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White Noise Approach to Feynman Path Integrals and Some of the Related Topics

By

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Abstract

In this paper the following will be discussed:

1. Why the white noise analysis is applied to the Feynman path integrals. Remind the idea of Dirac and Feynman and see how we can realize their idea from mathematics side.
2. White noise analysis, in particular the theory of generalized white noise functionals, is applied. We shall see how they are introduced so that it is fitting for the formulation of the Feynman integrals.
3. Actual method of applications. Examples.
4. Some of further developments in the present line.

§ 1. Introduction

1. The Idea. The basic idea of our approach to the path integrals in quantum dynamics is to apply the *white noise analysis* to the construction of the *quantum mechanical propagators*.

In fact, an attempt to give a correct interpretation to the Feynman integral, which had only formal significance, was one of the motivation of proposing the white noise analysis. Actually, there were two problems for us: one is how to realize a "flat" measure on the function space, thr function space itself should be clarified; another is how to understand the exponential functional of the action.

The path integral method is, as is well known, viewed as a third method of quantization, which is different from the formulation by W. Heisenberg and another one by E. Schrödinger. Our method of path integral, within the framework of white noise analysis, follows mainly the Feynman's method [3] in spirit. However, some other basic quantum mechanical considerations are taken into account.

While we are working the problem in question, we have come to realize that we should see the Dirac's ideas, which are mostly found in his textbook [2].

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2. We shall quickly explain our approach step by step.

i) What does a path means in quantum dynamics? In quantum dynamics, following the Lagrangian dynamics theory, there are many possible paths i.e. trajectories of a particle, and each trajectory may be viewed as a sum of the classical one and fluctuation. The classical path, denoted by y is, of course uniquely determined by the Lagrangian (with boundary values).

The so-called possible trajectories x in quantum dynamics can be expressed in the form

$$x = y + z,$$

where z is the fluctuation (see [4]).

Our first question is to determine the fluctuation z. We propose that z is a Gaussian Markov process, more precisely a Brownian bridge which is a liner function of a Brownian motion $B(t)$. We shall explain the reason why a Brownian bridge is fitting for a realization of fluctuation in the next section.

ii) To observe Feynman's expression of propagator, we first meet the action integral

$$A(t) = \int^t L(x, \dot{x}) ds,$$

The integrand involves a term $\dot{B}(s)^2$ to express the kinetic energy. The $\dot{B}(s)^2$ is not an ordinary random function. To come to the propagator we have even to exponentiate the action.

Thus we shall be concerned with analysis of generalized functions, actually functionals of $\dot{B}(t)$'s, that is *white noise*. §4 is provided to the background for the study of generalized functionals of white noise.

iii) Integration over the function space $X = \{x\}$, the space of trajectories, can be defined smoothly. To this end, we must specify a measure on X: this is now obvious, since we take a white nloise, the probability distribution μ has automatically been introduced.

There is one problem to be reminded. We expect the integral should be done with respect to the uniform measure on X. This problem can be solved also in §3.

iv) With these background we can come to the actual computation of the propagators. This can be done by using the white noise theory.

v) We can recognize that our approach can be applied to a pretty large class of Lagrangians.

vi) Our method of integration on function space can further be applied to other problems in physics as we shall see in the last two sections.

§ 2. Brownian Bridge and a Setup of the Propagator

First we have to explain why the Brownian bridge is involved in the class of quantum mechanical possible trajectories. In [2, §32] , Action principle, there is a statement that $B(t,s) = \int_t^s L(u) du$ satisfies a chain rule, by which we may imagine the formula for the transition probabilities of a Markov process. To fix the idea, we consider the case where the time interval is taken to be $[0,T]$. Now the term z that expresses the quantity of fluctuation can be a Markov process $X(t), 0 \le t \le T$. Further assumptions on $X(t)$ can be made as follows.

1) $X(t)$ is a Gaussian process, since it is a sort of noise [8].
2) As a usual requirement, the Gaussian process satisfies $E(X(t)) = 0$ and has the canonical representation by Brownian motion, namely

$$X(t) = \int_0^t F(t,u)\dot{B}(u)du$$

   and $X(0) = X(T) = 0$ (bridged).
3) $X(t)$ is a Gaussian 1-ple Markov process.
4) The normalized process $Y(t)$ enjoys the projective invariance under time-change.

Theorem 1. *The Brownian bridge $X(t)$ over the time interval $[0,T]$ is characterized by the above conditions* 1) – 4).

This theorem we have proved before and the proof is omitted here. We only note that the canonical representation of $X(t)$ is given by

$$X(t) = (T-t)\int_0^t \frac{1}{T-u}\dot{B}(u)du,$$

and the covariance $\Gamma(t,s)$ is

$$\Gamma(t,s) = \sqrt{\frac{s(T-t)}{t(T-s)}}, \quad s \le t.$$

Namely,

$$\Gamma(t,s) = \sqrt{(0,T;s,t)}, \quad s \le t,$$

where $(\cdot,\cdot;\cdot,\cdot)$ is the anharmonic ratio.

Remark. Heuristically speaking, it was 1981 when we proposed a white noise approach to path integrals to have quantum mechanical propagators (Hida-Streit paper presented 1981 Berlin Conference on Math-Phys. Later Streit-Hida [17]). We had, at that time, some idea in mind for the use of a Brownian bridge, and we had practically many good examples of integrand with various kinds of potentials, and satisfactory results have been obtained.

With this background we are ready to propose how to form quantum mechanical propagators. The possible quantum mechanical trajectories $x(t), t \in [0,T]$ are expressed in the form

$$x(t) = y(t) + \sqrt{\frac{\hbar}{m}}X(t),$$

where $X(t)$ is a Brownian bridge over the time interval $[0,T]$. The fluctuation z in the earlier expression is now taken to be a Brownian bridge.

Remind that the classical trajectory $y(t)$, $t \in [0,T]$, is uniquely determined by the variational principle for the action

$$A[x] = \int_0^T L(x,\dot{x})dt,$$

where the Lagrangian $L(x,\dot{x})$ in question is assumed to be of the form

$$L(x,\dot{x}) = \frac{1}{2}m(\dot{x})^2 - V(x).$$

The potential $V(x)$ is usually assumed to be regular, but later we can extend the theory to the case where V has certain singularity, even time-dependent.

The actual expression and computations of the propagator are given successively as follows:

We follow the Lagrangian dynamics. The possible trajectories are sample paths $y(s), s \in [0,t]$, expressed in the form

$$y(s) = x(s) + \sqrt{\frac{\hbar}{m}}B(s), \tag{1}$$

where the $B(t)$ is an ordinary Brownian motion. Hence the action S is expressed in the form in terms of quantum trajectory y:

$$A = \int_0^t L(y(s),\dot{y}(s))ds.$$

Note that the effect of forming a bridge is given by putting the delta-function $\delta_0(y(t)-y_2)$ as a factor of the integrand, where $y_2 = x(t)$.

Now we set

$$S(t_0,t_1) = \int_{t_0}^{t_1} L(t)dt. \tag{2}$$

and set

$$\exp\left[\frac{i}{\hbar}\int_{t_0}^{t_1} L(t)dt\right] = \exp\left[\frac{i}{\hbar}S(t_0,t_1)\right] = B(t_0,t_1).$$

Then, we have (see Dirac [2]), for $0 < t_1 < t_2 < \cdots < t_n < t$,

$$B(0,t) = B(0,t_1)\cdot B(t_1,t_2)\cdots B(t_n,t).$$

Theorem 2. *The quantum mechanical propagator $G(0,t;y_1,y_2)$ is given by the following average*

$$G(0,t;y_1,y_2) = \left\langle N\exp\Big(\frac{i}{\hbar}\int_0^t L(y,\dot{y})ds + \frac{1}{2}\int_0^t \dot{B}(s)^2 ds\Big)\delta_o(y(t)-y_2)\right\rangle, \tag{3}$$

where N is the amount of multiplicative renormalization. The average $\langle\ \rangle$ is understood to be the integral with respect to the white noise measure μ.

§ 3. Generalized White Noise Functionals Revisited

It is well-known that there are two classes of generalized white noise functionals; $(L^2)^-$ and $(S)^*$. We use them without discrimination except it is necessary to choose one of them specifically.

It seems better to explain the concept of "renormalization" which makes formal but important functionals of the $\dot{B}(t)$'s to be acceptable as generalized white noise functionals. To save time we refer the interpretation to the literatures [11] and [16].

We should note that there are generalized white noise functionals involved in the expectation in Theorem 2. For instance, there is involved the delta function, in fact the Donsker's delta function $\delta_o(y(t) - y_2)$, which is a generalized white noise functional. There is used a Gauss kernel of the form $\exp[c \int_0^t \dot{B}(s)^2 ds]$, the ideal case is $c = -\frac{1}{2}$. In general, if $c \neq \frac{1}{2}$, then it can be a generalized functional after having the multiplicative renormalization. Now we have the exceptional case, but it can be accepted by combining with other factor of an exponential; this is just the case. In reality, we combine it with the term that comes from the kinetic energy. The factor $\exp[\frac{1}{2} \int_0^t \dot{B}(s)^2 ds]$ serves as the flattening effect of the white noise measure. One may ask why the functional does so. An intuitive answer to this question is as follows: If we write a Lebesgue measure (exists only virtually) on E^* by dL, the white noise measure μ may be expressed in the form $\exp[-\frac{1}{2} \int_0^t \dot{B}(s)^2 ds]dL$. Hence, the the factor in question is put to make the measure μ to be a flat measure dL. In fact, this makes sense eventually.

Returning to the average (3) (in Theorem 2), which is an integral with respect to the white noise measure μ, it is important to note that the integrand (i.e. the inside of the angular bracket) is integrable, in other words, it is a bilinear form of a generalized functional and a test functional.

There have to follow short notes to be reminded. They are rather crucial. The formula (3) involves a product of functionals of the form like $\varphi(x) \cdot \delta(\langle x, f\rangle - a)$, $f \in L^2(R)$, $a \in \mathbf{C}$. To give a correct interpretation to the expectation of (3) with this choice, it should be checked that it can be regarded as a bilinear form of a pair of a test functional and a generalized functional. The following assertion answers to this question.

Theorem 3 (Streit et al [19]). *Let $\varphi(x)$ be a generalized white noise functional. Assume that the $\mathcal{T}$-transform $(\mathcal{T}\varphi)(\xi), \xi \in E$, of φ is extended to a functional of f in $L^2(R)$, in particular a function of $\xi + \lambda f$, and that $(\mathcal{T}\varphi)(\xi - \lambda f)$ is an integrable function of λ for any fixed ξ. If the transform $(\mathcal{T}\varphi)(\xi - \lambda f)$ is a U-functional, then the pointwise product $\varphi(x) \cdot \delta(\langle x, f\rangle - a)$ is defined and is a generalized white noise functional.*

Proof. First a formula for the δ-function is provided.

$$\delta_a(t) = \delta(t-a) = \frac{1}{2\pi} \int e^{ia\lambda} e^{-i\lambda x} d\lambda \quad \text{(in distribution sense).}$$

Hence, for $\varphi \in (S)^*$ and $f \in L^2(R)$ we have

$$
\begin{aligned}
\mathcal{T}(\varphi(x)\delta(\langle x,f\rangle - a))\xi) &= \frac{1}{2\pi}\int e^{ia\lambda}e^{-i\lambda\langle x,f\rangle}e^{i\langle x,\xi\rangle}\varphi(x)d\mu(x)d\Lambda \\
&= \frac{1}{2\pi}\int e^{ia\lambda}(\mathcal{T}\varphi)(\xi^{\lambda}f)d\lambda. \qquad (4)
\end{aligned}
$$

By assumption this determines a U-functional, which means the product $\varphi(x)\cdot\delta(\langle x,f\rangle - a)$ makes sense and it is a generalized white noise functional. □

Example 1. The above theorem can be applied to a Gauss kernel $\varphi_c(x) = N\exp[c\int x(t)^2dt]$, with $c \neq \frac{1}{2}$.
i) The case where c is real and $c < 0$. We have

$$
\begin{aligned}
(\mathcal{T}\varphi)(\xi - \lambda f) &= \exp[\frac{c}{1-2c}\int(\xi(t) - \lambda f(t))^2dt] \\
&= \exp[\frac{c}{1-2c}(\|\xi\|2 - 2\lambda(\xi,f) + \lambda^2\|f\|^2]).
\end{aligned}
$$

This is an integrable function of real λ. Hence, by the above Theorem 3, we have a generalized white noise functional.
ii) The case where $c = \frac{1}{2} + ia$, $a \in \mathbf{R} - \{0\}$.
The same expression as in i) is given.

Example 2. In the following case, exact values of the propagators can be obtained and, of course, they agree with the known results.
i) Free particle
ii) Harmonic oscillator.
iii) Potentials which are Fourier transforms of measures (the the Albeverio-Hohkron potential).

§4. Some of Further Developments and Related Topics

[I] In addition to Example 2, we have some more interesting potentials,including those are much singular and time depending. There are satisfactory results in the recent developments.

Example 3. Streit et al have obtained explicit formulae in the following cases:
1) a time depending Lagrangian of the form

$$
L(x(t),\dot{x}(t),t) = \frac{1}{2}m(t)\dot{x}(t)^2 - k(t)^2x(t)^2 - \dot{f}(t)x(t),
$$

where $m(t), k(t)$ and $f(t)$ are smooth funtions.

2) A singular potential $V(x)$ of the form

$$V(x) = \sum_{n} c^{-n^2} \delta_n(x), \quad c > 0,$$

and others.

[II] The Hopf equation.

There are many approaches to the Navier-Stokes equation.

$$u_{\alpha,t} + u_\beta u_{\alpha,\beta} = -p \cdot \alpha + \mu u_{\alpha,\beta\beta},$$

where $\alpha, \beta = 1, 2, 3$, and where the following notations are used:

$$f_{\alpha,t} = \frac{\partial f_\alpha}{\partial t},$$
$$f_{\alpha,\beta} = \frac{\partial f_\alpha}{\partial x_\beta},$$

and

$$f_{\alpha,\beta\gamma} = \frac{\partial^2 f_\alpha}{\partial x_\beta \partial x_\gamma}.$$

There is an approach to this equation by using the characteristic functional Φ of the measure $P^t(du)$ defined on the phase space $\{u = (u_1, u_2, u_3)\}$:

$$\Phi(\xi, t) = \int e^{i<\xi, u>} P^t(du).$$

E. Hopf shows that the characteristic functional $\Phi(\xi, , t)$ satisfies the following functional differential equation, called **Hopf equation**:

$$\frac{\partial \Phi}{\partial t} = \int_R \xi_\alpha(x) \Big[i \frac{\partial}{\partial x_\beta} \frac{\partial^2 \Phi}{\partial \xi_\beta(x) dx \partial \xi_\alpha(x) dx} + \mu \Delta_x \frac{\partial \Phi}{\partial \xi_\alpha(x) dx} - \frac{\partial \Pi}{\partial x_\alpha} \Big] dx$$

Studying this approach, we may think of the two matters. One is a similarity to the Feynman integral in the sense that both cases deal with functional obtained in the form

$$E(\exp[f(u)])$$

where $f(u)$ is a function of a path (trajectory) u. The expectation is taken with respect to the probability measure introduced on the path space.

As the second point, one may think of equations for $\Phi_n, n \geq 0$, that come from the Hopf equation and the Fock space expansion of generalized white noise functionals. In this case we expect that the calculus can be done in a similar manner to the white noise calculus. We may remind an interesting approach to the Navier-Stokes equation by A. Inoue (see [13]).

§ 5. Concluding Remarks

(1) There appears a particular quadratic form in the white noise analysis, i.e.

$$\int :\dot{B}(t)^2: dt.$$

There are somewhat general quadratic form

$$\int f(t) :\dot{B}(t)^2: dt + \int\int F(u,v) :\dot{B}(u)\dot{B}(v): dudv$$

which is called *normal functional*, the first term is called the singular part and the second term is the regular part. The two terms can be characterized from our viewpoint and play significant roles, respectively. Remind the role of singular part in the path integral.

(2) Our method of path integrals enables us to deal with the case of very irregular potentials to have the propagator, by L. Streit and others.

(3) Some other approaches: It is significant to see the results by C. C. Bernido and M. V. Carpio Bernido [1]. They are using our method of path integral to investigate the entanglement probabilities of two chain like macro-molecules where one polymer lies on a plane and the other perpendicular to it. The entanglement probabilities are calculated and the result shows a characteristic of the polymer.

We also should like to note that Masujima [14] has published a beautiful monograph collecting various approaches to path integrals,

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