, v]\) of the form

\[P_i(u, v) = \sum_{k, l \geq 0} p_{i;k,l} u^k v^l,\]

where \(p_{i;k,l} \in k_{d_i(2-a)-2d_i k-2d_i l}\) and \(p_{i;0,0}, p_{i;0,1}-, p_{i,2} \in k_0^\times\). We also take a matrix \((Q_{ij}(u, v))_{i,j \in I}\) in \(k[u, v]\) such that \(Q_{ij}(u, v) = Q_{ji}(v, u)\) and \(Q_{ij}(u, v)\) has the form

\[Q_{ij}(u, v) = \begin{cases} 0 & \text{if } i = j, \\ \sum_{k, l \geq 0} q_{ij;k,l} u^k v^l & \text{if } i \neq j. \end{cases}\]

where \(q_{ij;k,l} \in k_{-2(\alpha_i|\alpha_j)-2d_i k-2d_j l}\) and \(q_{i,j;\alpha_i|\alpha_j} \in k_0^\times\).

Definition 2.1. Let \((A, P, \Pi, \Pi^\vee)\) be a Borcherds-Cartan datum and \(\alpha \in Q^+\) with height \(m\).

The Khovanov-Lauda-Rouquier algebra \(R(\alpha)\) of weight \(\alpha\) associated with the data \((A, P, \Pi, \Pi^\vee)\), \((P_i)_{i \in I}\) and \((Q_{ij})_{i,j \in I}\) is the associative graded \(k\)-algebra generated by \(e(\nu) (\nu = (\nu_1, \ldots, \nu_m) \in I^\alpha), x_k (1 \leq k \leq m), \tau_t (1 \leq t \leq m-1)\) satisfying the following defining relations:

\[e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu \in I^\alpha} e(\nu) = 1, \quad x_ke(\nu) = e(\nu)x_k, \quad x_kx_l = x_lx_k,\]

\[\tau_t e(\nu) = e(s_t(\nu))\tau_t, \quad \tau_t\tau_s = \tau_s\tau_t \text{ if } |t-s| > 1,\]

\[\tau_t^2 e(\nu) = \begin{cases} \partial_t P_{\nu}(x_t, x_{t+1})\tau_t e(\nu) & \text{if } \nu_t = \nu_{t+1}, \\ Q_{\nu_t\nu_{t+1}}(x_t, x_{t+1})e(\nu) & \text{if } \nu_t \neq \nu_{t+1}, \end{cases}\]

\[(\tau_t x_k - x_{s(t)}\tau_t)e(\nu) = \begin{cases} -P_{\nu_t}(x_t, x_{t+1})e(\nu) & \text{if } k = t \text{ and } \nu_t = \nu_{t+1}, \\ P_{\nu_t}(x_t, x_{t+1})e(\nu) & \text{if } k = t + 1 \text{ and } \nu_t = \nu_{t+1}, \\ 0 & \text{otherwise}, \end{cases}\]

where \(\partial_t f = \frac{s_t f - f}{x_t - x_{t+1}}\), where \(w f(x_1, \ldots, x_m) = f(x_{w(1)}, \ldots, x_{w(m)})\) for \(w \in S_m\) and \(f(x_1, \ldots, x_m) \in k[x_1, \ldots, x_m]\).
\[(\tau_{t+1}\tau_{t+1} - \tau_{t}\tau_{t+1}\tau_{t})e(\nu) = \begin{cases} P_{\nu_{t}}(x_{t}, x_{t+2})\overline{Q}_{\nu_{t}, \nu_{t+1}}(x_{t}, x_{t+1}, ..., x_{t+1}, x_{t+2})\tau_{t+1}e(\nu) & \text{if } \nu_{t} = \nu_{t+1} = \nu_{t+2}, \\ 0 & \text{otherwise,} \end{cases}\]

where
\[
\overline{P}_{i}'(u, v, w) := \frac{P_{i}(v, u)P_{i}(u, w)}{(u-v)(u-w)} + \frac{P_{i}(u, w)P_{i}(v, w)}{(u-w)(v-w)} - \frac{P_{i}(u, v)P_{i}(v, w)}{(u-v)(v-w)},
\]
\[
\overline{P}_{i}''(u, v, w) := -\frac{P_{i}(u, v)P_{i}(u, w)}{(u-v)(u-w)} - \frac{P_{i}(u, w)P_{i}(w, v)}{(u-w)(v-w)} + \frac{P_{i}(u, v)P_{i}(v, w)}{(u-v)(v-w)},
\]
\[
\overline{Q}_{i,j}(u, v, w) := \frac{Q_{i,j}(u, v) - Q_{i,j}(w, v)}{u-w}.
\]

The algebra $R(\alpha)$ has the Z-grading given by
\[
\deg(e(\nu)) = 0, \quad \deg(x_{k}(\nu)) = 2d_{\nu_{k}}, \quad \deg(\tau_{t}(\nu)) = -\langle \alpha_{\nu_{t}}, \alpha_{\nu_{t+1}} \rangle,
\]
where $x_{k}(\nu) = x_{k}e(\nu)$ and $\tau_{t}(\nu) = \tau_{t}e(\nu)$ for $\nu \in I^{\alpha}$. A diagrammatic presentation of $R(\alpha)$ using planar diagrams with dots and strands is given in [7, 11, 12].

For $\lambda \in \mathbb{P}^{+}$ and $i \in I$, let us fix a polynomial $a_{i}^{\lambda}(x)$ of the form
\[
a_{i}^{\lambda}(x) = \sum_{k=0}^{\lambda(h_{i})}c_{i; k}^{\lambda}x^{\lambda(h_{i})-k},
\]
where $c_{i; k}^{\lambda} \in k_{2d_{i}k}$ and $c_{i; 0}^{\lambda} = 1$. Set $a^{\lambda}(x) = \sum_{\nu \in I}a_{\nu_{1}}^{\lambda}(x)e(\nu)$. Then the cyclotomic Khovanov-Lauda-Rouquier algebra $R^{\lambda}(\alpha)$ is defined to be the quotient algebra
\[
R^{\lambda}(\alpha) = R(\alpha) / R(\alpha)a^{\lambda}(x)R(\alpha).
\]

Let $R(\alpha)$-mod (resp. $R^{\lambda}(\alpha)$-mod) be the category of finite-dimensional graded left $R(\alpha)$-modules (resp. finite-dimensional graded left $R^{\lambda}(\alpha)$-modules). For a Z-graded module $M = \bigoplus_{k \in \mathbb{Z}} M_{k}$ and $t \in \mathbb{Z}$, let $M(t) = \bigoplus_{k \in \mathbb{Z}} M(t)_{k}$ be the Z-graded module obtained from $M$ by setting $M(t)_{k} := M_{t+k}$. The q-character $\text{ch}_{q}(M)$ and character $\text{ch}(M)$ of $M$ are defined by
\[
\text{ch}_{q}(M) = \sum_{\nu \in I^{\alpha}} \dim_{q}(e(\nu)M) \nu, \quad \text{ch}(M) = \sum_{\nu \in I^{\alpha}} \dim(e(\nu)M) \nu,
\]
where $\dim_{q}(N) := \sum_{i \in \mathbb{Z}} (\dim_{i}N)q^{i}$ for any graded module $N = \bigoplus_{i \in \mathbb{Z}} N_{i}$.

For $M, N \in R(\alpha)$-mod, let $\text{Hom}(M, N)$ be the set of homogeneous homomorphisms of degree 0, and let $\text{HOM}(M, N) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}(M, N(k))$. For $\beta_{1}, \ldots, \beta_{m} \in \mathbb{Q}^{+}$, we define
\[
e(\beta_{1}, \ldots, \beta_{m}) = \sum_{\nu_{j} \in I^{\beta_{j}}} e(\nu_{1} \cdots \nu_{m}),
\]
where $\nu_1 \cdots \nu_m$ is the concatenation of $\nu_k$'s. The natural embedding $R(\beta_1) \otimes \cdots \otimes R(\beta_m) \hookrightarrow R(\beta_1 + \cdots + \beta_m)$ gives the functors

$$\text{Ind}_{\beta_1, \ldots, \beta_m} : R(\beta_1) \otimes \cdots \otimes R(\beta_m)\text{-}\text{mod} \to R(\beta_1 + \cdots + \beta_m)\text{-}\text{mod},$$

$$\text{Res}_{\beta_1, \ldots, \beta_m} : R(\beta_1 + \cdots + \beta_m)\text{-}\text{mod} \to R(\beta_1) \otimes \cdots \otimes R(\beta_m)\text{-}\text{mod}$$

defined by $\text{Ind}_{\beta_1, \ldots, \beta_m} L := R(\beta_1 + \cdots + \beta_m) \otimes_{R(\beta_1) \otimes \cdots \otimes R(\beta_m)} L$ and $\text{Res}_{\beta_1, \ldots, \beta_m} M := e(\beta_1, \ldots, \beta_m)M$ for $L \in R(\beta_1) \otimes \cdots \otimes R(\beta_m)\text{-}\text{mod}$ and $M \in R(\beta_1 + \cdots + \beta_m)\text{-}\text{mod}$.

Let $\mathcal{B}(\infty)$ (resp. $\mathfrak{B}(\lambda)$) be the set of all isomorphism classes of irreducible graded $R(\beta)$-modules (resp. $R^\lambda(\beta)$-modules) for all $\beta \in Q^+$. Set 1 to be the 1-dimensional trivial $R(0)$-module. For $M \in R(\beta)$-mod, we define

$$\tilde{e}_i(M) = \text{soc}(\text{Res}_{\alpha_i, \beta - \alpha_i} M) \in R(\beta - \alpha_i)\text{-}\text{mod},$$

$$\tilde{f}_i(M) = \text{hd} \text{Ind}_{\alpha_i, \beta}(L(i) \otimes M) \in R(\beta + \alpha_i)\text{-}\text{mod},$$

$$\text{wt}(M) = \begin{cases} -\beta & \text{if } M \in R(\beta)\text{-}\text{mod}, \\ \lambda - \beta & \text{if } M \in R^\lambda(\beta)\text{-}\text{mod}, \end{cases}$$

$$\varphi_i(M) = \text{wt}(M)(h_i) - \epsilon_i(M).$$

where $L(i)$ is the 1-dimensional $R(\alpha_i)$-module with $\dim_q L(i) = 1$. Here, for an $R(\alpha)$-module $N$, $\text{soc}(N)$ (resp. $\text{hd}(N)$) is the maximal completely reducible submodule (resp. the maximal completely reducible quotient) of $N$. Then, when $a_{ii} \neq 0$ for all $i \in I$, the sets $\mathcal{B}(\infty)$ and $\mathfrak{B}(\lambda)$ with the above maps have crystal structures [7, 14].

**Theorem 2.2** ([7, 14]). If $a_{ii} \neq 0$ for all $i \in I$, then the crystal $\mathcal{B}(\infty)$, wt, $\tilde{e}_i, \tilde{f}_i, \epsilon_i, \varphi_i$ (resp. $\mathfrak{B}(\lambda)$, wt, $\tilde{e}_i, \tilde{f}_i, \epsilon_i, \varphi_i$) is isomorphic to the crystal $B(\infty)$ (resp. $B(\lambda)$).

We now assume that $a_{ii} \neq 0$ for all $i \in I$. Let $n = |I|$ be the rank of $U_q(\mathfrak{g})$, and let $I(i) (i = 1, \ldots, n)$ be subsets of $I = I_{(n+1)}$ such that $I(i) \subset I_{(k+1)}$ and $|I(i)| = k$ for all $k$. Let $U_k$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, (i \in I(i))$ and $q^h$ ($h \in P^\vee$) and let $B_k$ be the crystal obtained from $B(\infty)$ by forgetting the i-arrows for $i \notin I(i)$. Then $B_k$ can be understood as a $U_k$-crystal and every connected component of $B_k$ has a unique highest weight vector [1, Lemma 1.9].

Take an element $v \in B(\infty)$. Let $u_0 = v$ and let $u_k$ be the highest weight vector of the connected component $C_k$ of $B_k$ containing $v$ for $k = 1, \ldots, n$. By construction, there is a chain of injective maps

$$C_1 \hookrightarrow C_2 \hookrightarrow \cdots \hookrightarrow C_{n-1} \hookrightarrow B(\infty).$$

For $k = 1, \ldots, n$, let

$$N_k(v) = \tilde{f}_{i_{k,1}} \cdots \tilde{f}_{i_{k,k}} 1 \in \mathcal{B}(\infty).$$

(2.1)
where $\nu_k = (\nu_k,1, \ldots , \nu_k,t_k)$ is a sequence of $I$ such that $u_{k-1} = \tilde{f}_{\nu_{k,1}} \cdots \tilde{f}_{\nu_{k,t_k}} u_k$.

Hence, for each $v \in B(\infty)$, we obtain the corresponding $n$-tuple $(N_1(v), N_2(v), \ldots , N_n(v))$ of irreducible modules in $\mathfrak{B}(\infty)$. Then, using the same argument as in [1, Proposition 1.10], one can obtain the following proposition.

**Proposition 2.3.** [1, Proposition 1.10]

1. For $v \in B(\infty)$, $\text{hd Ind} (\mathfrak{B}_{k=1}^n N_k(v))$ is irreducible.
2. The map $\Phi : B(\infty) \rightarrow \mathfrak{B}(\infty)$ defined by
   $$\Phi(v) = \text{hd Ind} (\mathfrak{B}_{k=1}^n N_k(v)) \quad \text{for } v \in B(\infty)$$

is a crystal isomorphism.

3. **Crystal bases and irreducible representations**

In this section, we give a crystal basis theoretic construction of irreducible modules over Khovanov-Lauda-Rouquier algebras and their cyclotomic quotients for finite classical types. Though this section is based on our previous work [1], the description of irreducible modules in this section is different and more combinatorial than the description given in [1]. Throughout this section, we assume that $k = \mathbb{C}$ and $A$ is a generalized Cartan matrix of finite classical type $A_n$, $B_n$, $C_n$ and $D_n$.

Let $I = \{1, 2, \ldots , n\}$ and let $\Gamma$ be the following Dynkin diagram:

- $(A_n)$
  $$
  \begin{array}{c}
  1 \quad 2 \quad \cdots \quad n-1 \quad n \\
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  $$

- $(B_n)$
  $$
  \begin{array}{c}
  1 \quad 2 \quad \cdots \quad n-1 \quad n \\
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  $$

- $(C_n)$
  $$
  \begin{array}{c}
  1 \quad 2 \quad \cdots \quad n-1 \quad n \\
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  $$

- $(D_n)$
  $$
  \begin{array}{c}
  1 \quad 2 \quad \cdots \quad n-2 \quad n-1 \quad n \\
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  $$

We set $\mathfrak{B}_X (X = A, B, C, D)$ to be the crystal defined by

- $(A_n)$
  $$
  \begin{array}{c}
  1 \quad 1 \quad 2 \quad 2 \quad \cdots \quad n-1 \quad n \quad n+1 \\
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  $$

- $(B_n)$
  $$
  \begin{array}{c}
  1 \quad 1 \quad \cdots \quad n-1 \quad n \quad n \quad n \quad n-1 \quad \cdots \quad 1 \quad 1 \\
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  $$

- $(C_n)$
  $$
  \begin{array}{c}
  1 \quad 1 \quad \cdots \quad n-2 \quad n-1 \quad n-1 \quad n \quad n \quad n \quad n \quad n-1 \quad n-2 \quad \cdots \quad 1 \quad 1 \\
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  \begin{array}{c}
  \text{-----}
  \end{array}
  $$
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with the entries ordered by

$(A_n) \quad \tilde{f}_{\nu}^{k} = \tilde{f}_{\nu_{1}}^{k_{1}} \cdots \tilde{f}_{\nu_{m}}^{k_{m}}$

$(B_n) \quad \tilde{e}_{\nu}^{k} = \tilde{e}_{\nu_{1}}^{k_{1}} \cdots \tilde{e}_{\nu_{m}}^{k_{m}}$

For $\nu = (\nu_1, \ldots, \nu_m) \in I^m$ and $k = (k_1, \ldots, k_m) \in (\mathbb{Z}_{\geq 0})^m$, let $\tilde{f}_{\nu}^{k} = \tilde{f}_{\nu_{1}}^{k_{1}} \cdots \tilde{f}_{\nu_{m}}^{k_{m}}$ (resp. $\tilde{e}_{\nu}^{k} = \tilde{e}_{\nu_{1}}^{k_{1}} \cdots \tilde{e}_{\nu_{m}}^{k_{m}}$). If $k = (1, \ldots, 1)$, then we write $\tilde{f}_{\nu}$ (resp. $\tilde{e}_{\nu}$) for $\tilde{f}_{\nu}^{k}$ (resp. $\tilde{e}_{\nu}^{k}$).

For $b \in B_X$ and $X = B, C, D$, we set $b^\vee$ to be a unique element in $B_X$ such that $\operatorname{wt}(b^\vee) = -\operatorname{wt}(b)$. Let $b_X = \overline{n}$ ($X = A, B, C$), $b_X = n - \overline{1}$ ($X = D$). A multisegment $\underline{m}$ of $B_X$ is a multiset of segments of the crystal $B_X$ such that

(i) $h(s) \succeq b_X$ for $s \in \underline{m}$,

(ii) $t(s) \succeq h(s)\vee$ if $X = B, C, D$,

(iii) any two segments in $\underline{m}$ are comparable.

When no confusion can arise, we write $\underline{m} = \{s_1 \preceq s_2 \preceq s_3 \preceq \cdots \}$. For $i = 1, \ldots, n$, we set

$\underline{m}(i) = \{s \in \underline{m} \mid \tilde{e}_{i}(h(s)) \neq 0\}$.

Note that $\underline{m} = \bigcup_{i \in I} \underline{m}(i)$.

Let $\mathcal{M}_{X}$ be the set of all multisegments of $B_X$. We will show that $\mathcal{M}_{X}$ parameterizes the crystal $B(\infty)$. Let $\ell_A = n$, $\ell_B = \ell_C = 2n - 1$, $\ell_D = 2n - 2$ and let $\nu_X$ be the sequence of the
colors over the arrows of $B_X$, i.e.,
\[ \nu_A = (1, 2, \ldots, n-1, n), \]
\[ \nu_B = \nu_C = (1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1), \]
\[ \nu_D = (1, 2, \ldots, n-1, n, n-2, \ldots, 2, 1). \]

For a segment $s$, let $e(s) = (e_1, \ldots, e_{\ell_X}) \in Z^{\ell_X}$ be the \( \ell_X \)-tuple of $Z$ defined by
\[ h(s) = \tilde{f}_{\nu(s)}^{e(s)}, \quad e_j \neq 0 \text{ for some } 1 \leq j \leq n. \]

Here we consider $Z^{\ell_X}$ as an abelian group. Let $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ for $i = 1, \ldots, \ell_X$.

**Proposition 3.1.** Let $\mathcal{T} : \mathcal{M}_X \rightarrow B(\infty)$ be the map defined by
\[ \mathcal{T}(m) = \tilde{f}_{\nu_{X}}^{e_1(m)} \ldots \tilde{f}_{\nu_{X}}^{e_n(m)} \]
where
\[ e_i(m) = \begin{cases} \sum_{s \in M_{n+1-i}} e(s) & \text{if either } X = A, B, C \text{ or } i \neq 1, 2 (X = D), \\ n(n-1) & \text{if } i = 2 (X = D), \\ n & \text{if } i = 1 (X = D). \end{cases} \]

Then the map $\mathcal{T}$ is bijective.

**Proof.** We focus on the case $X = D$ since the remaining cases can be proved in a similar manner. Let $S$ be the set of $t = (t_1, \ldots, t_n) \in Z^{n(2n-2)}_{\geq 0}$ such that
\begin{enumerate}
\item[(a)] $t_j = (t_{j,1}, \ldots, t_{j,2n-2}) \in Z^{2n-2}_{\geq 0}$ for $j = 1, \ldots, n$,
\item[(b)] $t_{j,n+1-j} \geq t_{j,n+2-j} \geq \cdots \geq t_{j,n-1}, t_{j,n} \geq t_{j,n+1} \geq \cdots \geq t_{j,n-2+j}$ for $j = 3, \ldots, n$,
\item[(c)] if $j = 3, \ldots, n$, then $t_{j,k} = 0$ for either $k < n + 1 - j$ or $k > n - 2 + j$,
\item[(d)] $t_{2,k} = 0$ for $k \neq n - 1$ and $t_{1,k} = 0$ for $k \neq n$.
\end{enumerate}

It follows from [15, Section 7] that the map $S \rightarrow B(\infty)$ mapping $t = (t_1, \ldots, t_n) \in S$ to $\tilde{f}_{\nu_{D}}^{t_{1}} \ldots \tilde{f}_{\nu_{D}}^{t_{n}} 1 \in B(\infty)$ is bijective.

Let $m \in \mathcal{M}_X$ and write $e_j(m) = (e_{j,1}, \ldots, e_{j,2n-2})$ for $j = 1, \ldots, n$. By (ii) and (iii) of (3.1), the sequences $e_j(m)$ satisfy the above condition (b), (c) and (d). Hence the map $\phi : \mathcal{M}_X \rightarrow S$ given by
\[ \phi(m) = (e_{1}(m), \ldots, e_{n}(m)) \quad (m \in \mathcal{M}_X) \]
is well-defined.

On the other hand, let
\[ \tilde{\iota}_i = \begin{cases} \tilde{i} & \text{if } 1 \leq i \leq n, \\ 2n + 1 - i & \text{if } n + 1 \leq i \leq 2n. \end{cases} \]
and, for a segment $s$ and a multisegment $\mathfrak{m} \in \mathcal{M}_X$, let us denote by $\sigma(\mathfrak{m}, s)$ the multiplicity of $s$ in $\mathfrak{m}$. Let $\psi: \mathcal{S} \rightarrow \mathcal{M}_X$ be the map mapping $t \in \mathcal{S}$ to the multisegment $\psi(t) \in \mathcal{M}_X$ such that, if $i = 1, \ldots, n - 2$, then

$$
\sigma(\psi(t), \{\overline{i}, \overline{j+1}\}) = \begin{cases} t_{n+1-i,j} - t_{n+1-i,j+1} & \text{if } j \leq n - 3, \\
t_{n+1-i,n-2} - \max\{t_{n+1-i,n-1}, t_{n+1-i,n}\} & \text{if } j = n - 2, \\
\max\{0, t_{n+1-i,n-1} - t_{n+1-i,n}\} & \text{if } j = n - 1, \\
\max\{0, t_{n+1-i,n} - t_{n+1-i,n-1}\} & \text{if } j = n, \\
\min\{t_{n+1-i,n-1}, t_{n+1-i,n}\} - t_{n+1-i,n+1} & \text{if } j = n + 1, \\
t_{n+1-i,j-1} - t_{n+1-i,j} & \text{if } j \geq n + 2.
\end{cases}
$$

and, if $i = n - 1$, then

$$
\sigma(\psi(t), \{n-1, \overline{j+1}\}) = \begin{cases} \max\{0, t_{2,n-1} - t_{1,n}\} & \text{if } j = n - 1, \\
\max\{0, t_{1,n} - t_{2,n-1}\} & \text{if } j = n, \\
\min\{t_{1,n}, t_{2,n-1}\} & \text{if } j = n + 1.
\end{cases}
$$

Then it is straightforward to verify that $\phi \circ \psi = \text{id}_S$ and $\psi \circ \phi = \text{id}_{\mathcal{M}_X}$.

We now return to the Khovanov-Lauda-Rouquier algebras. Let $1$ be the trivial $R(0)$-module. For a segment $s$, we define

$$
\nabla(s) = \tilde{f}_{\nu_X}^{e(s)} 1 \in \mathfrak{B}(\infty).
$$

We give a explicit description of the module structure of $\nabla(s)$ as follows.

Let $l = \left| \text{wt}(h(s)) - \text{wt}(t(s)) \right|$. If one of the following holds: $h(s) \succ t(s)$ $(A_n, C_n)$, either $t(s) \succeq \overline{0}$ or $0 \succeq h(s)$ $(B_n)$, either $t(s) \succeq n - 1$ or $\overline{n-1} \succeq h(s)$ $(D_n)$, then the module $\nabla(s)$ is the $1$-dimensional module $\mathbb{C}v$ given by

$$
x_i v = 0, \quad \tau_j v = 0, \quad e(\nu)v = \begin{cases} v & \text{if } \nu = \nu(s), \\
0 & \text{otherwise},
\end{cases}
$$

where $\nu(s) \in I^l$ such that $h(s) = \tilde{f}_{\nu(s)} t(s)$.

If $h(s) \succ \overline{0} \succeq t(s)$ for type $B_n$, then $\nabla(s)$ is the $2$-dimensional module $\mathbb{C}u \oplus \mathbb{C}v$ with

$$
x_i u = 0, \quad \tau_j u = \begin{cases} v & \text{if } j = d, \\
0 & \text{otherwise},
\end{cases} \quad e(\nu)u = \begin{cases} u & \text{if } \nu = \nu(s), \\
0 & \text{otherwise},
\end{cases}
$$

$$
x_i v = \begin{cases} u & \text{if } i = d, \\
-u & \text{if } i = d + 1,
\end{cases} \quad \tau_j v = 0, \quad e(\nu)v = \begin{cases} u & \text{if } \nu = \nu(s), \\
0 & \text{otherwise},
\end{cases}
$$

where $\nu(s) \in I^l$ such that $h(s) = \tilde{f}_{\nu(s)} t(s)$, and $d$ is an integer such that $s_d(\nu(s)) = \nu(s)$.
If \( h(s) \geq n - 1 \) and \( n - 1 \geq t(s) \) for type \( D_n \), then the module \( \nabla(s) \) is the 2-dimensional module \( \mathbb{C}u \oplus \mathbb{C}v \) defined by

\[
x_i u = 0, \quad \tau_j u = \begin{cases} v & \text{if } j = d, \\ 0 & \text{otherwise}, \end{cases} \quad e(\nu)u = \begin{cases} \mathcal{Q}_{n-1,n}(x_{n-1}, x_n)u & \text{if } \nu = \nu^+, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
x_i v = 0, \quad \tau_j v = \begin{cases} u & \text{if } j = d, \\ 0 & \text{otherwise}, \end{cases} \quad e(\nu)v = \begin{cases} v & \text{if } \nu = \nu^-, \\ 0 & \text{otherwise}. \end{cases}
\]

where \( \nu^+, \nu^- \in \mathfrak{l}^t \) such that \( \nu^+ \neq \nu^- \) and \( h(s) = \tilde{f}_{\nu^+} + t(s) = \tilde{f}_{\nu^-} - t(s) \), and \( d \) is an integer such that \( s_d(\nu^+) = \nu^- \). Note that \( \mathcal{Q}_{n-1,n}(x_{n-1}, x_n) \in \mathbb{C}^* \). For the description above, the character \( \chi \nabla(s) \) is given as follows

\[
(3.2) \quad \chi \nabla(s) = \begin{cases} \nu^+ + \nu^- & \text{if } h(s) \geq n - 1, n - 1 \geq t(s) \quad (D_n), \\ 2\nu(s) & \text{if } h(s) > 0 > t(s) \quad (B_n), \\ \nu(s) & \text{otherwise}. \end{cases}
\]

For a multisegment \( m = \{s_1 \leq s_2 \leq s_3 \leq \cdots \} \), we define

\[
\nabla(m) = \nabla(s_1) \otimes \nabla(s_2) \otimes \nabla(s_3) \otimes \cdots.
\]

Then we have the following theorem.

**Theorem 3.2.** [1, Theorem 3.2]

1. For a multisegment \( m \), we have

\[
\text{hdInd} \nabla(m) = \tilde{f}_{e_1} \cdots \tilde{f}_{e_n} \mathbf{1},
\]

where \( e_i(m) = \begin{cases} \sum_{s \in \mathcal{m}(n+1-i)} e(s) & \text{if either } X = A, B, C \text{ or } i \neq 1, 2 \quad (X = D), \\ \#\mathcal{m}(n-1)e_{n-1} & \text{if } i = 2 \quad (X = D), \\ \#\mathcal{m}(n)e_n & \text{if } i = 1 \quad (X = D). \end{cases} \]

2. Let \( \Psi : \mathcal{M}_X \rightarrow \mathfrak{B}(\infty) \) be the map defined by

\[
\Psi(m) = \text{hdInd} \nabla(m) \quad \text{for } m \in \mathcal{M}_X.
\]

Then the map \( \Psi \) is bijective.

**Proof.** We give a sketch of the proof of [1, Theorem 3.2]. When \( X = D \), without loss of generality, we may assume that

\[
\sigma(m, \{n-1, n\}) \geq \sigma(m, \{n-1, n\}),
\]

where \( \sigma(m, s) \) is the multiplicity of \( s \) in \( m \). Note that \( \sum_{s \in \mathcal{m}(n-1)} e(s) = \#\mathcal{m}(n-1)e_{n-1} + \#\mathcal{m}(n)e_n \) if \( X = D \). It follows from [1, Lemma 4.3] and the definition of \( m(k) \) that

1. \( \varepsilon_i(\text{Ind} \nabla(m(k))) = 0 \) for \( i = n + 1 - k, n + 2 - k, \ldots, n \),
(b) \( \text{hdInd} \nabla(m(k)) = \begin{cases} \tilde{f}_{\nu_{X}^{n+1-k}}^{e_{1}} & \text{if } k = 1, \ldots, n \, (X = A, B, C), \\ \tilde{f}_{\nu_{X}^{n+1-k}}^{e_{2}} & \text{if } k = 1, \ldots, n-2 \, (X = D), \\ \tilde{f}_{\nu_{X}^{n+1-k}}^{e_{n}} & \text{if } k = n-1 \, (X = D). \end{cases} \)

Let \( I_{(k)} = \{ n+1-k, \ldots, n \} \) and let \( \mathcal{N}_{k} \) be the module given in (2.1). Then, by the crystal description given in [15], we have

\[
\text{hdInd} \nabla(m(k)) \simeq \begin{cases} \mathcal{N}_{n+1-k} & \text{if } k = 1, \ldots, n \, (X = A, B, C), \\ \text{Ind}(\mathcal{N}_{1} \otimes \mathcal{N}_{2}) & \text{if } k = n-1 \, (X = D). \end{cases}
\]

Combining Proposition 2.3 and [1, Lemma 1.8] with the above conditions (a) and (b), we obtain

\[
\text{hdInd} \nabla(m) \simeq \text{hdInd} \nabla(m),
\]

where \( n' = n \, (X = A, B, C) \) and \( n' = n-1 \, (X = D) \). Therefore, the assertion follows from Proposition 2.3. \( \square \)

From Proposition 3.1 and Theorem 3.2, we have the following corollary.

**Corollary 3.3.** The composition \( \Psi \circ \Upsilon^{-1} : B(\infty) \longrightarrow \mathfrak{B}(\infty) \) is a crystal isomorphism.

Let \( \lambda \in P^{+} \) be the dominant integral weight and let \( B(\lambda) \) be the crystal of the irreducible highest weight module \( V(\lambda) \). It was shown in [10] that there is a unique strict crystal embedding

\[
B(\lambda) \hookrightarrow B(\infty) \otimes T_{\lambda} \otimes C, \quad v_{\lambda} \mapsto 1 \otimes t_{\lambda} \otimes c
\]

where \( v_{\lambda} \) is the highest weight vector of \( B(\lambda) \). Here, \( T_{\lambda} = \{ t_{\lambda} \} \) (resp. \( C = \{ c \} \)) is a crystal with \( \text{wt}(t_{\lambda}) = \lambda, \epsilon_{i}(t_{\lambda}) = \phi_{i}(t_{\lambda}) = 0, \tilde{e}_{i}t_{\lambda} = \tilde{f}_{i}t_{\lambda} = -\infty \) (resp. \( \text{wt}(t_{\lambda}) = 0, \epsilon_{i}(t_{\lambda}) = \phi_{i}(t_{\lambda}) = 0, \tilde{e}_{i}t_{\lambda} = \tilde{f}_{i}t_{\lambda} = 0 \)). We denote by \( \iota_{\lambda} \) the composition of the strict embedding and the natural projection:

\[
B(\lambda) \hookrightarrow B(\infty) \otimes T_{\lambda} \otimes C \rightarrow B(\infty).
\]

Let \( \mathcal{M}_{X}(\lambda) = \Upsilon^{-1} \circ \iota_{\lambda}(B(\lambda)) \). By Proposition 3.1, the set \( B(\lambda) \) is in 1-1 correspondence to \( \mathcal{M}_{X}(\lambda) \) via \( \Upsilon^{-1} \circ \iota_{\lambda} \). Then the map \( \Upsilon_{\lambda} := (\Upsilon^{-1} \circ \iota_{\lambda})^{-1} : \mathcal{M}_{X}(\lambda) \rightarrow B(\lambda) \) is given by

\[
\Upsilon_{\lambda}(m) = \tilde{f}_{\nu_{X}^{n}}^{e_{1}} \cdots \tilde{f}_{\nu_{X}^{n}}^{e_{n}} v_{\lambda} \quad \text{for } m \in \mathcal{M}_{X}(\lambda),
\]

where \( e_{i}(m) = \begin{cases} \sum_{s \in m(n+1-i)} e(s) & \text{if either } X = A, B, C \text{ or } i \neq 1, 2 \, (X = D), \\ \#m(n-1)e_{n-1} & \text{if } i = 2 \, (X = D), \\ \#m(n)e_{n} & \text{if } i = 1 \, (X = D). \end{cases} \)
We remark that the set $\mathcal{M}_X (\lambda)$ can be described explicitly from the string parametrization of $B(\lambda)$ given in [15]. By Theorem 3.2 and Corollary 3.3, we have the following Corollary.

**Corollary 3.4.** Let $\lambda \in P^+$ be a dominant integral weight.

1. Let $\Psi^\lambda : \mathcal{M}_X (\lambda) \rightarrow \mathfrak{B}(\lambda)$ be the map defined by

$$\Psi^\lambda (m) = \text{hdInd}\nabla (m) \quad \text{for} \ m \in \mathcal{M}_X (\lambda).$$

Then the map $\Psi^\lambda$ is bijective.

2. The composition $\Psi^\lambda \circ \gamma_\lambda^{-1} : B(\lambda) \rightarrow \mathfrak{B}(\lambda)$ is a crystal isomorphism.

**Example 3.5.** Let $U_q (g)$ be of type $D_5$. Then the crystal $\mathfrak{B}_D$ is given as follows:

\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
  \node (4) at (2.5,0) {$4$};
  \node (5) at (3,0) {$5$};
  \node (6) at (1.5,1.5) {$\overline{1}$};
  \node (7) at (1.5,-1.5) {$\overline{5}$};
  \node (8) at (0.5,1.5) {$\overline{2}$};
  \node (9) at (0.5,-1.5) {$\overline{4}$};
  \node (10) at (2.5,1.5) {$\overline{2}$};
  \node (11) at (2.5,-1.5) {$\overline{2}$};
  \node (12) at (3,1.5) {$\overline{3}$};
  \node (13) at (3,-1.5) {$\overline{3}$};
  \node (14) at (4,0) {$1$};
  \draw[->] (1) -- (2);
  \draw[->] (2) -- (3);
  \draw[->] (3) -- (4);
  \draw[->] (4) -- (5);
  \draw[->] (5) -- (1);
  \draw[->] (6) -- (8);
  \draw[->] (8) -- (10);
  \draw[->] (10) -- (12);
  \draw[->] (12) -- (14);
  \draw[->] (7) -- (9);
  \draw[->] (9) -- (11);
  \draw[->] (11) -- (13);
  \draw[->] (13) -- (14);
\end{tikzpicture}
\end{center}

Note that $\nu_D = (1, 2, 3, 4, 5, 3, 2, 1)$. We choose the following segments $s_k \ (k = 1, \ldots, 6)$ of $\mathfrak{B}_X$:

- $s_1 := \{4, 5\}$,
- $s_2 := \{3, 5\}$,
- $s_3 := \{2, 4\}$,
- $s_4 := \{2, 4\}$,
- $s_5 := \{\overline{1}, \overline{4}\}$,
- $s_6 := \{\overline{1}, \overline{4}\}$

and let $m = \{s_1 \preceq s_2 \preceq s_3 \preceq s_4 \preceq s_5 \preceq s_6\}$ be the multisegment consisting of $s_k \ (k = 1, \ldots, 6)$. Note that

\[
\text{ch}\nabla (s_1) = (4), \quad \text{ch}\nabla (s_2) = (3, 5),
\]

\[
\text{ch}\nabla (s_3) = (2, 3), \quad \text{ch}\nabla (s_4) = (2, 3, 4, 5) + (2, 3, 5, 4).
\]

\[
\text{ch}\nabla (s_5) = (1, 2, 3), \quad \text{ch}\nabla (s_6) = (1, 2, 3, 4, 5) + (1, 2, 3, 5, 4).
\]

Since $m(1) = \{s_5, s_6\}$, $m(2) = \{s_3, s_4\}$, $m(3) = \{s_2\}$, $m(4) = \{s_1\}$ and $m(5) = \emptyset$, we have

\[
e_1 (m) = (2, 2, 1, 1, 0, 0, 0), \quad e_2 (m) = (0, 2, 2, 1, 1, 0, 0, 0),
\]

\[
e_3 (m) = (0, 0, 1, 0, 1, 0, 0, 0), \quad e_4 (m) = (0, 0, 0, 1, 0, 0, 0, 0),
\]

\[
e_5 (m) = (0, 0, 0, 0, 0, 0, 0, 0).
\]
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It follows from Theorem 3.2 that

\[
\text{hdInd}\nabla(m) \simeq \text{hdInd}(\nabla(g_1) \otimes \nabla(g_2) \otimes \nabla(g_3) \otimes \nabla(g_4) \otimes \nabla(g_5) \otimes \nabla(g_6)) \\
\simeq \tilde{f}_4 \tilde{f}_3 \tilde{f}_5 \tilde{f}_2 \tilde{f}_3 \tilde{f}_4 \tilde{f}_5 \tilde{f}_4 \tilde{f}_1 \tilde{f}_5 \tilde{f}_4 \tilde{f}_5 \tilde{f}_1.
\]

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