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A Note on the Integral Transforms on a Function Space I, II

By

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Abstract

In this paper we obtain some results of integration by parts formulas involving integral transforms of functionals of the form $F(y)=f(\langle\theta_1,y\rangle,\ldots,\langle\theta_n,y\rangle)$ for s-a.e. $y\in C_0[0,T]$, where $\langle\theta,y\rangle$ denotes the Riemann-Stieltjes integral $\int_0^T\theta(t)\,dy(t)$. Furthermore we obtain the various relationships that exist among the integral transform, the convolution product and the first variation for a class of functionals defined on $K(Q)$, the space of complex-valued continuous functions on $Q=[0,S]\times[0,T]$ which satisfy $x(s,0)=x(0,t)=0$ for all $0\le s\le S$ and $0\le t\le T$. Also we obtain Parseval's and Plancherel's relations for the integral transform of some functionals defined on $K(Q)$.

§ 1. Parts formulas involving integral transforms on function space

In a unifying paper [15], Lee defined an integral transform $\mathcal{F}_{\alpha,\beta}$ of analytic functionals on an abstract Wiener space. For certain values of the parameters α and β and for certain classes of functionals, the Fourier-Wiener transform [3], the Fourier-Feynman transform [4] and the Gauss transform are special cases of his integral transform $\mathcal{F}_{\alpha,\beta}$. In [6], Chang, Kim and Yoo established an interesting relationship between the integral transform and the convolution product for functionals on an abstract Wiener space. In this paper we establish several integration by parts formulas involving integral transforms, convolution products, and the first variations of functionals of the form $F(y)=f(\langle\theta_1,y\rangle,\ldots,\langle\theta_n,y\rangle)$ for s-a.e. $y\in C_0[0,T]$, where $\langle\theta,y\rangle$ denotes the Riemann-Stieltjes integral $\int_0^T\theta(t)\,dy(t)$.

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of all real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. $(C_0[0,T],\mathcal{M},m)$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]}F(x)\,m(dx). \tag{1.1}$$

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Let α and β be nonzero complex numbers. Next we state the definitions of the integral transform $\mathcal{F}_{\alpha,\beta}F$, the convolution product $(F*G)_\alpha$ and the first variation δF for functionals defined on $K = K[0,T]$, the space of complex-valued continuous functions defined on $[0,T]$ which vanish at $t = 0$.

Definition 1.1. Let F be a functional defined on K. Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F is defined by

$$(1.2)\qquad \mathcal{F}_{\alpha,\beta}(F)(y) \equiv \mathcal{F}_{\alpha,\beta}F(y) \equiv \int_{C_0[0,T]} F(\alpha x+\beta y)m(dx), \quad y \in K$$

if it exists [6, 12, 13, 15].

Definition 1.2. Let F and G be functionals defined on K. Then the convolution product $(F*G)_\alpha$ of F and G is defined by

$$(1.3)\qquad (F*G)_\alpha(y) \equiv \int_{C_0[0,T]} F\Big(\frac{y+\alpha x}{\sqrt{2}}\Big)G\Big(\frac{y-\alpha x}{\sqrt{2}}\Big)m(dx), \quad y \in K$$

if it exists[6, 10, 12, 19, 21].

Definition 1.3. Let F be a functional defined on K and let $w \in K$. Then the first variation δF of F is defined by

$$(1.4)\qquad \delta F(y|w) \equiv \frac{\partial}{\partial t}F(y+tw)|_{t=0}, \quad y \in K$$

if it exists [2, 5, 12, 17].

Let $\{\theta_1, \theta_2, \ldots\}$ be a complete orthonormal set of real-valued functions in $L_2[0,T]$ and assume that each θ_j is of bounded variation on $[0,T]$. Then for each $y \in K$ and $j \in \{1,2,\ldots\}$, the Riemann-Stieltjes integral $\langle\theta_j, y\rangle \equiv \int_0^T \theta_j(t)dy(t)$ exists. Furthermore

$$(1.5)\qquad |\langle\theta_j,y\rangle| = |\theta_j(T)y(T) - \int_0^T y(t)d\theta_j(t)| \le C_j\|y\|_\infty$$

with

$$(1.6)\qquad C_j = |\theta_j(T)| + \mathrm{Var}(\theta_j,[0,T]),$$

where $\mathrm{Var}(\theta_j,[0,T])$ denote the total variation of θ_j on $[0,T]$.

Next we describe the class of functionals which is related to this paper. For $0 \le \sigma < 1$, let E_σ be the space of all functionals $F : K \to \mathbb{C}$ of the form

$$(1.7)\qquad F(y) = f(\langle\vec{\theta},y\rangle) = f(\langle\theta_1,y\rangle,\ldots,\langle\theta_n,y\rangle)$$

for some positive integer n, where $f(\vec{\lambda}) = f(\lambda_1, \ldots, \lambda_n)$ is an entire function of the n complex variables $\lambda_1, \ldots, \lambda_n$ of exponential type; that is to say,

$$|f(\vec{\lambda})| \le A_F \exp\{B_F |\vec{\lambda}|^{1+\sigma}\} \tag{1.8}$$

for some positive constants A_F and B_F, where $|\vec{\lambda}|^{1+\sigma} = \sum_{j=1}^{n} |\lambda_j|^{1+\sigma}$.

In addition we use the notation

$$F_j(y) = f_j(\langle\vec{\theta}, y\rangle)$$

where $f_j(\vec{\lambda}) = \frac{\partial}{\partial \lambda_j} f(\lambda_1, \ldots, \lambda_n)$ for $j = 1, \ldots, n$.

Recently [12], Kim, Kim and Skoug established the results that if F and G are elements of E_σ then $\mathcal{F}_{\alpha,\beta}F$, $(F * G)_\alpha$, $\delta F(\cdot|w)$ and $\delta F(y|\cdot)$ are also elements of E_σ and examined various relationships holding among $\mathcal{F}_{\alpha,\beta}F$, $\mathcal{F}_{\alpha,\beta}G$, $(F * G)_\alpha$, δF and δG. For related work see [3, 6, 10, 12, 15, 17, 19, 21] and for a detailed survey of previous work see [18].

We introduce the following three existence theorems for the integral transform, the convolution product and the first variation of functionals in E_σ [12].

Theorem 1.4. *Let $F \in E_\sigma$ be given by (1.7). Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ exists, belongs to E_σ and is given by the formula*

$$\mathcal{F}_{\alpha,\beta}F(y) = h(\langle\vec{\theta}, y\rangle) \tag{1.9}$$

for $y \in K$, where

$$h(\vec{\lambda}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\alpha\vec{u} + \beta\vec{\lambda}) \exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\} d\vec{u} \tag{1.10}$$

where $\|\vec{u}\|^2 = \sum_{j=1}^{n} u_j{}^2$ and $d\vec{u} = du_1 \cdots du_n$.

Theorem 1.5. *Let $F, G \in E_\sigma$ be given by (1.7) with corresponding entire functions f and g, respectively. Then the convolution $(F * G)_\alpha$ exists, belongs to E_σ and is given by the formula*

$$(F * G)_\alpha(y) = k(\langle\vec{\theta}, y\rangle) \tag{1.11}$$

for $y \in K$, where

$$k(\vec{\lambda}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f\left(\frac{\vec{\lambda} + \alpha\vec{u}}{\sqrt{2}}\right) g\left(\frac{\vec{\lambda} - \alpha\vec{u}}{\sqrt{2}}\right) \exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\} d\vec{u}. \tag{1.12}$$

Theorem 1.6. *Let $F \in E_\sigma$ be given by (1.7) and let $w \in K$. Then*

$$\delta F(y|w) = p(\langle\vec{\theta}, y\rangle) \tag{1.13}$$

for $y \in K$, where

$$(1.14) \qquad p(\vec{\lambda}) = \sum_{j=1}^{n} \langle \theta_j, w \rangle f_j(\vec{\lambda}).$$

Furthermore, as a function of $y \in K$, $\delta F(y|w)$ is an element of E_σ.

Now we state some observations which we use later in this paper. First of all, equation (1.2) implies that

$$(1.15) \qquad \mathcal{F}_{\alpha,\beta} F(y/\sqrt{2}) = \mathcal{F}_{\alpha,\beta/\sqrt{2}} F(y)$$

for all $y \in K$. Next, a direct calculation using (1.4), (1.2), (1.13) and (1.15) shows that

$$(1.16) \qquad \begin{aligned} \delta \mathcal{F}_{\alpha,\beta} F(y/\sqrt{2}|w/\sqrt{2}) &= \delta \mathcal{F}_{\alpha,\beta/\sqrt{2}} F(y|w) \\ &= \frac{\beta}{\sqrt{2}} \sum_{j=1}^{n} \langle \theta_j, w \rangle \mathcal{F}_{\alpha,\beta/\sqrt{2}} F_j(y) \end{aligned}$$

for all y and w in K. Finally, by similar calculations, we obtain that

$$(1.17) \qquad \mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w))(y/\sqrt{2}) = \frac{\sqrt{2}}{\beta} \delta \mathcal{F}_{\alpha,\beta/\sqrt{2}} F(y|w)$$

for all y and w in K, and for all $y \in K$,

$$(1.18) \qquad \left(\mathcal{F}_{\alpha,\beta} F\right)_j(y) = \beta \mathcal{F}_{\alpha,\beta} F_j(y).$$

Let

$$A = \{y \in C_0[0,T] : y \text{ is absolutely continuous on } [0,T] \text{ with } y' \in L^2[0,T]\}.$$

We note that if we choose $z \in L_2[0,T]$ and define $w(t) = \int_0^t z(s)ds$ for $t \in [0,T]$, then w is an element of A, $w' = z$ a.e. on $[0,T]$, and for all $v \in L_2[0,T]$, $\langle v, w \rangle = (v, w') = (v, z)$, where $(v,z) = \int_0^T v(s)z(s)ds$.

The following theorem plays a key role throughout this paper. In this theorem the Wiener integral of the first variation of functional F is expressed in terms of the Wiener integral of F multiplied by a linear factor.

Theorem 1.7. *Let $F \in E_\sigma$ be given by* (1.7) *and $w \in A$, then*

$$(1.19) \qquad \int_{C_0[0,T]} \delta F(x|w) m(dx) = \int_{C_0[0,T]} F(x) \langle z, x \rangle m(dx)$$

where $w(t) = \int_0^t z(s)ds$ on $[0,T]$ for some $z \in L_2[0,T]$.

Proof. Let $w(t)=\int_0^t z(s)ds$ for some $z\in L^2[0,T]$. Using the Gram-Schmit process we can find an orthonormal set $\{\theta_1,\dots,\theta_n,\theta_{n+1}\}$ with $\theta_{n+1}=\frac{1}{\|z_{n+1}\|}z_{n+1}$, where

$$z_{n+1}=z-\sum_{j=1}^{n}(\theta_j,z)\theta_j.$$

Then by the Wiener integration formula

$$\begin{aligned}&\int_{C_0[0,T]}F(x)\langle z,x\rangle\, m(dx)\\ &=\int_{C_0[0,T]}f(\langle\vec{\theta},x\rangle)\Big(\sum_{j=1}^{n}(\theta_j,z)\langle\theta_j,x\rangle+\|z_{n+1}\|\langle\theta_{n+1},x\rangle\Big)\,m(dx)\\ &=(2\pi)^{-(n+1)/2}\int_{\mathbb{R}^{n+1}}f(\vec{u})\Big(\sum_{j=1}^{n}(\theta_j,z)u_j+\|z_{n+1}\|u_{n+1}\Big)\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2-\frac{1}{2}u_{n+1}^2\Big\}du_{n+1}d\vec{u}.\end{aligned}$$

If we evaluate the last integral with respect to u_{n+1}, we obtain

$$\int_{C_0[0,T]}F(x)\langle z,x\rangle\, m(dx)=(2\pi)^{-n/2}\sum_{j=1}^{n}(\theta_j,z)\int_{\mathbb{R}^n}f(\vec{u})u_j\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}d\vec{u}.$$

On the other hand, since $w\in A\subset K$, by Theorem 1.6

$$\delta F(x|w)=\sum_{j=1}^{n}\langle\theta_j,w\rangle f_j(\langle\vec{\theta},x\rangle)=\sum_{j=1}^{n}(\theta_j,z)f_j(\langle\vec{\theta},x\rangle).$$

Hence by the Wiener integration formula

$$\begin{aligned}\int_{C_0[0,T]}\delta F(x|w)m(dx)&=\sum_{j=1}^{n}(\theta_j,z)\int_{C_0[0,T]}f_j(\langle\vec{\theta},x\rangle)m(dx)\\ &=(2\pi)^{-n/2}\sum_{j=1}^{n}(\theta_j,z)\int_{\mathbb{R}^n}f_j(\vec{u})\exp\Big\{-\frac{1}{2}\|u\|^2\Big\}d\vec{u}.\end{aligned}$$

Note that for each $j=1,\dots,n$, the integration by parts formula yields

$$\begin{aligned}&\int_{\mathbb{R}}f_j(\vec{u})\exp\Big\{-\frac{1}{2}u_j^2\Big\}du_j\\ &=\lim_{b\to\infty}\lim_{a\to-\infty}\Big[f(\vec{u})\exp\Big\{-\frac{1}{2}u_j^2\Big\}\Big]_a^b+\int_{\mathbb{R}}f(\vec{u})u_j\exp\Big\{-\frac{1}{2}u_j^2\Big\}du_j.\end{aligned}$$

But since f is of exponential type, the double limit in the last equation is equal to 0 and so

$$\int_{\mathbb{R}}f_j(\vec{u})\exp\Big\{-\frac{1}{2}u_j^2\Big\}du_j=\int_{\mathbb{R}}f(\vec{u})u_j\exp\Big\{-\frac{1}{2}u_j^2\Big\}du_j.$$

Hence

$$\int_{C_0[0,T]} \delta F(x|w)\,m(dx) = (2\pi)^{-n/2}\sum_{j=1}^{n}(\theta_j, z)\int_{\mathbb{R}^n} f(\vec{u})u_j \exp\left\{-\frac{1}{2}\|u\|^2\right\} d\vec{u}$$

and this completes the proof. □

In our next theorem we obtain an integration by parts formula for the products of functionals in E_σ.

Theorem 1.8. *Let $F, G \in E_\sigma$ be given by* (1.7) *with corresponding entire functions f and g, respectively. Then for $w \in A$, we have the following integration by parts formula.*

$$\int_{C_0[0,T]} [F(y)\delta G(y|w) + \delta F(y|w)G(y)]m(dy) = \int_{C_0[0,T]} F(y)G(y)\langle z, y\rangle m(dy), \tag{1.20}$$

where $w(t) = \int_0^t z(s)\,ds$ for some $z \in L_2[0,T]$.

Proof. Define $H(y) = F(y)G(y)$ for $y \in K$ and let $h(\vec{\lambda}) = f(\vec{\lambda})g(\vec{\lambda})$. Then $H \in E_\sigma$ and

$$\begin{aligned}\delta H(y|w) &= \sum_{j=1}^{n}\langle\theta_j, w\rangle f_j(\langle\vec{\theta}, y\rangle)g(\langle\vec{\theta}, y\rangle) + f(\langle\vec{\theta}, y\rangle)\sum_{j=1}^{n}\langle\theta_j, w\rangle g_j(\langle\vec{\theta}, y\rangle)\\ &= \delta F(y|w)G(y) + F(y)\delta G(y|w).\end{aligned}$$

Thus equation (1.20) follows from Theorem 1.7. □

By choosing $G = F$ in Theorem 1.8, we obtain the following corollary.

Corollary 1.9. *Let $F \in E_\sigma$ be given by* (1.7). *Then for each $w \in A$,*

$$\int_{C_0[0,T]} F(y)\delta F(y|w)m(dy) = \frac{1}{2}\int_{C_0[0,T]} [F(y)]^2\langle z, y\rangle m(dy), \tag{1.21}$$

where $w(t) = \int_0^t z(s)\,ds$ for some $z \in L_2[0,T]$.

As we saw in Theorem 1.6 above if F belongs to E_σ, then $\delta F(y|w_1)$ also belongs to E_σ as a function of y. Thus if we replace $G(y)$ with $\delta F(y|w_1)$ in Theorem 1.8, then we have the following corollary.

Corollary 1.10. *Let $F \in E_\sigma$ be given by* (1.7). *Then for each $w_1, w_2 \in A$,*

$$\begin{aligned}&\int_{C_0[0,T]} [F(y)\delta^2 F(\cdot|w_1)(y|w_2) + \delta F(y|w_2)\delta F(y|w_1)]m(dy)\\ &= \int_{C_0[0,T]} F(y)\delta F(y|w_1)\langle z_2, y\rangle m(dy)\end{aligned} \tag{1.22}$$

where $w_i(t) = \int_0^t z_i(s)\,ds$ for some $z_i \in L_2[0,T], i = 1, 2$.

As we saw in Theorem 1.4 above if G belongs to E_σ, then $\mathcal{F}_{\alpha,\beta}G$ also belongs to E_σ. Thus if we replace G with $\mathcal{F}_{\alpha,\beta}G$ in Theorem 1.8, then we have the following corollary.

Corollary 1.11. *Let $F, G \in E_\sigma$ be given as in Theorem 1.8. Then for each $w \in A$,*

$$\begin{aligned}&\int_{C_0[0,T]} [F(y)\delta\mathcal{F}_{\alpha,\beta}G(y|w) + \delta F(y|w)\mathcal{F}_{\alpha,\beta}G(y)]\, m(dy)\\ &= \int_{C_0[0,T]} F(y)\mathcal{F}_{\alpha,\beta}G(y)\langle z, y\rangle m(dy),\end{aligned} \tag{1.23}$$

where $w(t) = \int_0^t z(s)ds$ for some $z \in L_2[0,T]$.

By replacing F and G by $\mathcal{F}_{\alpha,\beta}F$ and $\mathcal{F}_{\alpha,\beta}G$, respectively, in Theorem 1.8, we obtain the following corollary.

Corollary 1.12. *Let $F, G \in E_\sigma$ be as in Theorem 2.5. Then for each $w \in A$,*

$$\begin{aligned}&\int_{C_0[0,T]} [\mathcal{F}_{\alpha,\beta}F(y)\delta\mathcal{F}_{\alpha,\beta}G(y|w) + \delta\mathcal{F}_{\alpha,\beta}F(y|w)\mathcal{F}_{\alpha,\beta}G(y)] m(dy)\\ &= \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta}F(y)\mathcal{F}_{\alpha,\beta}G(y)\langle z, y\rangle m(dy),\end{aligned} \tag{1.24}$$

where $w(t) = \int_0^t z(s)ds$ for some $z \in L_2[0,T]$.

§ 2. Various integration formulas and examples

In this section we establish various integration formulas involving integral transforms, convolution products and first variations. Furthermore we give some examples to illustrate the integration formulas in this paper.

In [12], Kim, Kim and Skoug established various relationships holding among $\mathcal{F}_{\alpha,\beta}F$, $\mathcal{F}_{\alpha,\beta}G$, $(F * G)_\alpha$, δF and δG. From these relationships and the results in Section 1 above, we can establish various integration formulas.

From Theorem 1.7 above we know that the Wiener integral of the first variation of functional $F \in E_\sigma$ is expressed in terms of the Wiener integral of F multiplied by a linear factor. On the other hand, some of the formulas, for example, Formulas 3.3, 3.5, 4.1, 4.2, and 5.2 in [12] give us the expressions of the first variation of various functionals. Hence it is easy to obtain the following formulas (2.1) through (2.7) below. We just state the formulas without proofs.

Let $w \in A$ with $w(t) = \int_0^t z(s)ds$ for some $z \in L_2[0,T]$ throughout this section. The paper [12] was concerned with the class E_0. But as commented in Remark 5.6 of that paper, all the formulas in [12] still true for functionals in E_σ. Hence we will assume that $F \in E_\sigma$ in Formula 2.1 through Formula 2.6 and Corollary 2.7 below.

Formula 2.1. From Formula 3.3 of [7], we have

$$\beta\int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta}\delta F(\cdot|w)(y)m(dy) = \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta}F(y)\langle z,y\rangle m(dy). \tag{2.1}$$

Formula 2.2. From Formula 3.5 of [12], we have

$$\begin{aligned}&\sum_{j=1}^{n}\frac{\langle\theta_j,w\rangle}{\sqrt{2}}\int_{C_0[0,T]}[(F_j*G)_\alpha(y)+(F*G_j)_\alpha(y)]\,m(dy)\\ &=\int_{C_0[0,T]}(F*G)_\alpha(y)\langle z,y\rangle m(dy)\end{aligned} \tag{2.2}$$

and if $F=G$,

$$\sqrt{2}\sum_{j=1}^{n}\langle\theta_j,w\rangle\int_{C_0[0,T]}(F*F_j)_\alpha(y)m(dy)=\int_{C_0[0,T]}(F*F)_\alpha(y)\langle z,y\rangle m(dy). \tag{2.3}$$

Formula 2.3. From Formula 4.1 of [12], we have

$$\begin{aligned}&\int_{C_0[0,T]}\Big(\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)\delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(y|w)+\delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y|w)\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(y)\Big)m(dy)\\ &=\int_{C_0[0,T]}\Big(\mathcal{F}_{\alpha,\beta}F\Big(\frac{y}{\sqrt{2}}\Big)\delta\mathcal{F}_{\alpha,\beta}G\Big(\frac{y}{\sqrt{2}}\Big|\frac{w}{\sqrt{2}}\Big)+\delta\mathcal{F}_{\alpha,\beta}F\Big(\frac{y}{\sqrt{2}}\Big|\frac{w}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,\beta}G\Big(\frac{y}{\sqrt{2}}\Big)\Big)m(dy)\\ &=\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}(F*G)_\alpha(y)\langle z,y\rangle m(dy)\end{aligned} \tag{2.4}$$

and if $F=G$,

$$\begin{aligned}&\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}\left[F\Big(\frac{y}{\sqrt{2}}\Big)\right]\left[\delta\mathcal{F}_{\alpha,\beta}F\Big(\frac{y}{\sqrt{2}}\Big|\frac{w}{\sqrt{2}}\Big)\right]m(dy)\\ &=\frac{1}{2}\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}(F*F)_\alpha(y)\langle z,y\rangle m(dy).\end{aligned} \tag{2.5}$$

Formula 2.4. From Formula 4.2 of [12], we have

$$\begin{aligned}&\frac{\beta}{\sqrt{2}}\sum_{j=1}^{n}\langle\theta_j,w\rangle\int_{C_0[0,T]}[(\mathcal{F}_{\alpha,\beta}F_j*\mathcal{F}_{\alpha,\beta}G)_\alpha(y)+(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G_j)_\alpha(y)]m(dy)\\ &=\int_{C_0[0,T]}(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_\alpha(y)\langle z,y\rangle m(dy).\end{aligned} \tag{2.6}$$

Formula 2.5. From Formula 5.2 of [12] we have,

$$\begin{aligned}&\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}\left(\delta F(\cdot|w)G(\cdot)+F(\cdot)\delta G(\cdot|w)\right)\Big(\frac{y}{\sqrt{2}}\Big)m(dy)\\ &=\frac{\sqrt{2}}{\beta}\int_{C_0[0,T]}(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_{i\alpha/\beta}(y)\langle z,y\rangle m(dy).\end{aligned} \tag{2.7}$$

Using the equations (1.10) through (1.12) we obtain the following integration formula for the Wiener integral of the integral transform with respect to the first argument of the variation.

Formula 2.6. For $F \in E_\sigma$ we have

$$\int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}}(\delta F(\cdot|w))(y)m(dy) = \sum_{j=1}^{n} \langle \theta_j, w \rangle \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}} F_j(y)m(dy). \tag{2.8}$$

Proof. By (1.16) and Theorem 1.7 we have

$$\frac{\beta}{\sqrt{2}} \sum_{j=1}^{n} \langle \theta_j, w \rangle \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}} F_j(y)m(dy) = \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}} F(y)\langle z, y \rangle m(dy).$$

Similarly by (1.17) and Theorem 1.7 we have

$$\frac{\beta}{\sqrt{2}} \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}}(\delta F(\cdot|w))\Big(\frac{y}{\sqrt{2}}\Big) m(dy) = \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}} F(y)\langle z, y \rangle m(dy).$$

Thus we have the above formula (2.8). □

We next obtain an integration formula for functionals which is a product of elements of E_σ by some linear factors.

Corollary 2.7. *Let k be a natural number and let $z_j \in L_2[0,T]$ for $j = 1,2,\ldots,k+1$. Let $F \in E_\sigma$ and let*

$$F^{[k]}(y) = F^{[k-1]}(y)\langle z_k, y \rangle = F(y) \prod_{j=1}^{k} \langle z_j, y \rangle.$$

Then we have the following integral equation.

$$\begin{aligned} &\int_{C_0[0,T]} F^{[k+1]}(y)m(dy) \\ &= \int_{C_0[0,T]} \delta F^{[k-1]}(y|w_{k+1})\langle z_k, y \rangle m(dy) + \langle z_k, w_{k+1} \rangle \int_{C_0[0,T]} F^{[k-1]}(y)m(dy) \end{aligned} \tag{2.9}$$

where $F^{[0]} = F$ and $w_{k+1}(t) = \int_0^t z_{k+1}(s)ds$.

Proof. To prove this theorem we simply take the first variation of the $F^{[k]}(y) = F^{[k-1]}(y)\langle z_k, y \rangle$. Now we have

$$\begin{aligned} \delta F^{[k]}(y|w_{k+1}) &= \frac{\partial}{\partial t}(F^{[k-1]}(y + t w_{k+1})\langle z_k, y + t w_{k+1} \rangle)|_{t=0} \\ &= \delta F^{[k-1]}(y)\langle z_k, y \rangle + F^{[k-1]}(y)\langle z_k, w_{k+1} \rangle. \end{aligned}$$

Hence we have

$$\int_{C_0[0,T]} \delta F^{[k]}(y|w_{k+1})m(dy)$$
$$= \int_{C_0[0,T]} \delta F^{[k-1]}(y)\langle z_k, y\rangle m(dy) + \langle z_k, w_{k+1}\rangle \int_{C_0[0,T]} F^{[k-1]}(y)m(dy)$$

But by Theorem 1.7,

$$\int_{C_0[0,T]} \delta F^{[k]}(y|w_{k+1})m(dy) = \int_{C_0[0,T]} \delta F^{[k]}\langle z_{k+1}, y\rangle m(dy) = \int_{C_0[0,T]} F^{[k+1]}(y)m(dy)$$

and this completes the proof. □

We finish this section by giving some examples for the illustration of the integration by parts formulas.

Example 2.8. Let $F(y) = \sum_{j=1}^{n} \langle \theta_j, y\rangle$ which is an element of E_σ, then we have $\delta F(y|w) = F(w)$, where $w(t) = \int_0^t z(s)ds \in A$. Thus we can obtain

$$\int_{C_0[0,T]} \delta F(y|w)m(dy) = \int_{C_0[0,T]} F(w)m(dy) = \sum_{j=1}^{n} \langle \theta_j, w\rangle = \sum_{j=1}^{n} \int_0^T \theta_j(s)z(s)ds.$$

Since the constant functional $G \equiv 1$ belongs to E_σ and its first variation equals to zero, Theorem 1.8 yields the following.

$$\sum_{j=1}^{n} \int_{C_0[0,T]} \langle \theta_j, y\rangle \langle z, y\rangle m(dy) = \sum_{j=1}^{n} \int_0^T \theta_j(s)z(s)ds. \tag{2.10}$$

Example 2.9. Let $G(y) = \exp\{\sum_{j=1}^{n} \langle \theta_j, y\rangle\}$ which is an element of E_σ, then we have

$$\delta G(y|w) = \sum_{j=1}^{n} \langle \theta_j, w\rangle \exp\Big\{\sum_{j=1}^{n} \langle \theta_j, y\rangle\Big\} = \sum_{j=1}^{n} \langle \theta_j, w\rangle G(y),$$

where $w(t) = \int_0^t z(s)ds \in A$. Hence by the Wiener integration formula

$$\begin{aligned}\int_{C_0[0,T]} \delta G(y|w)m(dy) =& \sum_{j=1}^{n} \langle \theta_j, w\rangle \int_{C_0[0,T]} \sum_{j=1}^{n} \exp\{\langle \theta_j, y\rangle\} m(dy) \\ =& e^{n/2} \sum_{j=1}^{n} \langle \theta_j, w\rangle.\end{aligned}$$

From Theorem 1.8 we obtain the following Wiener integral.

$$\int_{C_0[0,T]} \exp\Big\{\sum_{j=1}^{n} \langle \theta_j, y\rangle\Big\} \langle z, y\rangle m(dy) = e^{n/2} \sum_{j=1}^{n} \langle \theta_j, w\rangle. \tag{2.11}$$

Example 2.10. Let $H(y) = \sum_{j=1}^{n}[\langle\theta_j,y\rangle]^2$ which is an element of E_σ, then we have,

$$\delta H(y|w) = 2\sum_{j=1}^{n}\langle\theta_j,w\rangle\langle\theta_j,y\rangle = 2\sum_{j=1}^{n}(\theta_j,z)\langle\theta_j,y\rangle.$$

where $w(t) = \int_0^t z(s)ds \in A$. Thus we can obtain the following by Wiener integration formula.

$$\int_{C_0[0,T]}\delta H(y|w)m(dy) = 2\sum_{j=1}^{n}(\theta_j,z)\int_{C_0[0,T]}\langle\theta_j,y\rangle m(dy) = 0.$$

By Theorem 1.8 we have the following.

$$\sum_{j=1}^{n}\int_{C_0[0,T]}[\langle\theta_j,y\rangle]^2\langle z,y\rangle m(dy) = 0. \tag{2.12}$$

Example 2.11. Let $L(y) = \left[\sum_{j=1}^{n}\langle\theta_j,y\rangle\right]^2$ which is an element of E_σ, then we have,

$$\delta\mathcal{F}_{\alpha,\beta}L(y|w) = 2\beta^2\sum_{j=1}^{n}\langle\theta_j,w\rangle\sum_{j=1}^{n}\langle\theta_j,y\rangle.$$

From the Wiener integration formula, we have the followings.

$$\int_{C_0[0,T]}\delta\mathcal{F}_{\alpha,\beta}L(y|w)m(dy) = 2\beta^2\sum_{j=1}^{n}\langle\theta_j,w\rangle\int_{C_0[0,T]}\sum_{j=1}^{n}\langle\theta_j,y\rangle m(dy) = 0$$

and

$$\begin{aligned}
&\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}L(y)\langle z,y\rangle m(dy)\\
&=\int_{C_0[0,T]}\left[n\alpha^2 + [\beta\sum_{j=1}^{n}\langle\theta_j,y\rangle]^2\right]\langle z,y\rangle\ m(dy)\\
&=0+\beta^2\int_{C_0[0,T]}\left[\sum_{j=1}^{n}\langle\theta_j,y\rangle\right]^2\langle z,y\rangle\ m(dy).
\end{aligned}$$

From the Corollary 1.11 we have the following.

$$\int_{C_0[0,T]}\left[\sum_{j=1}^{n}\langle\theta_j,y\rangle\right]^2\langle z,y\rangle\ m(dy) = 0. \tag{2.13}$$

Examples 2.8 through 2.11 are interesting to note that we can obtain the Wiener integrals on the left hand side of (2.10) through (2.13) by using Theorem 1.8 or Corollary 1.11 rather than direct calculation using Wiener integration formula.

§ 3. Integral transforms of functionals on function space of two variables

Recently [12] Kim, Kim and Skoug studied the relationships that exist among the integral transform, the convolution product and the first variation for functionals defined on $K[0,T]$, the space of complex-valued continuous functions on $[0,T]$ which vanish at zero. In this paper we extend the results in [12] for functionals of two variables.

Let $Q=[0,S]\times[0,T]$ and let $C(Q)$ denote Yeh-Wiener space; that is, the space of all real-valued continuous functions $x(s,t)$ on Q with $x(s,0)=x(0,t)=0$ for all $0\le s\le S$ and $0\le t\le T$. Yeh [18] defined a Gaussian measure m_Y on $C(Q)$ (later modified in [20]) such that as a stochastic process $\{x(s,t):(s,t)\in Q\}$ has mean $E[x(s,t)]=0$ and covariance $E[x(s,t)x(u,v)]=\min\{s,u\}\min\{t,v\}$.

Let $\mathcal{M}$ denote the class of all Yeh-Wiener measurable subsets of $C(Q)$ and we denote the Yeh-Wiener integral of a Yeh-Wiener integrable functional F by

$$\int_{C(Q)} F(x)\,m_Y(dx). \tag{3.1}$$

Let $K(Q)$ be the space of complex-valued continuous functions defined on Q and satisfying $x(s,0)=x(0,t)=0$ for all $0\le s\le S$ and $0\le t\le T$. Let α and β be nonzero complex numbers. Next we state the definitions of the integral transform $\mathcal{F}_{\alpha,\beta}F$, the convolution product $(F*G)_\alpha$ and the first variation δF for functionals defined on $K(Q)$.

Definition 3.1. Let F be a functional defined on $K(Q)$. Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F is defined by

$$\mathcal{F}_{\alpha,\beta}F(y)=\int_{C(Q)} F(\alpha x+\beta y)\,m_Y(dx),\quad y\in K(Q) \tag{3.2}$$

if it exists [6,12,13,15].

It is obvious that (3.2) implies that

$$\mathcal{F}_{\alpha,\beta}F\Big(\frac{y}{\sqrt{2}}\Big)=\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)$$

for all $y\in K(Q)$.

Definition 3.2. Let F and G be functionals defined on $K(Q)$. Then the convolution product $(F*G)_\alpha$ of F and G is defined by

$$(F*G)_\alpha(y)=\int_{C(Q)} F\Big(\frac{y+\alpha x}{\sqrt{2}}\Big)G\Big(\frac{y-\alpha x}{\sqrt{2}}\Big)\,m_Y(dx),\quad y\in K(Q) \tag{3.3}$$

if it exists [6,10,12,19,21].

Definition 3.3. Let F be a functional defined on $K(Q)$ and let $w \in K(Q)$. Then the first variation δF of F is defined by

$$\delta F(y|w) = \frac{\partial}{\partial t} F(y+tw)|_{t=0}, \quad y \in K(Q) \tag{3.4}$$

if it exists [2,5,14,16].

Now we introduce a concept of the function of bounded variation of two variables, and an integration by parts formula for a Riemann-Stieltjes integral for functions of two variables. The concept of bounded variation for a function of two variables is surprisingly complex. There are several nonequivalent definitions. The paper [7] by Clarkson and Adams is useful in sorting out many of the relationships between the various definitions. In this paper we will use the definition used by Hardy and Krouse [1,11] which we now review.

Let $R = [a,b] \times [c,d]$ and let P be a partition of R given by

$$a = s_0 < s_1 < \cdots < s_n = b, \quad c = t_0 < t_1 < \cdots < t_m = d.$$

A function $f(s,t)$ is said to be of bounded variation on R in the sense of Hardy and Krouse provided the following three conditions hold.

(a) There is a constant k such that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} |f(s_i,t_j) - f(s_i,t_{j-1}) - f(s_{i-1},t_j) + f(s_{i-1},t_{j-1})| \leq k \tag{3.5}$$

for all partition P.

(b) For each $t \in [c,d]$, $f(\cdot,t)$ is a function of bounded variation on $[a,b]$.

(c) For each $s \in [a,b]$, $f(s,\cdot)$ is a function of bounded variation on $[c,d]$.

The total variation $\mathrm{Var}(f,R)$ of f over R is defined to be the supremum of the sums in (3.5) over all partitions P of R. $\mathrm{Var}(f(\cdot,t),[a,b])$ and $\mathrm{Var}(f(s,\cdot),[c,d])$ will denote the total variations of $f(\cdot,t)$ on $[a,b]$ and $f(s,\cdot)$ on $[c,d]$, respectively, as functions of single variable.

The definition of bounded variation used by Hardy and Krouse has the important property that if g is continuous on R and f is of bounded variation on R then the Riemann-Stieltjes integrals $\int_R g(s,t)\,df(s,t)$ and $\int_R f(s,t)\,dg(s,t)$ both exist and are related by the following integration by parts formula [9].

Theorem 3.4. *Let $R = [a,b] \times [c,d]$. Let $g(s,t)$ be a function of bounded variation in the sense of Hardy and Krouse and let $f(s,t)$ be a continuous function on R. Then the following integration by parts formula holds.*

$$\begin{aligned}\int_R g(s,t)\,df(s,t) =& [f(s,t)g(s,t)]_R - \int_c^d [f(s,t)\,dg(s,t)]_a^b \\ & - \int_a^b [f(s,t)\,dg(s,t)]_c^d + \int_R f(s,t)\,dg(s,t)\end{aligned} \tag{3.6}$$

where

$$[f(s,t)g(s,t)]_R = f(b,d)g(b,d) - f(a,d)g(a,d) - f(b,c)g(b,c) + f(a,c)g(a,c),$$

$$[f(s,t)\,dg(s,t)]_a^b = f(b,t)\,dg(b,t) - f(a,t)\,dg(a,t)$$

for each $t \in [c,d]$, *and*

$$[f(s,t)\,dg(s,t)]_c^d = f(s,d)\,dg(s,d) - f(s,c)\,dg(s,c)$$

for each $s \in [a,b]$.

Let $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be an orthonormal set of real-valued functions in $L_2(Q)$. Furthermore assume that each θ_j is of bounded variation in the sense of Hardy and Krouse on Q. Then for each $y \in K(Q)$ and $j = 1, 2, \ldots$, the Riemann-Stieltjes integral $\langle \theta_j, y\rangle \equiv \int_Q \theta_j(s,t)\,dy(s,t)$ exists. Furthermore

$$\begin{aligned} |\langle \theta_j, y\rangle| =& |\theta_j(S,T)y(S,T) - \int_0^T y(S,t)\,d\theta_j(S,t) \\ & - \int_0^S y(s,T)\,d\theta_j(s,T) + \int_Q y(s,t)\,d\theta_j(s,t)| \le C_j\|y\|_\infty \end{aligned} \tag{3.7}$$

with

$$C_j = |\theta_j(S,T)| + \mathrm{Var}(\theta_j(S,\cdot),[0,T]) + \mathrm{Var}(\theta_j(\cdot,T),[0,S]) + \mathrm{Var}(\theta_j,Q) < \infty. \tag{3.8}$$

In Section 4 below, we show that if F and G are elements of $E_\sigma(Q)$ then $\mathcal{F}_{\alpha,\beta}F(\cdot)$, $(F*G)_\alpha(\cdot)$, $\delta F(\cdot|w)$ and $\delta F(y|\cdot)$ exist and are also elements of $E_\sigma(Q)$. Also we examine all relationships involving exactly two of the three concepts of "integral transform", "convolution product" and "first variation" for functionals in $E_\sigma(Q)$. Furthermore we obtain Parseval's and Plancherel's relations for functionals in $E_0(Q)$. For related work, see [3, 6, 10, 12, 14, 15, 16, 19, 21] and for a detailed survey of previous work, see [17].

We finish this section by introducing a well-known Yeh-Wiener integration formula for functionals $f(\langle\vec{\theta},x\rangle)$:

$$\int_{C(Q)} f(\langle\vec{\theta},x\rangle)\,m_Y(dx) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(\vec{u})\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}\,d\vec{u} \tag{3.9}$$

where $\|\vec{u}\|^2 = \sum_{j=1}^n u_j^2$ and $d\vec{u} = du_1\cdots du_n$.

§ 4. Integral transform, convolution product and first variation of functionals in $E_\sigma(Q)$

In our first theorem, we show that if F is an element of $E_\sigma(Q)$, then the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F exists and is an element of $E_\sigma(Q)$.

Theorem 4.1. *Let $F \in E_\sigma(Q)$ be given by* (1.7). *Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ exists, belongs to $E_\sigma(Q)$ and is given by the formula*

$$\mathcal{F}_{\alpha,\beta}F(y) = h(\langle\vec{\theta}, y\rangle) \tag{4.1}$$

for $y \in K(Q)$, where

$$h(\vec{\lambda}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(\alpha\vec{u}+\beta\vec{\lambda})\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}\,d\vec{u}. \tag{4.2}$$

Proof. For each $y \in K(Q)$, using the Yeh-Wiener integration formula (3.9) we obtain

$$\begin{aligned}\mathcal{F}_{\alpha,\beta}F(y) &= \int_{C(Q)} f(\alpha\langle\vec{\theta},x\rangle + \beta\langle\vec{\theta},y\rangle)\,m_Y(dx)\\ &=(2\pi)^{-n/2}\int_{\mathbb{R}^n} f(\alpha\vec{u}+\beta\langle\vec{\theta},y\rangle)\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}\,d\vec{u}\\ &=h(\langle\vec{\theta},y\rangle)\end{aligned}$$

where h is given by (4.2). By [8, Theorem 3.15], $h(\vec{\lambda})$ is an entire function. Moreover by the inequality (1.8) we have

$$|h(\vec{\lambda})| \le (2\pi)^{-n/2}\int_{\mathbb{R}^n} A_F\exp\Big\{B_F\sum_{j=1}^{n}|\alpha u_j+\beta\lambda_j|^{1+\sigma} - \frac{1}{2}\|\vec{u}\|^2\Big\}\,d\vec{u}.$$

But since

$$|\alpha u_j+\beta\lambda_j|^{1+\sigma} \le |2\alpha u_j|^{1+\sigma} + |2\beta\lambda_j|^{1+\sigma},$$

we have

$$|h(\vec{\lambda})| \le A_{\mathcal{F}_{\alpha,\beta}F}\exp\Big\{B_{\mathcal{F}_{\alpha,\beta}F}\sum_{j=1}^{n}|\lambda_j|^{1+\sigma}\Big\},$$

where

$$A_{\mathcal{F}_{\alpha,\beta}F} = (2\pi)^{-n/2}A_F\Big(\int_{\mathbb{R}}\exp\Big\{B_F|2\alpha u|^{1+\sigma} - \frac{u^2}{2}\Big\}\,du\Big)^n < \infty$$

and $B_{\mathcal{F}_{\alpha,\beta}F} = B_F(2|\beta|)^{1+\sigma}$. Hence $\mathcal{F}_{\alpha,\beta}F \in E_\sigma(Q)$. □

In our next theorem we show that the convolution product of functionals from $E_\sigma(Q)$ exists and is an element of $E_\sigma(Q)$. We may assume that F and G in Theorem 4.2 below can be expressed using the same positive integer n. For details see Remark 1.4 of [12].

Theorem 4.2. *Let $F, G \in E_\sigma(Q)$ be given by* (1.7) *with corresponding entire functions f and g. Then the convolution $(F * G)_\alpha$ exists, belongs to $E_\sigma(Q)$ and is given by the formula*

$$(F * G)_\alpha(y) = k(\langle\vec{\theta}, y\rangle) \tag{4.3}$$

for $y \in K(Q)$, where

$$k(\vec{\lambda}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f\Big(\frac{\vec{\lambda}+\alpha\vec{u}}{\sqrt{2}}\Big) g\Big(\frac{\vec{\lambda}-\alpha\vec{u}}{\sqrt{2}}\Big) \exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}\, d\vec{u}. \tag{4.4}$$

Proof. For each $y \in K(Q)$, using the Yeh-Wiener integration formula (3.9) we obtain

$$\begin{aligned}(F * G)_\alpha(y) &= \int_{C(Q)} f\Big(\frac{\langle\vec{\theta},y\rangle + \alpha\langle\vec{\theta},x\rangle}{\sqrt{2}}\Big) g\Big(\frac{\langle\vec{\theta},y\rangle - \alpha\langle\vec{\theta},x\rangle}{\sqrt{2}}\Big) m_Y(dx)\\ &=(2\pi)^{-n/2} \int_{\mathbb{R}^n} f\Big(\frac{\langle\vec{\theta},y\rangle + \alpha\vec{u}}{\sqrt{2}}\Big) g\Big(\frac{\langle\vec{\theta},y\rangle - \alpha\vec{u}}{\sqrt{2}}\Big) \exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}\, d\vec{u}\\ &=k(\langle\vec{\theta},y\rangle)\end{aligned}$$

where k is given by (4.4). By [8, Theorem 3.15], $k(\vec{\lambda})$ is an entire function and

$$|k(\vec{\lambda})| \le (2\pi)^{-n/2} \int_{\mathbb{R}^n} A_F A_G \exp\Big\{(B_F + B_G)\sum_{j=1}^{n}\Big(\frac{|\lambda_j| + |\alpha u_j|}{\sqrt{2}}\Big)^{1+\sigma} - \frac{1}{2}\|\vec{u}\|^2\Big\}\, d\vec{u}.$$

By the same method as in Theorem 4.1, we have

$$|k(\vec{\lambda})| \le A_{(F*G)_\alpha} \exp\Big\{B_{(F*G)_\alpha} \sum_{j=1}^{n} |\lambda_j|^{1+\sigma}\Big\},$$

where $B_{(F*G)_\alpha} = (B_F + B_G)2^{(1+\sigma)/2}$ and

$$A_{(F*G)_\alpha} = (2\pi)^{-n/2} A_F A_G \Big(\int_{\mathbb{R}} \exp\Big\{(B_F + B_G)(\sqrt{2}|\alpha u|)^{1+\sigma} - \frac{u^2}{2}\Big\} du\Big)^n < \infty.$$

Hence $(F * G)_\alpha \in E_\sigma(Q)$. $\square$

In Theorem 4.3 below, we fix $w \in K(Q)$ and consider $\delta F(y|w)$ as a function of y, while in Theorem 4.4 below, we fix $y \in K(Q)$ and consider $\delta F(y|w)$ as a function of w.

Theorem 4.3. *Let $F \in E_\sigma(Q)$ be given by* (1.7) *and let $w \in K(Q)$. Then*

$$\delta F(y|w) = p(\langle\vec{\theta},y\rangle) \tag{4.5}$$

for $y \in K(Q)$, where

$$p(\vec{\lambda}) = \sum_{j=1}^{n} \langle\theta_j, w\rangle f_j(\vec{\lambda}). \tag{4.6}$$

Furthermore, as a function of $y \in K(Q)$, $\delta F(y|w)$ is an element of $E_\sigma(Q)$.

Proof. For $y \in K(Q)$,

$$\begin{aligned}\delta F(y|w) =& \frac{\partial}{\partial t} f(\langle\vec{\theta}, y\rangle + t\langle\vec{\theta}, w\rangle)|_{t=0} \\ =& \sum_{j=1}^{n} \langle\theta_j, w\rangle f_j(\langle\vec{\theta}, y\rangle) = p(\langle\vec{\theta}, y\rangle)\end{aligned}$$

where p is given by (4.6). Since $f(\vec{\lambda})$ is an entire function, $f_j(\vec{\lambda})$ and so $p(\vec{\lambda})$ are entire functions. By the Cauchy integral formula we have

$$f_j(\lambda_1, \cdots, \lambda_j, \cdots, \lambda_n) = \frac{1}{2\pi i} \int_{|\zeta - \lambda_j| = 1} \frac{f(\lambda_1, \cdots, \zeta, \cdots, \lambda_n)}{(\zeta - \lambda_j)^2} d\zeta.$$

By the inequality (1.10), for any ζ with $|\zeta - \lambda_j| = 1$, we have

$$\begin{aligned}\left|\frac{f(\lambda_1, \cdots, \zeta, \cdots, \lambda_n)}{(\zeta - \lambda_j)^2}\right| \le& A_F \exp\{B_F[|\lambda_1|^{1+\sigma} + \cdots + |\zeta|^{1+\sigma} + \cdots + |\lambda_n|^{1+\sigma}]\} \\ \le& A_F \exp\left\{2^{1+\sigma} B_F \left[\sum_{j=1}^{n} |\lambda_j|^{1+\sigma} + 1\right]\right\}.\end{aligned}$$

Hence

$$|f_j(\vec{\lambda})| \le A_F \exp\{2^{1+\sigma} B_F\} \exp\left\{2^{1+\sigma} B_F \sum_{j=1}^{n} |\lambda_j|^{1+\sigma}\right\},$$

and so

$$|p(\vec{\lambda})| \le \sum_{j=1}^{n} |\langle\theta_j, w\rangle| |f_j(\vec{\lambda})| \le A_{\delta F(\cdot|w)} \exp\left\{B_{\delta F(\cdot|w)} \sum_{j=1}^{n} |\lambda_j|^{1+\sigma}\right\}$$

where

$$A_{\delta F(\cdot|w)} = A_F \exp\{2^{1+\sigma} B_F\} \|w\|_\infty \sum_{j=1}^{n} C_j < \infty$$

with C_j given by (3.8) and $B_{\delta F(\cdot|w)} = 2^{1+\sigma} B_F$. □

Theorem 4.4. *Let $y \in K(Q)$ and let $F \in E_\sigma(Q)$ be given by* (1.7). *Then*

$$\delta F(y|w) = q(\langle\vec{\theta}, w\rangle) \tag{4.7}$$

for $w \in K(Q)$, where

$$q(\vec{\lambda}) = \sum_{j=1}^{n} \lambda_j f_j(\langle\vec{\theta}, y\rangle). \tag{4.8}$$

Furthermore, as a function of w, $\delta F(y|w)$ is an element of $E_\sigma(Q)$.

Proof. Equations (4.7) and (4.8) are immediate from the first part of the proof of Theorem 4.3. Clearly $q(\vec{\lambda})$ is an entire function. Next, using the estimation for $|f_j|$ we saw in the proof of Theorem 4.3 above we obtain,

$$\begin{aligned}|q(\vec{\lambda})| \le& \sum_{j=1}^{n} |\lambda_j f_j(\langle\vec{\theta},y\rangle)| \\ \le& A_F \exp\{2^{1+\sigma}B_F\}\exp\{2^{1+\sigma}B_F\|y\|_\infty^{1+\sigma}(C_1^{1+\sigma}+\cdots+C_n^{1+\sigma})\}\sum_{j=1}^{n}|\lambda_j|.\end{aligned}$$

Since $t \le \exp\{t^{1+\sigma}\}$ for all $t \ge 0$,

$$\sum_{j=1}^{n}|\lambda_j| \le \exp\Big\{\Big(\sum_{j=1}^{n}|\lambda_j|\Big)^{1+\sigma}\Big\} \le \exp\Big\{n^{1+\sigma}\sum_{j=1}^{n}|\lambda_j|^{1+\sigma}\Big\}$$

and so

$$|q(\vec{\lambda})| \le A_{\delta F(y|\cdot)} \exp\Big\{B_{\delta F(y|\cdot)}\sum_{j=1}^{n}|\lambda_j|^{1+\sigma}\Big\}$$

where $B_{\delta F(y|\cdot)} = n^{1+\sigma}$ and

$$A_{\delta F(y|\cdot)} = A_F \exp\{2^{1+\sigma}B_F\}\exp\{2^{1+\sigma}B_F\|y\|_\infty^{1+\sigma}(C_1^{1+\sigma}+\cdots+C_n^{1+\sigma})\} < \infty.$$

Hence, as a function of w, $\delta F(y|w) \in E_\sigma(Q)$. $\square$

Now, we establish all of the various relationships involving exactly two of the three concepts of "integral transform", "convolution product" and "first variation" for functionals belonging to $E_\sigma(Q)$. The seven distinct relationships, as well as alternative expressions for some of them, are given by equations (4.9) through (4.15) below.

In view of Theorem 4.1 through Theorem 4.4 above, all of the functionals that occur in this section are elements of $E_\sigma(Q)$. For example, let F and G be any functionals in $E_\sigma(Q)$. Then by Theorem 4.2, the functional $(F*G)_\alpha$ belongs to $E_\sigma(Q)$, and hence by Theorem 4.1, the functional $\mathcal{F}_{\alpha,\beta}(F*G)_\alpha$ also belongs to $E_\sigma(Q)$. By similar arguments, all of the functionals that arise in equations (4.9) through (4.19) below, exist and belong to $E_\sigma(Q)$.

Once we have shown the existence theorems (Theorems 4.1 through 4.4 above), the proofs of the Formulas 4.5 through 4.11 below are exactly the same as those in [12]. Hence we just state the formulas without proofs.

Formula 4.5. The integral transform of the convolution product equals the product of the integral transforms:

$$\mathcal{F}_{\alpha,\beta}(F*G)_\alpha(y) = \mathcal{F}_{\alpha,\beta}F\Big(\frac{y}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,\beta}G\Big(\frac{y}{\sqrt{2}}\Big) = \mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(y) \tag{4.9}$$

for all y in $K(Q)$.

Formula 4.6. A formula for the convolution product of the integral transform of functionals from $E_\sigma(Q)$:

$$
\begin{aligned}
&(\mathcal{F}_{\alpha,\beta}F * \mathcal{F}_{\alpha,\beta}G)_\alpha(y) \\
&= (2\pi)^{-3n/2}\int_{\mathbb{R}^{3n}} f\Big(\alpha\vec{r} + \frac{\beta}{\sqrt{2}}\langle\vec{\theta},y\rangle + \frac{\beta\alpha}{\sqrt{2}}\vec{u}\Big) \qquad (4.10)\\
&\qquad g\Big(\alpha\vec{s} + \frac{\beta}{\sqrt{2}}\langle\vec{\theta},y\rangle - \frac{\beta\alpha}{\sqrt{2}}\vec{u}\Big)\exp\Big\{-\frac{\|\vec{u}\|^2 + \|\vec{r}\|^2 + \|\vec{s}\|^2}{2}\Big\}\,d\vec{u}\,d\vec{r}\,d\vec{s}
\end{aligned}
$$

for all y in $K(Q)$.

Formula 4.7. The integral transform with respect to the first argument of the variation equals $1/\beta$ times the variation of the integral transform:

$$\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w))(y) = \frac{1}{\beta}\delta\mathcal{F}_{\alpha,\beta}F(y|w) = \sum_{j=1}^{n}\langle\theta_j, w\rangle\mathcal{F}_{\alpha,\beta}F_j(y) \tag{4.11}$$

for all y and w in $K(Q)$.

Formula 4.8. The transform with respect to the second argument of the variation equals β times the variation of the functional:

$$\mathcal{F}_{\alpha,\beta}(\delta F(y|\cdot))(w) = \beta\delta F(y|w) \tag{4.12}$$

for all y and w in $K(Q)$.

Formula 4.9. A formula for the first variation of the convolution product of functionals from $E_\sigma(Q)$:

$$\delta(F*G)_\alpha(y|w) = \sum_{j=1}^{n}\frac{\langle\theta_j,w\rangle}{\sqrt{2}}[(F_j*G)_\alpha(y) + (F*G_j)_\alpha(y)] \tag{4.13}$$

for all y and w in $K(Q)$.

Formula 4.10. A formula for the convolution product, with respect to the first argument of the variation, of the variation of functionals from $E_\sigma(Q)$:

$$(\delta F(\cdot|w) * \delta G(\cdot|w))_\alpha(y) = \sum_{j=1}^{n}\sum_{l=1}^{n}\langle\theta_j,w\rangle\langle\theta_l,w\rangle(F_j*G_l)_\alpha(y) \tag{4.14}$$

for all y and w in $K(Q)$.

Formula 4.11. A formula for the convolution product, with respect to the second argument of the variation, of the variation of functionals from $E_\sigma(Q)$:

$$(\delta F(y|\cdot) * \delta G(y|\cdot))_\alpha(w) = \frac{1}{2}\delta F(y|w)\delta G(y|w) - \frac{\alpha^2}{2}\sum_{j=1}^{n}F_j(y)G_j(y) \tag{4.15}$$

for all y and w in $K(Q)$.

Finally, letting $G=F$ in equations (4.9), (4.13), (4.14) and (4.15) above, yields the formulas

$$\mathcal{F}_{\alpha,\beta}(F*F)_\alpha(y)=[\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)]^2, \tag{4.16}$$

$$\delta(F*F)_\alpha(y|w)=\sqrt{2}\sum_{j=1}^{n}\langle\theta_j,w\rangle(F*F_j)_\alpha(y), \tag{4.17}$$

$$(\delta F(\cdot|w)*\delta F(\cdot|w))_\alpha(y)=\sum_{j=1}^{n}\sum_{l=1}^{n}\langle\theta_j,w\rangle\langle\theta_l,w\rangle(F_j*F_l)_\alpha(y), \tag{4.18}$$

and

$$(\delta F(y|\cdot)*\delta F(y|\cdot))_\alpha(w)=\frac{1}{2}[\delta F(y|w)]^2-\frac{\alpha^2}{2}\sum_{j=1}^{n}[F_j(y)]^2 \tag{4.19}$$

for all y and w in $K(Q)$.

Furthermore we can obtain the following Parseval's and Plancherel's relation.

Let $H_0=H_0(Q)$ be the space of real-valued functions f on Q which are absolutely continuous and whose derivative Df is in $L_2(Q)$. The inner product on H_0 is given by

$$\langle f,g\rangle=\int_Q(Df)(s)(Dg)(s)\,ds.$$

Then H_0 is a real separable infinite dimensional Hilbert space. Let $B_0=B_0(Q)$ be the Yeh-Wiener space $C(Q)$ and equip B_0 with the sup norm. Then (H_0,B_0,m_Y) is an example of an abstract Wiener space.

We restrict our attention, in this subsection, to the space $E_0(Q)$ rather than $E_\sigma(Q)$. Now it is well known, see for example [6,15], that for all $F\in E_0(Q)$, all $y\in K(Q)$ and all complex numbers a,b and c,

$$\int_{C(Q)}\int_{C(Q)}F(ax+by+cw)\,m_Y(dx)\,m_Y(dy)=\int_{C(Q)}F(\sqrt{a^2+b^2}z+cw)\,m_Y(dz) \tag{4.20}$$

and that

$$\mathcal{F}_{\alpha,\beta}(\mathcal{F}_{\alpha',\beta'}F)(y)=F(y)=\mathcal{F}_{\alpha',\beta'}(\mathcal{F}_{\alpha,\beta}F)(y) \tag{4.21}$$

provided $\beta\beta'=1$ and $\alpha^2+(\beta\alpha')^2=0$.

Theorem 4.12. *Let $F,G\in E_0(Q)$ and let α' be a complex number such that $\alpha^2+(\beta\alpha')^2=0$. Then Parseval's relation*

$$\int_{C(Q)}\mathcal{F}_{\alpha,\beta}F\Big(\frac{\alpha'y}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,\beta}G\Big(\frac{\alpha'y}{\sqrt{2}}\Big)\,m_Y(dy)=\int_{C(Q)}F\Big(\frac{\alpha y}{\sqrt{2}}\Big)G\Big(-\frac{\alpha y}{\sqrt{2}}\Big)\,m_Y(dy) \tag{4.22}$$

holds. In particular, if $\beta=i$, we have

$$\int_{C(Q)}\mathcal{F}_{\alpha,i}F\Big(\frac{\alpha y}{\sqrt{2}}\Big)G\Big(\frac{\alpha y}{\sqrt{2}}\Big)\,m_Y(dy)=\int_{C(Q)}F\Big(\frac{\alpha y}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,i}G\Big(\frac{\alpha y}{\sqrt{2}}\Big)\,m_Y(dy). \tag{4.23}$$

Moreover, formula (4.23) *induces Plancherel's relation of the form*

$$\int_{C(Q)} \left| \mathcal{F}_{\alpha,i} F\left(\frac{\alpha y}{\sqrt{2}}\right) \right|^2 m_Y(dy) = \int_{C(Q)} \left| F\left(\frac{\alpha y}{\sqrt{2}}\right) \right|^2 m_Y(dy). \tag{4.24}$$

Proof. From Formula 4.5 and Definition 3.1, it follows that the left hand side of (4.22) is equal to

$$\begin{aligned} &\int_{C(Q)} \mathcal{F}_{\alpha,\beta}(F * G)_\alpha(\alpha' y) m_Y(dy) \\ &= \int_{C(Q)} \int_{C(Q)} (F * G)_\alpha(\alpha x + \beta \alpha' y) m_Y(dx) m_Y(dy). \end{aligned}$$

But by (4.20) and the fact that $\alpha^2 + (\beta\alpha')^2 = 0$, the last integral is equal to $(F * G)_\alpha(0)$, which is equal to the right hand side of (4.22).

From (4.21) we know that $\mathcal{F}_{\alpha,i}(\mathcal{F}_{\alpha,-i}G)(y) = G(y)$ and so we have

$$\begin{aligned} &\int_{C(Q)} \mathcal{F}_{\alpha,i} F\left(\frac{\alpha y}{\sqrt{2}}\right) G\left(\frac{\alpha y}{\sqrt{2}}\right) m_Y(dy) \\ &= \int_{C(Q)} \mathcal{F}_{\alpha,i} F\left(\frac{\alpha y}{\sqrt{2}}\right) \mathcal{F}_{\alpha,i}(\mathcal{F}_{\alpha,-i}G)\left(\frac{\alpha y}{\sqrt{2}}\right) m_Y(dy) \\ &= \int_{C(Q)} F\left(\frac{\alpha y}{\sqrt{2}}\right) \mathcal{F}_{\alpha,-i} G\left(-\frac{\alpha y}{\sqrt{2}}\right) m_Y(dy), \end{aligned}$$

where the second equality is obtained by (4.22). But it is easy to see that $\mathcal{F}_{\alpha,-i}G(-\alpha y/\sqrt{2}) = \mathcal{F}_{\alpha,i}G(\alpha y/\sqrt{2})$ and this completes the proof of (4.23).

Finally, since $\overline{\mathcal{F}_{\alpha,i}F(\alpha y/\sqrt{2})} = \mathcal{F}_{\overline{\alpha},-i\overline{\alpha}/\alpha}\overline{F}(\alpha y/\sqrt{2})$, by (4.23) we have

$$\int_{C(Q)} \left| \mathcal{F}_{\alpha,i} F\left(\frac{\alpha y}{\sqrt{2}}\right) \right|^2 m_Y(dy) = \int_{C(Q)} F\left(\frac{\alpha y}{\sqrt{2}}\right) \mathcal{F}_{\alpha,i}\mathcal{F}_{\overline{\alpha},-i\overline{\alpha}/\alpha}\overline{F}\left(\frac{\alpha y}{\sqrt{2}}\right) m_Y(dy).$$

But by (4.20), it is easy to see that $\mathcal{F}_{\alpha,i}\mathcal{F}_{\overline{\alpha},-i\overline{\alpha}/\alpha}\overline{F}(\alpha y/\sqrt{2}) = \overline{F(\alpha y/\sqrt{2})}$ and this completes the proof of (4.24). □

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