数理解析研究所講究録
第 1795 巻 2012 年 224-233

KKR TYPE BIJECTION FOR $E_6^{(1)}$: ALGORITHMS AND EXAMPLES

MASATO OKADO* AND NOBUMASA SANO

ABSTRACT. We review the KKR type bijection for the exceptional affine algebra $E_6^{(1)}$ obtained in our previous paper [4] by giving concise explanations on algorithms and more examples.

1. AFFINE ALGEBRA $E_6^{(1)}$ AND KR CRYSTAL $B^{1,1}$

We consider the exceptional affine Lie algebra of type $E_6^{(1)}$. We label the nodes of the Dynkin diagram as in Figure 1 following [2]. Let I be the index set of the Dynkin nodes, and let $\alpha_i, \alpha_i^\vee, \Lambda_i\,(i \in I)$ be simple roots, simple coroots, fundamental weights, respectively. Following [2] we denote the projection of Λ_i onto the weight space of E_6 by $\overline{\Lambda}_i$ $(i \in I_0)$ and set $\overline{P} = \bigoplus_{i\in I_0} \mathbb{Z}\overline{\Lambda}_i, \overline{P}^+ = \bigoplus_{i\in I_0} \mathbb{Z}_{\geq 0}\overline{\Lambda}_i$. Let $(C_{ij})_{i,j\in I}$ stand for the Cartan matrix. For $i,j \in I$, $i \sim j$ means that the nodes i and j are adjacent in the Dynkin diagram. Then $C_{ij} = (\alpha_i|\alpha_j) = 2$ if $i = j$, $= -1$ if $i \sim j$, and $= 0$ otherwise.

Let us consider the level 0 fundamental representaion of $U_q'(E_6^{(1)})$ or simplest Kirillov-Reshetikhin module corresponding to the node 1. It is known that it has a crystal basis, denoted by $B^{1,1}$. Following [1] we show the crystal graph of $B^{1,1}$ in Figure 2. Here vertices in the crystal graph signify elements of $B^{1,1}$ and $b \xrightarrow{i} b'$

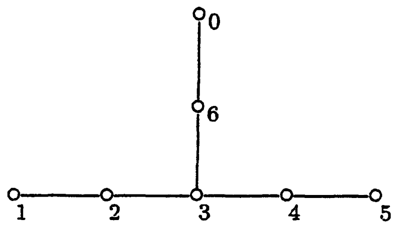


FIGURE 1. Dynkin diagram for $E_6^{(1)}$

Date: April 7, 2012

*Department of Mathematical Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan
E-mail: okado@sigmath.es.osaka-u.ac.jp

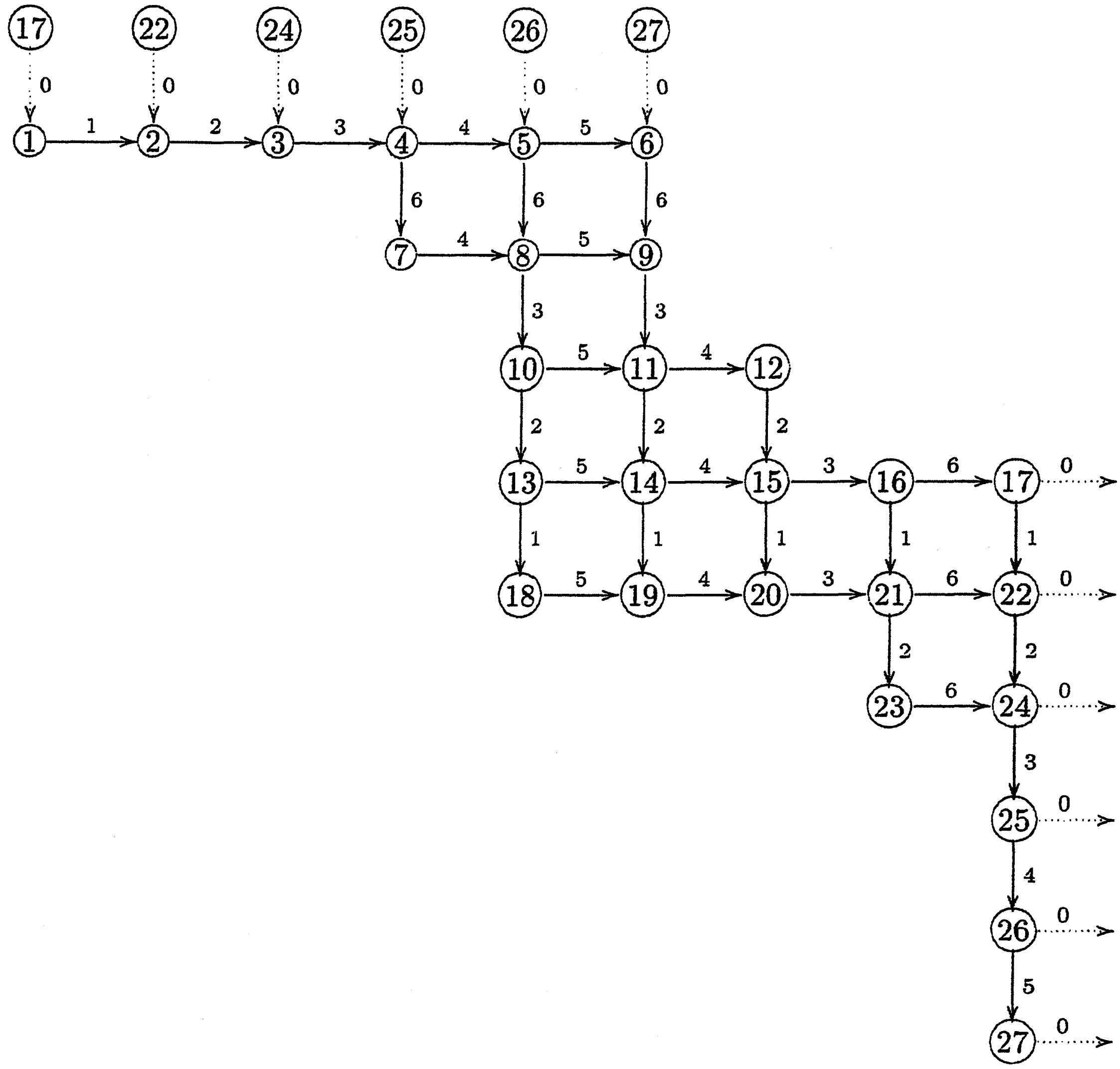


FIGURE 2. Crystal graph for $B^{1,1}$

stands for the relation

$$f_i b = b', \text{ or equivalently } b = e_i b'$$

where e_i and f_i are Kashiwara operators. $f_i b = 0$ (resp. $e_i b = 0$) signifies that there is no arrow of color i sourcing from (resp. sink into) b. Standard notations are in order. Let B be a crystal. For $b \in B$ we set

$$\varepsilon_i(b) = \max\{m \in \mathbb{Z}_{\geq 0} \mid e_i^m b \neq 0\}, \qquad \varphi_i(b) = \max\{m \in \mathbb{Z}_{\geq 0} \mid f_i^m b \neq 0\},$$
$$\varepsilon(b) = \sum_i \varepsilon_i(b)\Lambda_i, \qquad \varphi(b) = \sum_i \varphi_i(b)\Lambda_i,$$
$$\mathrm{wt}(b) = \varphi(b) - \varepsilon(b).$$

Let B_1, B_2 be crystals. Then the set $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ is endowed with the structure of crystal by

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes e_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \tag{1.1}$$

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes f_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \tag{1.2}$$

$0 \otimes b$ and $b \otimes 0$ should be understood as 0. One checks that ε_i, φ_i and wt are given by

$$\begin{aligned} \varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_1) + \varepsilon_i(b_2) - \varphi_i(b_1)), \\ \varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varphi_i(b_1) + \varphi_i(b_2) - \varepsilon_i(b_2)), \\ \mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2). \end{aligned}$$

Tensor product of crystals is associative. In order to compute the action of Kashiwara operators e_i, f_i on multiple tensor products, it is convenient to use the rule called signature rule. See [3, §2.1].

2. Path

In what follows we set $B = B^{1,1}$. The set of classically restricted paths in $B^{\otimes L}$ of weight $\lambda \in \overline{P}^+$ is by definition

$$\mathcal{P}(\lambda, L) = \{b \in B^{\otimes L} \mid \mathrm{wt}(b) = \lambda \text{ and } e_i b = 0 \text{ for all } i \in I_0\}.$$

Using the tensor product rule (1.1) one can check that the following two conditions are equivalent for $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B^{\otimes L}$ and $\lambda \in \overline{P}^+$.

(i) b is a classically restricted path of weight $\lambda \in \overline{P}^+$.
(ii) $b_1 \otimes \cdots \otimes b_{L-1}$ is a classically restricted path of weight $\lambda - \mathrm{wt}(b_L)$, and $\varepsilon_i(b_L) \leq \langle \lambda - \mathrm{wt}(b_L), \alpha_i^\vee \rangle$ for all $i \in I_0$.

Example 2.1. The following three elements of $B^{\otimes 6}$ are classically restricted paths. The dot $\cdot$ signifies $\otimes$.

(1) $b =$ ① · ② · ㉓ · ⑰ · ⑱ · ① $\quad \mathrm{wt}(b) = \overline{\Lambda}_1 + \overline{\Lambda}_5$
(2) $b =$ ① · ② · ③ · ⑯ · ② · ㉔ $\quad \mathrm{wt}(b) = \overline{\Lambda}_3$
(3) $b =$ ① · ② · ③ · ④ · ⑫ · ㉔ $\quad \mathrm{wt}(b) = \overline{\Lambda}_3$

One checks that b in (1) is a classically restricted path as follows. Firstly, ① is the unique classically restricted path in B. ① $\otimes$ ② is classically restricted by the above criterion with $L = 2$, since ① is classically restricted and $\varepsilon($②$) = \Lambda_0 + \Lambda_1$, $\mathrm{wt}($①$) = \Lambda_1 - \Lambda_0$. ① $\otimes$ ② $\otimes$ ㉓ is classically restricted with $L = 3$, since ① $\otimes$ ② is classically restricted and $\varepsilon($㉓$) = \Lambda_2$, $\mathrm{wt}($① $\otimes$ ②$) = \Lambda_2 - 2\Lambda_0$. We continue these checks until $L = 6$.

The element of $B^{\otimes L}$ has a grading called energy. To define it we introduce a local energy function $H : B \otimes B \to \mathbb{Z}$. Since the crystal graph of $B \otimes B$ is connected, it is defined uniquely, up to a global additive constant, by

$$H(e_i(b \otimes b')) = H(b \otimes b') + \begin{cases} 1 & \text{if } i = 0 \text{ and } e_0(b \otimes b') = e_0 b \otimes b' \\ -1 & \text{if } i = 0 \text{ and } e_0(b \otimes b') = b \otimes e_0 b' \\ 0 & \text{otherwise.} \end{cases}$$

We normalize the additive constant by the condition

$$H(\textcircled{1}\otimes\textcircled{1})=0.$$

More specifically, the value of H is calculated as follows. Let B_0 be the crystal graph obtained by forgetting arrows of color 0 from B. One knows that the crystal graph of $B_0\otimes B_0$ decomposes into three connected components as

$$B_0\otimes B_0=B(2\overline{\Lambda}_1)\oplus B(\overline{\Lambda}_1+\overline{\Lambda}_2)\oplus B(\overline{\Lambda}_1+\overline{\Lambda}_5),$$

where $B(\lambda)$ stands for the highest weight E_6-crystal of highest weight $\lambda\in\overline{P}^+$ and the highest weight vector of each component is given by $\textcircled{1}\otimes\textcircled{1},\textcircled{1}\otimes\textcircled{2},\textcircled{1}\otimes\textcircled{18}$. H is constant on each component, and takes the value $0,-1,-2$, respectively. One can confirm it from the fact that $e_0(\textcircled{1}\otimes\textcircled{1})=\textcircled{1}\otimes\textcircled{17}$ and $e_0(\textcircled{1}\otimes\textcircled{2})=\textcircled{1}\otimes\textcircled{22}$ belong to the second and third component. In fact, the value of H is given by

$$H(b\otimes c)=\begin{cases}-2 & \text{if } b\otimes c\in S_1\\ 0 & \text{if } b\otimes c\in S_2\\ -1 & \text{otherwise}\end{cases},$$

where

$$\begin{aligned}S_1=&\{\textcircled{1}\otimes\textcircled{j}\mid j\geq 18\}\sqcup\{\textcircled{2}\otimes\textcircled{j}\mid j\geq 23\}\sqcup\{\textcircled{3}\otimes\textcircled{j}\mid j\geq 25\}\\ &\sqcup\{\textcircled{4}\otimes\textcircled{j},\textcircled{7}\otimes\textcircled{j}\mid j\geq 26\}\sqcup\{\textcircled{i}\otimes\textcircled{27}\mid i=5,8,10,13,18\},\\ S_2=&\{\textcircled{i}\otimes\textcircled{j}\mid \textcircled{i}\text{ can be reached by following some (possibly zero) arrows from }\textcircled{j}\}.\end{aligned}$$

With this H the energy function D is defined by

$$D(b_1\otimes\cdots\otimes b_L)=\sum_{j=1}^{L-1}(L-j)\;H(b_j\otimes b_{j+1}).$$

Example 2.2. The energies of classically restricted paths in Example 2.1 are given by

(1) $D(b)=5(-1)+4(-2)+3(-1)+2(-1)+1\cdot 0=-18$,
(2) $D(b)=5(-1)+4(-1)+3(-1)+2\cdot 0+1(-2)=-14$,
(3) $D(b)=5(-1)+4(-1)+3(-1)+2(-1)+1(-1)=-15$.

3. Rigged configuration

We first provide the definition of a rigged configuration that is valid for any simply-laced affine type $\mathfrak{g}$ and datum L, and then restrict $\mathfrak{g}$ and L to $E_6^{(1)}$ and the case corresponding to paths we consider in this paper. Fix $\lambda\in\overline{P}^+$ and a matrix $L=(L_i^{(a)})_{a\in I_0,i\in\mathbb{Z}_{>0}}$ of nonnegative integers, almost all zero. Let $\nu=(m_i^{(a)})_{a\in I_0,i\in\mathbb{Z}_{>0}}$ be another such matrix. Say that ν is an admissible configuration if it satisfies

$$\sum_{\substack{a\in I_0\\ i\in\mathbb{Z}_{>0}}} i\,m_i^{(a)}\alpha_a=\sum_{\substack{a\in I_0\\ i\in\mathbb{Z}_{>0}}} i\,L_i^{(a)}\overline{\Lambda}_a-\lambda$$

and

$$p_i^{(a)}\geq 0\qquad\text{for all } a\in I_0 \text{ and } i\in\mathbb{Z}_{>0},$$

where

$$p_i^{(a)}=\sum_{j\in\mathbb{Z}_{>0}}\left(L_j^{(a)}\min(i,j)-\sum_{b\in I_0}C_{ab}\min(i,j)m_j^{(b)}\right).\tag{3.1}$$

$p_i^{(a)}$ is called a vacancy number.

Let $\nu = (m_i^{(a)})_{a\in I_0, i\in\mathbb{Z}_{>0}}$ be an admissible configuration. We identify ν with a sequence of partitions $\{\nu^{(a)}\}_{a\in I_0}$ such that $\nu^{(a)} = (1^{m_1^{(a)}} 2^{m_2^{(a)}} \cdots)$. For a partition μ and $i \in \mathbb{Z}_{>0}$, define

$$Q_i(\mu) = \sum_j \min(\mu_j, i),$$

the area of the corresponding Young diagram μ in the first i columns. Then setting $Q_i^{(a)} = Q_i(\nu^{(a)})$ the vacancy number (3.1) is rewritten as

$$p_i^{(a)} = \sum_{j\in\mathbb{Z}_{>0}} L_j^{(a)} \min(i,j) + \sum_{b\sim a} Q_i^{(b)} - 2Q_i^{(a)}, \tag{3.2}$$

where $b \sim a$ stands for $C_{ba} = -1$ as defined in §1.

Let $J = \{J^{(a,i)}\}_{(a,i)\in I_0\times\mathbb{Z}_{>0}}$ be a double sequence of partitions. Then a rigged configuration is a pair (ν, J) of an admissible configuration ν and $J = \{J^{(a,i)}\}$ such that $J^{(a,i)}$ is a partition contained in the $m_i^{(a)} \times p_i^{(a)}$ rectangle. The set of rigged configurations for fixed λ and L is denoted by $\mathrm{RC}(\lambda, L)$.

We define the charge of a rigged configuration. First, define the charge of an admissible configuration ν by

$$\begin{aligned} c(\nu) =& \frac{1}{2} \sum_{a,b\in I_0} \sum_{j,k\in\mathbb{Z}_{>0}} C_{ab} \min(j,k) m_j^{(a)} m_k^{(b)} \\ &- \sum_{a\in I_0} \sum_{j,k\in\mathbb{Z}_{>0}} \min(j,k) L_j^{(a)} m_k^{(a)}. \end{aligned}$$

Using (3.1) $c(\nu)$ is rewritten as

$$c(\nu) = -\frac{1}{2}\left(\sum_{a\in I_0, i\in\mathbb{Z}_{>0}} p_i^{(a)} m_i^{(a)} + \sum_{a\in I_0, j,k\in\mathbb{Z}_{>0}} \min(j,k) L_j^{(a)} m_k^{(a)} \right). \tag{3.3}$$

We then define the charge of a rigged configuration (ν, J) by

$$c(\nu, J) = c(\nu) + |J| \tag{3.4}$$

where $|J| = \sum_{(a,i)\in I_0\times\mathbb{Z}_{>0}} |J^{(a,i)}|$.

To return to our case, we use the Cartan matrix (C_{ab}) of $E_6^{(1)}$ and set

$$L_i^{(a)} = L\delta_{a1}\delta_{i1} \qquad (a \in I_0, i \in \mathbb{Z}_{>0}) \tag{3.5}$$

corresponding to considering paths in $(B^{1,1})^{\otimes L}$. Under this constraint, the formulas (3.2) and (3.3) are rewritten as

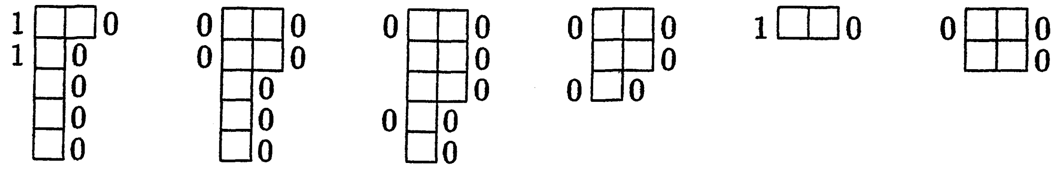
$$p_i^{(a)} = L\delta_{a1} + \sum_{b\sim a} Q_i^{(b)} - 2Q_i^{(a)},$$

$$c(\nu) = -\frac{1}{2}\left(\sum_{a\in I_0, i\in\mathbb{Z}_{>0}} p_i^{(a)} m_i^{(a)} + L \sum_{k\in\mathbb{Z}_{>0}} m_k^{(1)} \right).$$

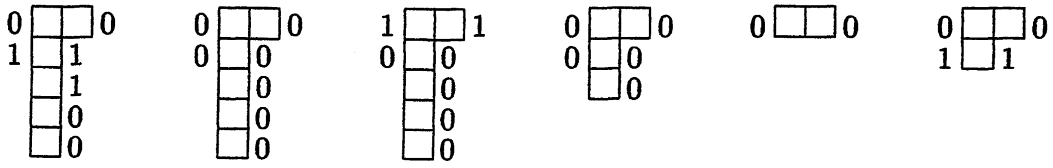
By abuse of notation, the set $\mathrm{RC}(\lambda, L)$ for $E_6^{(1)}$ with the restriction (3.5) is denoted by $\mathrm{RC}(\lambda, L)$.

Example 3.1. Three rigged configurations in $\mathrm{RC}(\overline{\Lambda}_1+\overline{\Lambda}_5,6), \mathrm{RC}(\overline{\Lambda}_3,6), \mathrm{RC}(\overline{\Lambda}_3,6)$ are illustrated below.

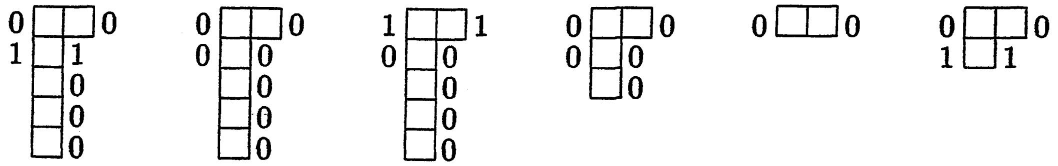
(1)



(2)



(3)



The partitions $\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(6)}$ are depicted from left to right as Young diagrams. In the second example, 0 and 1 in $\nu^{(1)}$ on the left signify $p_2^{(1)} = 0$ and $p_1^{(1)} = 1$. Looking on the right we see $J_2^{(1)} = (0), J_1^{(1)} = (1,1,0,0)$. From (3.3) we have $c(\nu) = -18$, hence $c(\nu, J) = -14$. $c(\nu) = -18$ is the same for the other two. $c(\nu, J) = -18, -15$ for the first and third example.

4. The bijection Φ

In the previous two sections we have introduced two combinatorial objects $\mathcal{P}(\lambda, L)$, $\mathrm{RC}(\lambda, L)$ for $\lambda \in \overline{P}^+, L \in \mathbb{Z}_{\geq 0}$. In [4] we showed

Theorem (Theorem 3.2 of [4]). *There exists a bijection* $\Phi : \mathrm{RC}(\lambda, L) \to \mathcal{P}(\lambda, L)$ *satisfying*

$$c(\nu, J) = D(\Phi(\nu, J)). \tag{4.1}$$

Leaving the proof to [4], we shall describe the bijection Φ and its inverse Φ^{-1}. The bijection Φ is defined by giving a more fundamental procedure denoted by δ. For $(\nu, J) \in \mathrm{RC}(\lambda, L)$ δ produces a smaller rigged configuration $(\tilde{\nu}, \tilde{J})$ in $\mathrm{RC}(\rho, L-1)$ and an element b of B. The weight ρ is in fact given by $\rho = \lambda - \mathrm{wt}(b)$. Write b_L for b. We then apply δ to $(\tilde{\nu}, \tilde{J})$, obtaining b_{L-1}. We eventually arrive at the empty rigged configuration ($\nu^{(a)} = \emptyset$ for all $a \in I_0$) in $\mathrm{RC}(0,0)$ that should correspond to the empty path in $\mathcal{P}(0,0)$, giving $b_L, b_{L-1}, \dots, b_1 \in B$. Then the image of (ν, J) by Φ is determined as $b_1 \otimes \cdots \otimes b_{L-1} \otimes b_L \in \mathcal{P}(\lambda, L) \subset B^{\otimes L}$.

The inverse bijection Φ^{-1} is defined in a completely similar manner. Given a rigged configuration $(\tilde{\nu}, \tilde{J})$ in $\mathrm{RC}(\rho, L-1)$ and an element b of B satisfying $\varepsilon_i(b) \leq \langle \rho, \alpha_i^\vee \rangle$ for all $i \in I_0$, we introduce a procedure $\tilde{\delta}$ to construct a new rigged configuration (ν, J) in $\mathrm{RC}(\rho + \mathrm{wt}(b), L)$. If $b_1 \otimes b_2 \otimes \cdots b_L \in \mathcal{P}(\lambda, L)$ is given, we start from the empty rigged configuration and apply $\tilde{\delta}$ L times, thereby obtaining a rigged configuration in $\mathrm{RC}(\lambda, L)$.

Before giving the algorithms of δ and $\tilde{\delta}$, we recall that B_0 is the crystal graph (see Figure 2) obtained by forgetting arrows of color 0 from B. We call a row in $\nu^{(a)}$ singular if its rigging (number on the right in Example 3.1) is equal to the corresponding vacancy number $p_i^{(a)}$ (number on the left).

4.1. **Algorithm δ.** For a rigged configuration (ν, J) the outputs $(\tilde{\nu}, \tilde{J})$ and b by δ are are given as follows. First set $b = ①$ and $\ell_0 = 1$. Viewing the crystal graph of B_0 (Figure 2), repeat the following process for $j = 1, 2, \dots$ until stopped. From b proceed by one step through an arrow of color a. Find the minimal integer $i \geq \ell_{j-1}$ such that $\nu^{(a)}$ has a singular row of length i and set $\ell_j = i$, reset b to be the sink of the arrow. A singular row selected previously during the process should not be selected again. If there is no such integer, then set $\ell_j = \infty$ and stop. If there are two arrows sourcing from b, compare the minimal integers and take the smaller one. If the integers are the same, either one can be taken. The output of the algorithm does not depend on the choice. Then b is the one just obtained and $\tilde{\nu}$ is obtained from (ν, J) by removing the rightmost box from every singular row selected by the above procedure. The new rigging $\tilde{J}$ is defined to be the same if the corresponding row is not selected, and declared to be singular if selected.

4.2. **Algorithm $\tilde{\delta}$.** For a given rigged configuration $(\tilde{\nu}, \tilde{J})$ and $b \in B$ satisfying $\varepsilon_i(b) \leq \langle \rho, \alpha_i^\vee \rangle$ for all $i \in I_0$, the inverse algorithm $\tilde{\delta}$ of δ is described as follows. From $b \in B$ go back the arrow in the crystal graph B_0. Suppose the color of the arrow is a. Let the maximal length of the singular row in $\nu^{(a)}$ be $\tilde{\ell}_0$. Repeat the following process for $j = 1, 2, \dots$ until we arrive at ①. Find the maximal integer $i \leq \tilde{\ell}_{j-1}$ such that $\nu^{(a)}$ has a singular row of length i and set $\tilde{\ell}_j = i$, reset b to be the source of the arrow. A singular row selected previously during the process should not be selected again. If there is no singular row in $\nu^{(a)}$, then assume there is a singular row of length 0 in $\nu^{(a)}$ and set $\tilde{\ell}_j = 0$. If there are two arrows ending at b, compare the maximal integers and take the larger one. If the integers are the same, either one can be taken. The output of the algorithm does not depend on the choice. The new configuration ν is given by adding a box to every singular row selected by the above procedure. The new rigging J is defined to be the same if the corresponding row is not selected, and declared to be singular if selected.

4.3. **Examples.** We apply δ successively to the rigged configurations in Example 3.1. We see that each rigged configuration corresponds under Φ to the classically restricted path in Example 2.1. The charge of the rigged configuration agrees with the energy of the corresponding path as guaranteed by our theorem. See Example 2.2.

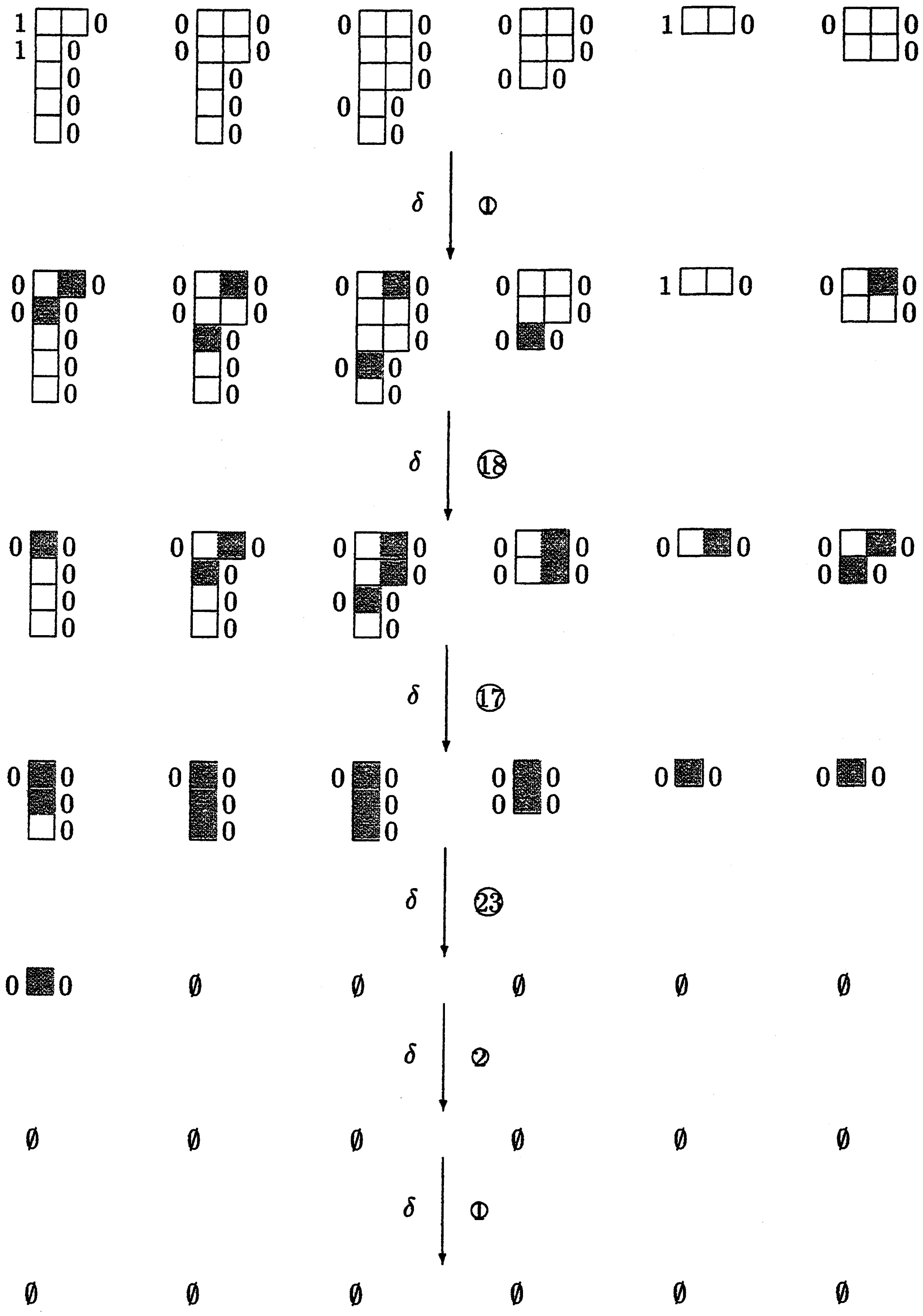
Let us look at the algorithm in detail in the first example. The top rigged configuration has no singular row in $\nu^{(1)}$, hence $\ell_1 = \infty$ and $b = ①$. Since no box is removed by the first δ, the next rigged configuration is the same as the top one. However, the vacancy numbers change. Applying δ one proceeds in B_0 as

$$① \to ② \to ③ \to ④ \to ⑤ \to ⑧ \to ⑩ \to ⑬ \to ⑱$$

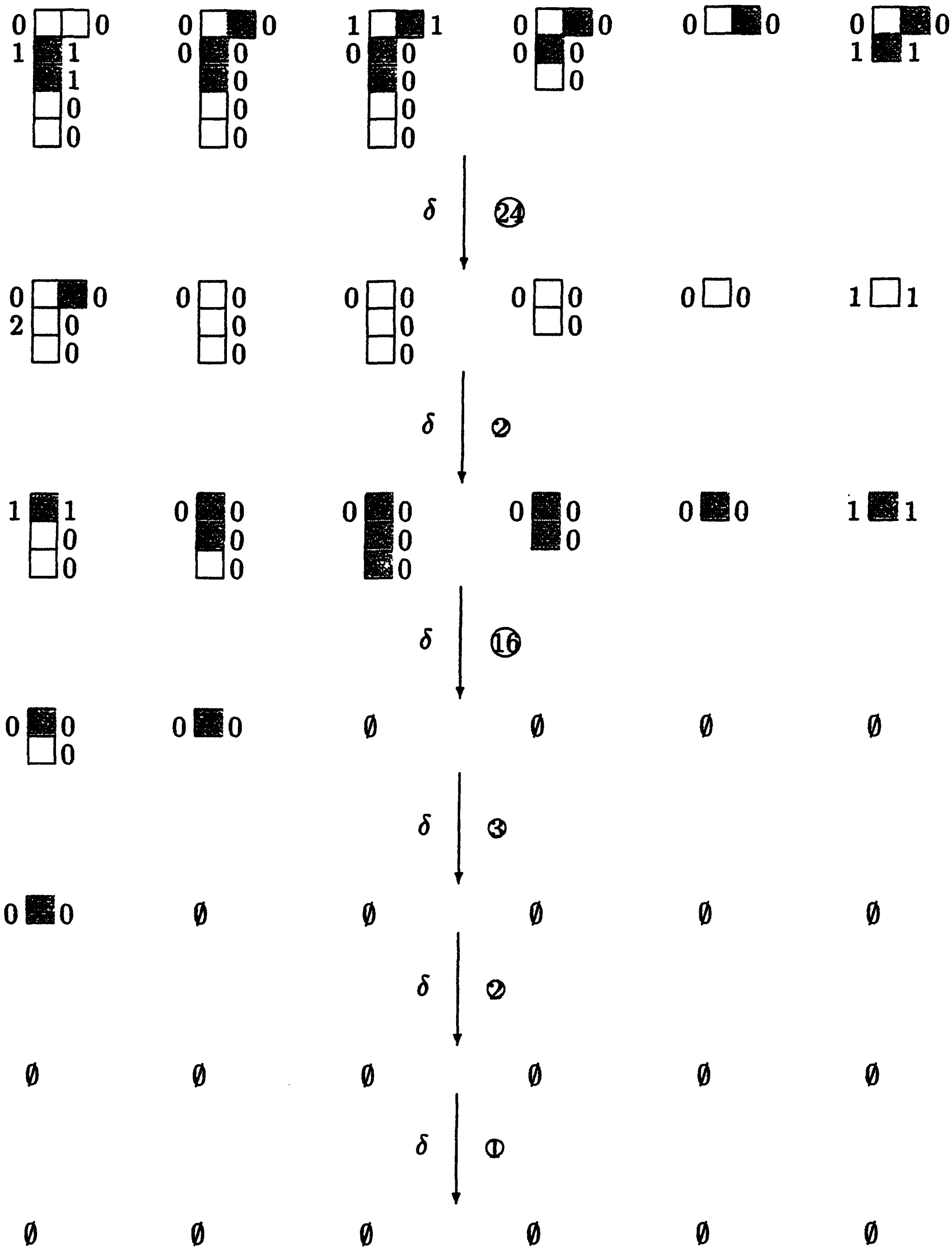
with $\ell_j = 1$ for $1 \leq j \leq 4$, $\ell_j = 2$ for $5 \leq j \leq 8$, $\ell_j = \infty$ for $j = 9$, obtaining $b = ⑱$. The boxes in the selected rows are shaded.

The third example differs from the second one just by a rigging in $\nu^{(1)}$.

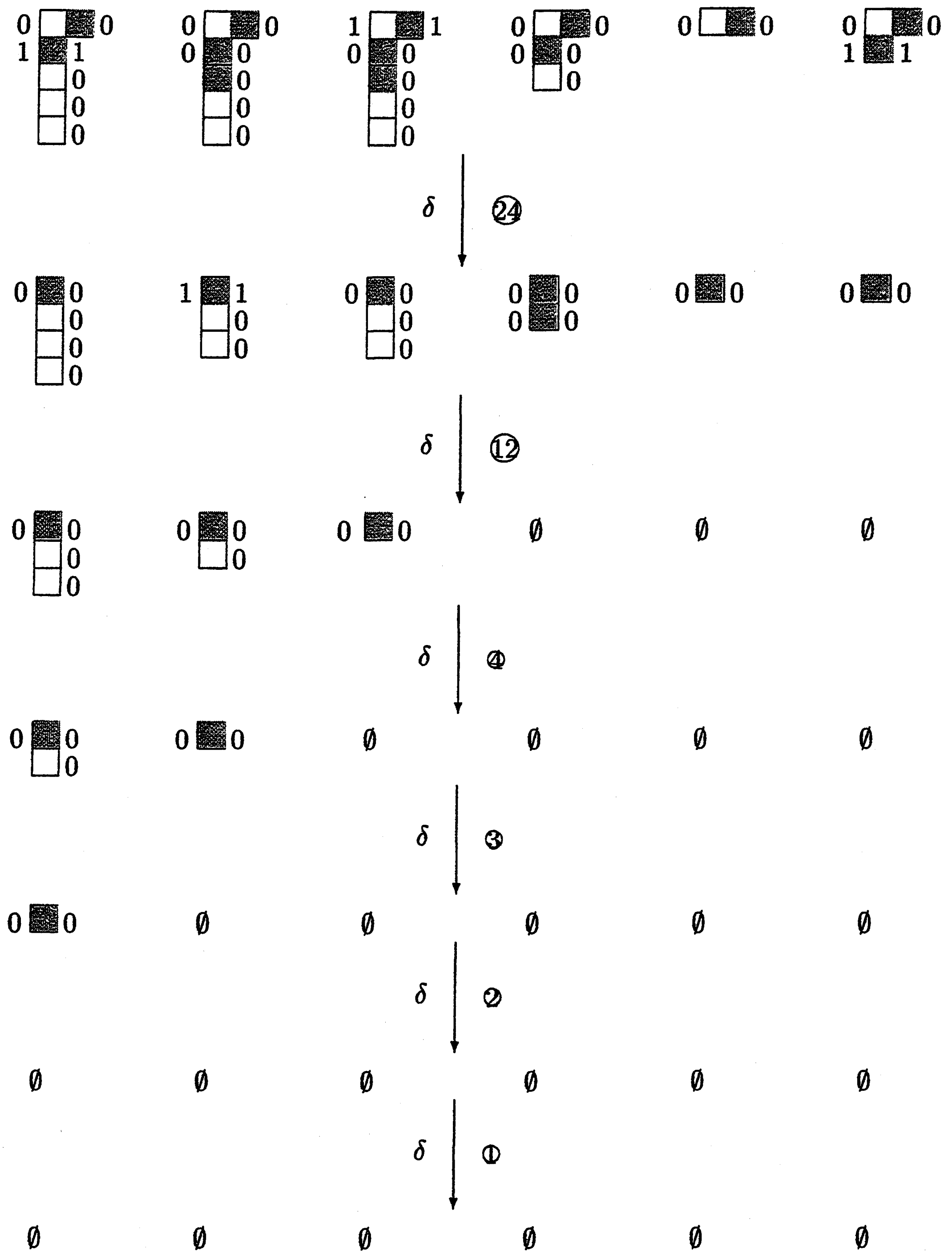
(1)



(2)



(3)



REFERENCES

[1] B. Jones and A. Schilling, *Affine structures and a tableau model for E_6 crystals*, J. Algebra **324** (2010) 2512–2542.

[2] V. Kac, Infinite dimensional Lie algebras, 3rd ed., Cambridge University Press, 1990.

[3] M. Okado, *$X = M$ conjecture*, Combinatorial Aspect of Integrable Systems, MSJ Memoirs **17**, 43-73 (2007).

[4] M. Okado and N. Sano, *KKR type bijection for the exceptional affine algebra $E_6^{(1)}$*, Contemp. Math. **565**, (2012); http://dx.doi.org/10.1090/conm/565/11181.