数理解析研究所講究録
第 1797 巻 2012 年 51-85

Introduction to Analogue of Wiener Measure Space and Its Applications

By

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Abstract

This talk is the improvement of our survey paper [42]. The contents of this talk consist of the following:

(1) The definitions, notations and some well-known facts which are needed to understand this talk.
(2) Complex-valued, measure-valued and operator-valued analogue of Wiener measure and their examples.
(3) The translation theorem of analogue of Wiener measure and its applications.
(4) The integration formula of $\exp\{\alpha\|x\|_{\infty}\}$.
(5) The integration formula of $\exp\{\lambda\int_0^t x(s)^2 ds\}$.
(6) The measure of the set of all analogue of Wiener paths staying below a continuous differentiable function.
(7) Relationship among the Bartle integral and the conditional expectations.
(8) The simple formula for conditional expectation.
(9) The measure-valued Feynman-Kac formula.
(10) Volterra integral equation for the measure-valued Feynman-Kac formula.
(11) Dobrakov's integral with respect to the operator-valued analogue of Wiener measure.
(12) The operational calculus of analogue of Wiener functional.
(13) The theories of Fourier-Feynman transform.

§1. Preliminaries

In this section, we present some notation, definitions and well-known facts which are needed to understand the subsequent sections.

2010 Mathematics Subject Classification(s): Primary 28C20; Secondary 28C35
Key Words: analogue of Wiener measure, Bartle integral, the Bochner integral, Dobrakov integral, Volterra integral equation, conditional expectation, measure-valued Feynman-Kac formula, integral transform
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(A) Let $\mathbb{R}$ be the real number field and $\mathbb{C}$ the complex number field. For a natural number n, let $\mathbb{R}^n$ be the n-times product space of $\mathbb{R}$. Let $\mathcal{B}(\mathbb{R})$ be the set of all Borel measurable subsets of $\mathbb{R}$ and m_L the Lebesgue measure on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = i$ and $\alpha_4 = -i$.

(B) For a positive real number a, b, let $C[a,b]$ be the space of all real-valued continuous functions on a closed bounded interval $[a,b]$ with the supremum norm $\|\cdot\|_\infty$. By the Stone-Weierstrass theorem,

(1.1) $$(C[a,b], \|\cdot\|_\infty) \text{ is a real separable Banach space.}$$

Let $\mathcal{M}(\mathbb{R})$ be the space of all finite complex-valued countably additive measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For $p \in \mathbb{R}$, let δ_p be the Dirac measure concentrated at p with total mass one. For $\mu \in \mathcal{M}(\mathbb{R})$ and for $E \in \mathcal{B}(\mathbb{R})$, the total variation $|\mu|(E)$ on E is defined by

(1.2) $$|\mu|(E) = \sup \sum_{i=1}^{n} |\mu(E_i)|,$$

where the supremum is taken over all finite sequences $\langle E_i \rangle$ of disjoint sets in $\mathcal{B}(\mathbb{R})$. Then $|\mu|$ is in $\mathcal{M}(\mathbb{R})$ and, by the Jordan decomposition theorem [16, p. 307, (19.13) Theorem], there are unique non-negative measures $\mu_j \in \mathcal{M}(\mathbb{R})$ $(j = 1,2,3,4)$ such that

(1.3) $$\mu = \sum_{j=1}^{4} \alpha_j \mu_j .$$

By [10, Theorem 4.1.7], $(\mathcal{M}(\mathbb{R}), |\cdot|(\mathbb{R}))$ is a complex Banach space.

Let $\mathcal{RM}(\mathbb{R})$ be the space of all finite complex-valued measures μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which are absolutely continuous with respect to m_L, that is, the Radon-Nikodim derivative $\dfrac{d|\mu|}{dm_L}$ exists.

(C) Let $(X, \mathcal{B}, \mu)$ be a measure space. For a positive real number p, let $\mathcal{L}^p(X,\mu)$ be the space of complex-valued μ-measurable functions f on X such that $|f|^p$ is $|\mu|$-integrable. Let $\mathcal{L}^\infty(X,\mu)$ be the space of complex-valued μ-measurable functions f on X which are $|\mu|$-essentially bounded. The elements of $L^p(X,\mu)$ and $L^\infty(X,\mu)$ are equivalence classes of functions in $\mathcal{L}^p(X,\mu)$ and $\mathcal{L}^\infty(X,\mu)$, respectively, with the equivalence relation being defined by $|\mu|$-a.e. Since $\mathcal{RM}(\mathbb{R})$ is isomorphic to $L^1(\mathbb{R}, m_L)$, $\mathcal{RM}(\mathbb{R})$ is a Banach space and the dual space $\mathcal{RM}(\mathbb{R})^*$ of $\mathcal{RM}(\mathbb{R})$ is isomorphic to $L^\infty(\mathbb{R}, m_L)$. For $x^* \in \mathcal{RM}(\mathbb{R})^*$, there is a function θ in $L^\infty(\mathbb{R}, m_L)$ such that $x^*(\mu) = \int_{\mathbb{R}} \theta(s)\, d\mu(s)$ for $\mu \in \mathcal{RM}(\mathbb{R})$.

Let $\mathbb{B}$ be a complex Banach space and $\mathbb{B}^*$ the dual space of $\mathbb{B}$. For a $\mathbb{B}$-valued countably additive measure ν on $(X, \mathcal{B})$ and for $E \in \mathcal{B}$, the semivariation $\|\nu\|(E)$ of ν on E is given by

(1.4) $$\|\nu\|(E) = \sup\{|x^*\nu|(E) \mid x^* \in \mathbb{B}^* \text{ and } \|x^*\|_{\mathbb{B}^*} \le 1\}$$

where $|x^*\nu|(E)$ is the total variation on E of the complex-valued measure $x^*\nu$.

(D) Let $\mathbb{B}$ be a complex Banach space and $(X,\mathcal{B},\mu)$ a complex measure space. A function $f\colon X\to\mathbb{B}$ is said to be μ-measurable if there exists a sequence $\langle f_n\rangle$ of $\mathbb{B}$-valued simple functions with

(1.5) $$\lim_{n\to\infty}\|f_n-f\|_{\mathbb{B}}=0\quad |\mu|\text{-a.e.}$$

A function f is said to be μ-weakly measurable if x^*f is μ-measurable for each $x^*\in\mathbb{B}^*$. By the Pettis' measurability theorem [11],

(1.6) f is μ-measurable if and only if f is $|\mu|$-essentially separably valued and f is μ-weakly measurable.

We say that f is μ-Bochner integrable if there exists a sequence $\langle f_n\rangle$ of $\mathbb{B}$-valued simple functions such that $\langle f_n\rangle$ converges to f in the norm sense in $\mathbb{B}$ for $|\mu|$-a.e. and

$$\lim_{n\to\infty}\int_X\|f(t)-f_n(t)\|_{\mathbb{B}}\,d|\mu|(t)=0.$$

In this case, $(\mathrm{Bo})-\int_X f(t)\,d\mu(t)$ is defined by

(1.7) $$(\mathrm{Bo})-\int_X f(t)\,d\mu(t)=\lim_{n\to\infty}\int_X f_n(t)\,d\mu(t),$$

where the limit means the limit in the norm sense. By [11], [4, p. 45, Theorem 2],

(1.8) f is μ-Bochner integrable if and only if $\int_X\|f(t)\|_{\mathbb{B}}\,d|\mu|(t)$ is finite.

By [52, Corollary 2],

(1.9) if U is a bounded linear operator on $\mathbb{B}$ into a Banach $\mathbb{B}_1$ and f is a $\mathbb{B}$-valued μ-Bochner integrable function, then Uf is a $\mathbb{B}_1$-valued μ-Bochner integrable function, and

$$(\mathrm{Bo})-\int_X (Uf)(t)\,d\mu(t)=U((\mathrm{Bo})-\int_X f(t)\,d\mu(t)).$$

Theorem 1.1. *Let $(X,\mathcal{B},\mu)$ be a complex measure space and $f\colon X\to\mathcal{M}(\mathbb{R})$ a μ-Bochner integrable function. Then for $E\in\mathcal{B}(\mathbb{R})$, $[f(t)](E)$ is a complex-valued μ-integrable function of t and*

(1.10) $$[(\mathrm{Bo})-\int_X f(t)\,d\mu(t)](E)=\int_X[f(t)](E)\,d\mu(t).$$

Remark. Consider a function H on $[0,1]\times[0,1]$ defined by $H(x,y)=\chi_{[0,x]}(y)$. Then H is $m_L\times m_L$-integrable on $[0,1]\times[0,1]$, so by the Fubini theorem, $H(x,y)$ is an m_L-integrable function of x for all y and $H(x,\cdot)$ is in $L^\infty([0,1],m_L)$ for all $x\in[0,1]$. But $H(x,\cdot)$ has no essentially separable range, so $H(x,\cdot)$ is not m_L-Bochner integrable. Hence, in generally, the equality (1.10) is not true in the theory of Bochner integral.

(E) Let $\mathbb{B}$ be a complex Banach space and $(Y, \mathcal{C}, \nu)$ a $\mathbb{B}$-valued measure space. Let g be a complex-valued $\|\nu\|$-measurable function on Y, that is, there exists a sequence $\langle g_n \rangle$ of complex-valued simple functions with $\lim_{n\to\infty} |g_n - g| = 0$ $\|\nu\|$-a.e. We say that g is ν-Bartle integrable if there exists a sequence $\langle g_n \rangle$ of simple functions such that $\langle g_n \rangle$ converges to g $\|\nu\|$-a.e. and the sequence $\langle \int_Y g_n(s) d\nu(s) \rangle$ is Cauchy in the norm sense. In this case, (Ba) $-\int_Y g(s) d\nu(s)$ is defined by

$$\text{(Ba)} - \int_Y g(s)\, d\nu(s) = \lim_{n\to\infty} \int_Y g_n(s)\, d\nu(s), \tag{1.11}$$

where the limit means the limit in the norm sense. By [13, Theorem 8],

(1.12) if f is a ν-measurable function which is $\|\nu\|$-essentially bounded, then f is ν-Bartle integrable and

$$\|\text{(Ba)} - \int_Y f(s) d\nu(s)\|_{\mathbb{B}} \leq (\|\nu\| - \operatorname{ess\,sup}|f(s)|)\|\nu\|(Y).$$

By [27, Theorem 2.4],

(1.13) g is ν-Bartle integrable if and only if for each $x^* \in \mathbb{B}^*$, g is $x^*\nu$-integrable, and for each $E \in \mathcal{C}$, there is an element (Ba) $-\int_E g(s)\, d\nu(s)$ in $\mathbb{B}$ such that

$$x^*[\text{(Ba)} - \int_E g(s)\, d\nu(s)] = \int_E g(s)\, dx^*\nu(s) \quad \text{for } x^* \in \mathbb{B}^*.$$

By [13, Theorem 8],

(1.14) if U is a bounded linear operator from $\mathbb{B}$ into a Banach space $\mathbb{B}_1$ and g is ν-Bartle integrable, then g is $U\nu$-Bartle integrable. In this case

$$U[\text{(Ba)} - \int_Y g(s)\, d\nu(s)] = \text{(Ba)} - \int_Y g(s)\, dU\nu(s).$$

By [13, Theorem 10],

(1.15) if $\langle f_n \rangle$ is a sequence of ν-Bartle integrable functions which converges $\|\nu\|$-a.e. to f and if g is a ν-Bartle integrable function such that $|f_n(s)| \leq g(s)$ $\|\nu\|$-a.e. s for all natural numbers n then f is ν-Bartle integrable and for $E \in \mathcal{C}$

$$\text{(Ba)} - \int_E f(s)\, d\nu(s) = \lim_{n\to\infty} \text{(Ba)} - \int_E f_n(s)\, d\nu(s).$$

(F) Let $\mathbb{B}$ be a complex Banach space. Let $(X,\mathcal{B})$ and $(Y,\mathcal{C})$ be two measurable spaces and let $\mathcal{B}\otimes\mathcal{C}$ the σ-algebra of sets in the space $X\times Y$ generated by the family of rectangles $E\times F$ for all E in $\mathcal{B}$ and F in $\mathcal{C}$. Let μ be a complex-valued measure on $(X,\mathcal{B})$ and ν a $\mathbb{B}$-valued measure on $(Y,\mathcal{C})$. For G in $\mathcal{B}\otimes\mathcal{C}$, let

$$(\mu\times\nu)(G)=(\mathrm{Ba})-\int_Y\Big[\int_X\chi_G(u,v)\,d\mu(u)\Big]\,d\nu(v). \tag{1.16}$$

By n [20, Proposition 2], using the dominated convergence theorem in [21], Kluvanek proved that $\mu\times\nu$ is a $\mathbb{B}$-valued measure on $\mathcal{B}\otimes\mathcal{C}$ and for $G\in\mathcal{B}\otimes\mathcal{C}$,

$$\begin{aligned}(\mu\times\nu)(G)&=(\mathrm{Ba})-\int_Y\Big[\int_X\chi_G(u,v)\,d\mu(u)\Big]\,d\nu(v)\\&=(\mathrm{Bo})-\int_X\Big[(\mathrm{Ba})-\int_Y\chi_G(u,v)\,d\nu(v)\Big]\,d\mu(u)\end{aligned} \tag{1.17}$$

holds. Moreover, in [20, Proposition 3], he showed that

$$x^*(\mu\times\nu)=\mu\times(x^*\nu) \tag{1.18}$$

for all $x^*\in\mathbb{B}^*$.

When both measures μ and ν are complex-valued, a sufficient condition for validity of the Fubini theorem is the integrability of the function with respect to $\mu\times\nu$. But, if ν is a vector measure then the integrability of the function with respect to $\mu\times\nu$ is no longer a sufficient condition for the validity of the Fubini theorem. Indeed, we can find a counter example for this fact in [20].

Theorem 1.2. *Let $\mathbb{B}$ be a separable complex Banach space, $(X,\mathcal{B},\mu)$ a complex-valued measure space and $(Y,\mathcal{C},\nu)$ a $\mathbb{B}$-valued measure space. Let $f\colon X\times Y\to\mathbb{C}$ be $\mathcal{B}\otimes\mathcal{C}$-measurable and $\mu\times\nu$-Bartle integrable. Then*

(1.19) *for $\|\nu\|$-a.e. v, $f(u,v)$ is a μ-integrable function of u*

(1.20) *$\int_X f(u,v)\,d\mu(u)$ is ν-Bartle integrable and*

$$(\mathrm{Ba})-\int_{X\times Y}f(u,v)\,d\mu\times\nu(u,v)=(\mathrm{Ba})-\int_Y\Big[\int_X f(u,v)\,d\mu(u)\Big]\,d\nu(v). \tag{1.21}$$

Moreover, if for $|\mu|$-a.e. u, $f(u,v)$ is a ν-Bartle integrable function of v and $(\mathrm{Ba})-\int_Y f(u,v)\,d\nu(v)$ *is μ-Bochner integrable then*

$$\begin{aligned}(\mathrm{Ba})-\int_{X\times Y}f(u,v)\,d\mu\times\nu(u,v)&=(\mathrm{Ba})-\int_X\Big[(\mathrm{Ba})-\int_Y f(u,v)\,d\nu(v)\Big]\,d\mu(u)\\&=(\mathrm{Ba})-\int_Y\Big[\int_X f(u,v)\,d\mu(u)\Big]\,d\nu(v).\end{aligned} \tag{1.22}$$

(G) Let $\mathbb{X}$ and $\mathbb{Y}$ be (real or complex) Banach spaces and denote by $L(\mathbb{X},\mathbb{Y})$ the Banach space of all bounded linear operator from $\mathbb{X}$ to $\mathbb{Y}$. Let T be a non-empty set and $\mathcal{B}$ a σ-algebra of subsets of T. We say that a set function $m\colon \mathcal{B} \to L(\mathbb{X},\mathbb{Y})$ is an operator-valued countably additive in the strong operator topology if for every x in $\mathbb{X}$ the set function $\mathcal{B} \ni E \mapsto m(E)x \in \mathbb{X}$ is a countable additive vector measure. We define a non negative set function $\widehat{m}$, which is called the semivariation of the measure m by equality

$$\widehat{m}(E) = \sup\{\|\sum_{i=1}^{n} m(E\cap E_i)x_i|\ |E_i \in \mathcal{B},\ x_i \in \mathbb{X} \text{ with } |x_i| \leq 1 \text{ for } i = 1,2,\ldots,n \text{ and } E_i \cap E_j = \emptyset \text{ for } i \neq j\}.$$

We say that E is an integrable subset in $\mathcal{B}$ if the semivariation $\widehat{m}(E)$ of E is finite. Let $\mathcal{K}$ be the set of all integrable subsets of T. From [12], we have following theorem.

Theorem 1.3 (*-Theorem). *Let $\mathbb{Y}$ contains no subspace isomorphic to the space c_0 (for example let $\mathbb{Y}$ be a weakly complete Banach space). Then the semivariation $\widehat{m}$ is continuous on $\mathcal{K}$, that is, if $\langle E_n\rangle$ is a sequence of decreasing subsets in $\mathcal{K}$ with $\lim_{n\to\infty} E_n = \emptyset$ then $\lim_{n\to\infty} \widehat{m}(E_n) = 0$.*

A $\mathcal{K}$-simple function on T with values in $\mathbb{X}$ is called the simple integrable function. For any simple integrable function $\psi = \sum_{k=1}^{n} x_k \chi_{E_k}$, let $\int_E \psi dm = \sum_{k=1}^{n} m(E_k \cap E)x_k$.

A function $f\colon T \to \mathbb{X}$ is called measurable if there is a sequence $\langle f_n\rangle$ of simple integrable functions such that $\lim_{n\to\infty} f_n(t) = f(t)$ for each $t \in T$.

A measurable function $f\colon T \to \mathbb{X}$ is said to be Dobrakov integrable if there is a sequence $\langle f_n\rangle$ of simple integrable functions converging almost everywhere $\widehat{m}$ to f. In this case, the integral of the function f on a set E in $\mathcal{K}$ is defined by the equality

$$(\mathrm{D})\text{-}\int_E f dm = \lim_{n\to\infty} \int_E f_n dm.$$

Here this limit is uniform with respect to $E \in \mathcal{K}$.

(H) Let $(\Omega,\mathcal{B},\mu)$ be a measure space. Let $X\colon \Omega \to \mathbb{R}^{n+1}$ be a measurable function and F a $\mathbb{C}$-valued integrable function on $(\Omega,\mathcal{B},\mu)$. Let $P_X(A) = \mu(X^{-1}(A))$ for $A \in \mathcal{B}(\mathbb{R}^{n+1})$. Then P_X is a measure on $\mathcal{B}(\mathbb{R}^{n+1})$. By the Radon-Nikodym theorem, there is a function $E^{\mu}(F|X)$, unique up to μ-null sets such that

$$\int_{X^{-1}(A)} F d\mu = \int_A E^{\mu}(F|X) dP_X$$

for $A \in \mathcal{B}(\mathbb{R}^{n+1})$. This function $E^{\mu}(F|X)$ is called the conditional expectation of F given X.

(I) Let φ be in $\mathcal{M}(\mathbb{R})$ and η be a complex-valued Borel measure on $[a,b]$. A complex-valued Borel measurable function θ on $[a,b]\times\mathbb{R}$ is said to belong to $L_{\varphi;\infty,1;\eta}$ (or $L^t_{\varphi;\infty,1;\eta}$) if

$$\|\theta\|_{\varphi;\infty,1;\eta} = \int_{[a,b]} \|\theta(s,\cdot)\|_{\varphi;\infty}\, d|\eta|(s) \tag{1.23}$$

is finite, where

$$\|\theta(0,\cdot)\|_{\varphi;\infty} = \inf\{\lambda>0 \mid |\varphi|(\{\xi\in\mathbb{R} \mid |\theta(0,\xi)|>\lambda\})=0\},$$
$$\|\theta(s,\cdot)\|_{\varphi;\infty} = \inf\{\lambda>0 \mid m_L(\{\xi\in\mathbb{R} \mid |\theta(s,\xi)|>\lambda\})=0\} \qquad (0<s\le t).$$

If θ is bounded Borel measurable then θ is in $L_{\varphi;\infty,1;\eta}$.

(J) For $\theta\in L^\infty(\mathbb{R},m_L)$, we consider an operator M_θ from $\mathcal{RM}(\mathbb{R})$ into itself by

$$[M_\theta(\mu)](E) = \int_E \frac{d\mu}{dm_L}(\xi)\theta(\xi)\, dm_L(\xi) \tag{1.24}$$

for $E\in\mathcal{B}(\mathbb{R})$ and $\mu\in\mathcal{RM}(\mathbb{R})$. Then

$$\frac{dM_\theta(\mu)}{dm_L}(\xi) = \frac{d\mu}{dm_L}(\xi)\theta(\xi), \tag{1.25}$$

so M_θ is well-defined. Since

$$|M_\theta(\mu)|(\mathbb{R}) \le \int_{\mathbb{R}} \left|\frac{d\mu}{dm_L}(\xi)\right| |\theta(\xi)|\, dm_L(\xi) \le \|\theta\|_\infty |\mu|(\mathbb{R}),$$

M_θ is a bounded linear operator.

For $s>0$, let

$$P_s(E) = \int_E \frac{1}{\sqrt{2\pi s}} \exp\{-\frac{u^2}{2s}\}\, dm_L(u) \tag{1.26}$$

for $E\in\mathcal{B}(\mathbb{R})$. For $s>0$, we consider an operator S_s from $\mathcal{RM}(\mathbb{R})$ into itself defined by

$$[S_s(\mu)](E) = (\mu * P_s)(E) = \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \Big[\int_E \exp\{-\frac{(u-v)^2}{2s}\} dm_L(u)\Big]\, d\mu(v). \tag{1.27}$$

Then

$$\frac{dS_s(\mu)}{dm_L}(\xi) = \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \exp\{-\frac{(\xi-v)^2}{2s}\}\, d\mu(v),$$

so S_s is well-defined. It is not hard to show that S_s is a bounded linear operator and the operator norm $\|S_s\|$ of S_s is less than or equals one.

Let s_1 and s_2 be two positive real numbers. Then by the Chapman-Kolmogorov equation in [19] and the classical Fubini theorem, we have

$$S_{s_1}\circ S_{s_2} = S_{s_1+s_2}. \tag{1.28}$$

For $s > 0$, $\varphi \in \mathcal{M}(\mathbb{R})$, a Borel measurable $|\varphi|$-essentially bounded function θ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $E \in \mathcal{B}(\mathbb{R})$, let

$$[T(s,\varphi,\theta)](E) = \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \Big[\int_E \theta(v) \exp\{-\frac{(u-v)^2}{2s}\} dm_L(u) \Big] d\varphi(v). \tag{1.29}$$

Then $T(s,\varphi,\theta) \in \mathcal{RM}(\mathbb{R})$ and

$$\frac{dT(s,\varphi,\theta)}{dm_L}(u) = \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \theta(v) \exp\{-\frac{(u-v)^2}{2s}\} d\varphi(v). \tag{1.30}$$

(K) Let φ is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $F : C[a,b] \to \mathbb{R}$ a measurable function. For all $\lambda > 0$, if the integral $\int_{C[a,b]} F(\lambda^{-1}x)\, d\omega_\varphi(x)$ exists, then we denote

$$\int_{C[a,b]} F(\lambda^{-1}x) d\omega_\varphi(x) = J(\lambda)$$

If there exists a function $J^*(\lambda)$ analytic in the half-plane $\mathbb{C}^+$ such that $J(\lambda) = J^*(\lambda)$ for almost all real $\lambda > 0$, then we write

$$\int_{C[a,b]}^{\text{an anw}_\lambda} F(x) d\omega_\varphi(x) = J^*(\lambda)$$

and we call that $J^*(\lambda)$ is the analytic analogue of Wiener integral of F over $C[a,b]$ with parameter λ, and for non-zero real number q, if the limit $\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} J^*(\lambda)$ exists, then we set

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} J^*(\lambda) = \int_{C[a,b]}^{\text{an anf}_q} F(x) d\omega_\varphi(x)$$

and we say that the limit is the analytic analogue of Feynman integral of F.

Notation. For $\lambda \in \mathbb{C}^+$ and $y \in C[a,b]$ let

$$(T_{\text{an},\lambda}F)(y) = \int_{C[a,b]}^{\text{an anw}_\lambda} F(x+y) d\omega_\varphi(x),$$

and given a number p such that $1 \le p \le \infty$, p and p' will always be related by $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\{H_n\}$ and H be analogue of Wiener measurable functions such that for each $\rho > 0$,

$$\lim_{n \to \infty} \int_{C[a,b]} |H_n(\rho y) - H(\rho y))|^2 d\omega_\varphi(y) = 0.$$

Then we write

$$\lim_{n \to \infty} (w^2_{\varphi,s}) H_n \overset{\text{an}}{\approx} H \tag{1.31}$$

and we call H the scale invariant limit in the mean of order 2 of H_n over $C[a,b]$. We define a similar definition for any real number instead of n. Let q be non-zero real number. For $1 < p \leq 2$ we define the L^p analytic Fourier-Feynman transform of F, which we denote by $T^{(p)}_{\mathrm{an},q}F$, by the formula

$$(T^{(p)}_{\mathrm{an},q}F)(y) = \lim_{\substack{\lambda\in\mathbb{C}^+ \\ \lambda\longrightarrow -iq}} (w^{p'}_{\varphi,s})(T_{\mathrm{an},\lambda}F)(y)$$

whenever this limit exists. Let F be a functional on analogue of Wiener space such that $(T_{an,\lambda}F)(y)$ exists in $\mathbb{C}^+$ for s-almost every y. We define the L^1 analytic analogue of Fourier-Feynman transform of F, which we denote by $T^{(1)}_{\mathrm{an},q}F$, as that functional (if it exists) on analogue of Wiener space such that

$$(T^{(1)}_{\mathrm{an},q}F)(y) = \lim_{\substack{\lambda\in\mathbb{C}^+ \\ \lambda\longrightarrow -iq}} (T_{\mathrm{an},\lambda}F)(y)$$

for s-almost every y. For each natural number n and a partition $a = t_0 < t_1 < \cdots < t_n = b$, let $\mathcal{A}_n$ be the collection of functions $F\colon C[0,t] \to \mathbb{R}$ satisfying (1) and (2) below:

(1) f is a measurable function on $\mathbb{R}^{n+1}$.

(2) $F(x) \overset{\mathrm{an}}{\approx} f(x(t_0), x(t_1), \ldots, x(t_n))$.

(L) Let $(-1)!! = 1!! = 1$, $(2n)!! = (2n)(2n-2)\cdots 2$, $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$ for a natural number n. Let $\prod_{p=k}^{n} c_p = c_k c_{k+1}\cdots c_n$ if $n \geq k$ and $\prod_{p=k}^{n} c_p = 1$ if $n < k$.

By the elementary calculus for integral and the properties of Gamma functions, for a positive real number A and for a non-negative integer m, we have the following equality.

$$\begin{aligned}(1.32)\qquad \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi A}} u^m \exp\{-\frac{(u-u_o)^2}{2A}\} dm_L(u) &= \sum_{k=0}^{[\frac{m}{2}]} \binom{m}{2k} A^k (2k-1)!! u_0^{m-2k} \\ &= \sum_{k=0}^{[\frac{m}{2}]} \frac{m! A^k}{(m-2k)!(2k)!!} u_0^{m-2k}.\end{aligned}$$

Here $[\cdot]$ is the Gauss symbol.

Using Dirichilet's integral in [14] and the change of variables theorem, we can show the following equality.

$$(1.33)\qquad \int_{\Delta^t_n} \prod_{j=1}^{n} (s_j - s_{j-1})^{k_j} d(\prod_{j=1}^{n} m_L)(s_1, s_2, \ldots, s_n) = t^{n+\sum_{j=1}^{n} k_j} \frac{\prod_{j=1}^{n} k_j!}{(n+\sum_{j=1}^{n} k_j)!}.$$

where $k_1, k_2, \ldots, k_n$ are all non-negative integers, $\Delta^t_n = \{(s_1, s_2, \ldots, s_n) \mid 0 < s_1 < s_2 < \cdots < s_n \leq t\}$ and $s_0 = 0$.

For a natural number n, let

$$(1.34)\qquad \sum_{k,n}{}' p(k_1,k_2,\ldots,k_n) = \sum_{k_n=0}^{1}\sum_{k_{n-1}=0}^{2-k_n}\sum_{k_{n-2}=0}^{3-k_n-k_{n-1}}\cdots\sum_{k_1=0}^{n-\sum_{j=2}^{n}k_j} p(k_1,k_2,\ldots,k_n).$$

For $1 \le u \le n-1$, k_{n-u} moves from 0 to $(u+1) - \sum_{p=1}^{u} k_{n-(p-1)}$, so

$$(u+2) - \sum_{p=1}^{u+1} k_{n-(p-1)} = [(u+1) - (\sum_{p=1}^{u} k_{n-(p-1)}) - k_{n-u}] + 1 \ge 1.$$

Hence $2-k_n$, $3-(k_n+k_{n-1}),\ldots,n-\sum_{p=2}^{n}k_p$ are all large than or equal 1 which implies that $\sum_{k,n}' p(k_1,k_2,\ldots,k_n)$ is well-defined.

§ 2. The complex-Valued Analogue of Wiener Measure ω_φ

In this section, we will introduce a complex-valued analogue of Wiener measure ω_φ on $C[a,b]$ and we will give some examples of it.

Let n be a non-negative integer. For $\vec{t} = (t_0,t_1,\ldots,t_n)$ with $a = t_0 < t_1 < \cdots < t_n \le b$, let $J_{\vec{t}}\colon C[a,b] \to \mathbb{R}^{n+1}$ be a function with

$$J_{\vec{t}}(x) = (x(t_0),x(t_1),\ldots,x(t_n)).$$

For $B_j \in \mathcal{B}(\mathbb{R})$ $(j=0,1,2,\ldots,n)$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ of $C[a,b]$ is called an interval and let $\mathcal{I}$ be the set of all intervals. For a non-negative finite Borel measure φ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, let

$$m_\varphi(J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)) = \int_{B_0}\Big[\int_{\prod_{j=1}^{n} B_j} W(n+1;\vec{t};u_0,u_1,\ldots,u_n)\,d\prod_{j=1}^{n} m_L(u_1,\ldots,u_n)\Big]\,d\varphi(u_0)$$

where

$$W(n+1;\vec{t};u_0,u_1,\ldots,u_n) = \Big(\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\Big)\exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\Big\}.$$

Then the set $\mathcal{B}(C[a,b])$ of all Borel subsets in $C[a,b]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique positive measure ω_φ on $(C[a,b],\mathcal{B}(C[a,b]))$ such that $\omega_\varphi(I) = m_\varphi(I)$ for all I in $\mathcal{I}$.

For $\varphi \in \mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi = \sum_{j=1}^{4}\alpha_j\varphi_j$, let $\omega_\varphi = \sum_{j=1}^{4}\alpha_j\omega_{\varphi_j}$. We say that ω_φ is the complex-valued analogue of Wiener measure on $(C[a,b],\mathcal{B}(C[a,b]))$, associated with φ. If φ is a Dirac measure δ_0 at the origin in $\mathbb{R}$ then ω_φ is the classical Wiener measure.

By the change of variables formula, we can easily prove the following theorem.

Theorem 2.1 (The Wiener Integration Formula). *If $f\colon \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function then the following equality holds.*

$$\int_{C[a,b]} f(x(t_0), x(t_1), \ldots, x(t_n))\, d\omega_\varphi(x)$$
$$\stackrel{*}{=} \int_{\mathbb{R}^{n+1}} f(u_0, u_1, \ldots, u_n) W(n+1; \vec{t}; u_0, u_1, \ldots, u_n)\, d(\prod_{j=1}^{n} m_L \times \varphi)((u_1, u_2, \ldots, u_n), u_0)$$

where $\stackrel{*}{=}$ means that if one side exists then both sides exist and the two values are equal.

Remark. Let $\varphi \in \mathcal{M}(\mathbb{R})$.

(1) It is not hard to show that ω_φ has no atoms.

(2) $\omega_\varphi(C[a,b]) = \varphi(\mathbb{R})$.

(3) Let $J_t\colon C[a,b] \to \mathbb{C}$ be a function with $J_t(x) = x(t)$. Then for E in $\mathcal{B}(\mathbb{R})$, $\omega_\varphi(J_t^{-1}(E)) = [S_t(\varphi)](E)$.

Example 2.2. Let $\varphi \in \mathcal{M}(\mathbb{R})$.

(1) Let $I = \{x \in C[0,t] \mid x(0) \in B\}$ where B is in $\mathcal{B}(\mathbb{R})$. Then $\omega_\varphi(I) = \varphi(B)$.

(2) Suppose that $f(u) = u$ is φ-integrable. Then for $0 \le s \le t$,

$$\int_{C[0,t]} x(s)\, d\omega_\varphi(x) = \int_{\mathbb{R}} u\, d\varphi(u).$$

If $\varphi = \delta_p$ then $\int_{C[0,t]} x(s)\, d\omega_\varphi(x) = p$ and if φ has a normal distribution with mean α and variation σ^2 then $\int_{C[0,t]} x(s)\, d\omega_\varphi(x) = \alpha$.

(3) Suppose that $g(u) = u^2$ is φ-integrable. Then for $0 \le s \le t$,

$$\int_{C[0,t]} x(s)^2\, d\omega_\varphi(x) = \int_{\mathbb{R}} u^2 d\varphi(u) + s\varphi(\mathbb{R}).$$

If $\varphi = \delta_p$ then $\int_{C[0,t]} x(s)^2 d\omega_\varphi(x) = p^2 + s$ and if φ has a normal distribution with mean α and variance σ^2 then

$$\int_{C[0,t]} x(s)^2 d\omega_\varphi(x) = \alpha^2 + \sigma^2 + s.$$

(4) Let $\mathcal{F}(\varphi)$ be the Fourier transform of a measure φ, that is, $[\mathcal{F}(\varphi)](\xi) = \int_{\mathbb{R}} \exp\{i\xi u\}\, d\varphi(u)$. Then for $0 \le s \le t$,

$$\int_{C[0,t]} \exp\{i\xi x(s)\}\, d\omega_\varphi(x) = \exp\{-\frac{s\xi^2}{2}\}[\mathcal{F}(\varphi)](\xi).$$

If $\varphi = \delta_p$ then $\int_{C[0,t]} \exp\{i\xi x(s)\}\, d\omega_\varphi(x) = \exp\{-\frac{s\xi^2}{2} + ip\xi\}$ and if φ has a normal distribution with mean α and variance σ^2 then

$$\int_{C[0,t]} \exp\{i\xi x(s)\}\, d\omega_\varphi(x) = \exp\{-\frac{(s+\sigma^2)\xi^2}{2} + i\alpha\xi\}.$$

Let $0 < s \leq t$ be given and let $J_s\colon C[0,t] \to \mathbb{R}$ be a function with $J_s(x) = x(s)$. We assume that $\langle \varphi_n \rangle$ converges to φ weakly. By calculation similar as in this example, since $\langle \mathcal{F}(\varphi_n) \rangle$ converges to $\mathcal{F}(\varphi)$ pointwise, $\langle \mathcal{F}(\omega_{\varphi_n}(J_s^{-1}(\cdot))) \rangle$ converges to $\mathcal{F}(\omega_\varphi(J_s^{-1}(\cdot)))$ pointwise, so by the continuity theorem in [1, Theorem 12-5A, p 273], $\langle \omega_{\varphi_n}(J_s^{-1}(\cdot)) \rangle$ converges to $\omega_\varphi(J_s^{-1}(\cdot))$ weakly.
(5) We assume that $k(u) = u^2$ is φ-integrable. For $0 \leq s_1, s_2 \leq t$,

$$\int_{C[0,t]} x(s_1)x(s_2)\, d\omega_\varphi(x) = (\min\{s_1, s_2\})\varphi(\mathbb{R}) + \int_{\mathbb{R}} u^2 d\varphi(u).$$

If $\varphi = \delta_p$ then $\int_{C[0,t]} x(s_1)x(s_2)\, d\omega_\varphi(x) = \min\{s_1, s_2\} + p^2$ and if φ has a normal distribution with mean α and variance σ^2.

$$\int_{C[0,t]} x(s_1)x(s_2)\, d\omega_\varphi(x) = \min\{s_1, s_2\} + \alpha^2 + \sigma^2.$$

(6) For $0 \leq s_1 < s_2 \leq s_3 < s_4 \leq t$ and for $\alpha, \beta \in \mathbb{R}$, using the change of variable formula, we have

$$\begin{aligned}
&\varphi(\mathbb{R})\omega_\varphi(\{x \in C[0,t] \mid x(s_2) - x(s_1) \leq \alpha \text{ and } x(s_4) - x(s_3) \leq \beta\}) \\
&\quad = \omega_\varphi(\{x \in C[0,t] \mid x(s_2) - x(s_1) \leq \alpha\}) \cdot \omega_\varphi(\{x \in C[0,t] \mid x(s_4) - x(s_3) \leq \beta\}).
\end{aligned}$$

Hence, if φ is a probability measure then $x(s_2) - x(s_1)$ and $x(s_4) - x(s_3)$ are independent.

Theorem 2.3. *For $\varphi \in \mathcal{M}(\mathbb{R})$, $|\omega_\varphi| = \omega_{|\varphi|}$ on $(C[a,b], \mathcal{B}(C[a,b]))$.*

We consider a set $\mathcal{A} = \{E \in \mathcal{B}(C[a,b]) \mid |\omega_\varphi|(E) = \omega_{|\varphi|}(E)\}$. Then we have $\mathcal{I} \subset \mathcal{A}$. Since $|\omega_\varphi|$ and $\omega_{|\varphi|}$ are both measures on $(C[a,b], \mathcal{B}(C[a,b]))$, $|\omega_\varphi| = \omega_{|\varphi|}$ on $\mathcal{B}(C[a,b])$.

Theorem 2.4. *If a sequence $\langle \varphi_n \rangle$ of non-negative finite measures, converges to φ in the sense of total variation norm then a sequence $\langle \omega_{\varphi_n} \rangle$ converges to ω_φ in the total variation norm.*

From [2], we can find a sequence $\langle P_n \rangle$ of measures on $C[a,b]$ such that $\langle P_n \rangle$ does not converges to P weakly even though every finite dimensional measures of P_n converges to some finite dimensional measure of P weakly. Here, we want to find the conditions such that $\langle \omega_{\varphi_n} \rangle$ converges to ω_φ weakly whenever $\langle \varphi_n \rangle$ converges to φ weakly.

Lemma 2.5. *Let $X\colon [a,b]\times C[a,b]\to\mathbb{R}$ be a function with $X(s,x)=x(s)$. Then for $a<t_1\le b$ and for $\epsilon>0$,*

$$\omega_\varphi(\{x \mid \sup\{x(s)-x(a)|a\le s\le t_1\}\ge\epsilon\})\le\frac{1}{\epsilon}\sqrt{\frac{2t_1}{\pi}}\exp\{-\frac{\epsilon^2}{2t_1}\}.$$

Lemma 2.6. *For $\epsilon>0$ and $\lambda>0$,*

$$\omega_\varphi(\{x \mid \sup_{0\le t\le\epsilon}|x(t)-x(\epsilon)|\le\lambda\})=\omega_\varphi(\{x \mid \sup_{0<t<\frac{\epsilon}{2}}|x(t)-x(0)|\le\lambda\})^2.$$

Corollary 2.7.

$$\omega_\varphi(\{x \mid \sup_{0\le s\le t_1}|x(s)-x(t_1)|>\lambda\})\le\frac{1}{\lambda}\sqrt{\frac{t_1}{\pi}}e^{-\frac{\lambda^2}{t_1}}(2-\frac{1}{\lambda}\sqrt{\frac{t_1}{\pi}}e^{-\frac{\lambda^2}{t_1}}).$$

Corollary 2.8. *For each positive ϵ and η, there exists a δ with $0<\delta<1$ such that for s_1,s_2 in $[a,b]$*

$$\omega_\varphi(\{x \mid \sup_{|s_1-s_2|<\delta}|x(s_1)-x(s_2)|\ge\epsilon\})\le\eta.$$

From [2], we find the following theorem.

Theorem 2.9. *The sequence $\langle P_n\rangle$ of probability measures on $C[a,b]$ is tight, that is, for positive ϵ there exists a compact set K such that $P_n(K)>1-\epsilon$ for all natural number n, if and only if*

(i) *for each positive η, there exists an α such that $P_n(\{x||x(a)|>\alpha\})\le\eta$ for all n and*
(ii) *for each positive ϵ and η, there exists a δ with $0<\delta<1$ and a natural number n_0 such that for $n\ge n_0$,*

$$P_n(\{x \mid \sup_{|s_1-s_2|<\delta}|x(s_1)-x(s_2)|\ge\epsilon\})\le\eta.$$

From [2], we can find a sequence $\langle P_n\rangle$ of measures on $C[a,b]$ such that $\langle P_n\rangle$ does not converges to P weakly even though every finite dimensional measures of P_n converges to some finite dimensional measure of P weakly. Here, we want to find the conditions such that $\langle\omega_{\varphi_n}\rangle$ converges to ω_φ weakly whenever $\langle\varphi_n\rangle$ converges to φ weakly.

Theorem 2.10. *Let P_n, P be probability measures on $(C[a,b],\mathcal{B}(C[a,b]))$. If the finite dimensional distributions of P_n converge weakly to those of P, and if $\langle P_n\rangle$ is tight, then $\langle P_n\rangle$ converges to P weakly.*

Theorem 2.11. *Suppose $\langle\varphi_n\rangle$ is tight. Then $\langle\omega_{\varphi_n}\rangle$ is also tight.*

Lemma 2.12. *Let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ be bounded continuous. Let $\vec{t}=(t_0,t_1,\dots,t_n)$ be a vector in $\mathbb{R}^{n+1}$ with $t_0=a<t_1<\cdots<t_n\le b$ and $J_{\vec{t}}\colon C[a,b]\to\mathbb{R}^n$ a function with $J_{\vec{t}}(x)=(x(t_0),x(t_1),\dots,x(t_n))$. Suppose $\langle\varphi_n\rangle$ converges to φ weakly. Then*

$$\lim_{m\to\infty}\int_{\mathbb{R}^{n+1}}f(u_0,u_1,\dots,u_n)d\omega_{\varphi_m}J_{\vec{t}}^{-1}(x)=\lim_{m\to\infty}\int_{C[a,b]}f(J_{\vec{t}}(x))d\omega_{\varphi_m}(x)$$

$$= \lim_{m\to\infty} \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(u_0,u_1,\ldots,u_n) \frac{\exp\{-\sum_{j=1}^{n} \frac{(u_j-u_{j-1})^2}{2(t_j-t_{j-1})}\}}{\prod_{j=1}^{n} \sqrt{2\pi(t_j-t_{j-1})}} d\prod_{j=1}^{n} m_L(u_1,\ldots,u_n)\, d\varphi_m(u_0)$$

$$= \int_{C[a,b]} f(J_{\vec{t}}(x))\, d\omega_\varphi(x).$$

Theorem 2.13. *If $\langle\varphi_n\rangle$ is tight and $\langle\varphi_n\rangle$ converges to φ weakly, then $\langle\omega_{\varphi_n}\rangle$ converges to ω_φ weakly.*

Remark. The referee point out the following facts: for $y \in C[a,b]$, there are $\alpha \in \mathbb{R}$ and $x \in C_0[a,b]$ with $y = \alpha + x$ where $\alpha = y(a)$ and $x = y - \alpha \in C_0[a,b]$. Let $\psi\colon C[a,b] \to \mathbb{R}\oplus C_0[a,b]$ be a function with $\psi(y) = (\alpha, x)$ as in above. Then $\|y\|_\infty \leq |\alpha| + \|x\|_\infty \equiv \|(\alpha,x)\| = \|\psi(y)\|$. By Two norm theorem [26], ψ is a homeomophism. So we have $\omega_\varphi = (\varphi \times m_\omega)\circ\psi^{-1}$. Using this facts, we can easily prove the following corollary.

Corollary 2.14. *Let f be in $L^1(\mathbb{R})$ and set $\varphi(E) = \int_E f(x) dm_L(x)$ where $f > 0$ and E is a Borel subset of $\mathbb{R}$. For any integrable function F,*

$$\text{(2.1)} \quad \int_{C[a,b]\times C[a,b]} F(x,y)\, d\omega_\varphi \times \omega_\varphi(x,y)$$
$$= \int_{C[a,b]\times C[a,b]} F(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)\, d\omega_\varphi \times \omega_\varphi(x,y),$$

for all real number θ if and only if the function $f(x)$ has the form Ae^{-ax^2} where A and a are positive constants.

§ 3. A Translation Theorem on $(C[a,b], \mathcal{B}(C[a,b]), \omega_\varphi)$ and the Paley-Wiener-Zygmund Integral

It is well-known fact that there is no quasi-invariant probability measure on the infinite dimensional vector space [49]. So, there is no quasi-invariant probability measure on $C_0[a,b]$ or $C[a,b]$. In 1944, under the some assumptions, Cameron and Martin established a translation theorem on $(C_0[a,b], m_w)$ in [5]. In this section, we will prove a translation theorem on $(C[a,b], \omega_\varphi)$ under the similar assumptions to Cameron's assumptions. From these concepts, we will show that the Paley-Wiener-Zygmund integral is well-defined ω_φ -a.e.

By either the similar method as in the proof of Cameron and Martin's translation theorem on $C_0[a,b]$ in [5] or Remark 2, we can prove the following theorem.

Theorem 3.1 (The Translation Theorem on $(C[a,b], \mathcal{B}(C[a,b]), \omega_\varphi)$). *Let $h \in C[a,b]$ and of bounded variation. Let $\alpha \in \mathbb{R}$ and set $x_0(s) = \int_a^s h(u)\, dm_L(u) + \alpha$ for $a \leq s \leq b$. Let $L\colon C[a,b] \to$*

$C[a,b]$ be a function with $L(x)=x+x_0$ and φ a probability measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Let φ_α be a measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ such that $\varphi_\alpha(B)=\varphi(B+\alpha)$ for $B\in\mathcal{B}(\mathbb{R})$ and $\varphi_\alpha \ll \varphi$. Then if F is ω_φ-integrable then $F(x+x_0)$ is ω_φ-integrable of x and

$$\int_{C[a,b]} F(y)\,d\omega_\varphi(y) = e^{-\frac{1}{2}\|h\|_2^2}\int_{C[a,b]} F(x+x_0)\,e^{-\int_a^b h(u)dx(u)}\frac{d\varphi_\alpha}{d\varphi}(x(0))\,d\omega_\varphi(x).$$

Putting $F\equiv 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Under the assumptions in Theorem* 3.1,

$$\int_{C[a,b]} \exp\{-\int_a^b h(u)\,dx(u)\}\,d\omega_\varphi(x) = \exp\{-\frac{1}{2}\|h\|_2^2\}.$$

Replacing h by λh in Corollary 3.2, by the uniqueness theorem for analytic extension in the theory of complex analysis, we have the following corollary.

Corollary 3.3. *Under the assumptions in Theorem* 3.1, *for all $\lambda\in\mathbb{C}$,*

$$\int_{C[a,b]} \exp\{-\lambda\int_a^b h(u)\,dx(u)\}\,d\omega_\varphi(x) = \exp\{-\frac{\lambda^2}{2}\|h\|_2^2\}.$$

Theorem 3.4. *Consider a random variable $X\colon C[a,b]\to\mathbb{R}$ with $X(x)=\int_a^b h(u)\,dx(u)$ under the assumptions in Theorem* 3.1. *Then X has a normal distribution with the mean zero and the variation $\|h\|_2^2$.*

By the same method as in the proof of [50, Theorem 29.7], we can prove the following theorem.

Theorem 3.5. *Let $\{h_1,h_2,\dots,h_n\}$ be an orthonormal system such that each h_i is of bounded variation. For $i=1,2,\dots,n$, let $X_i(x)=\int_a^b h_i(s)\,dx(s)$. Then $X_1,X_2,\dots,X_n$ are independent, each X_i has the standard normal distribution. Moreover, if $f\colon\mathbb{R}^n\to\mathbb{R}$ is Borel measurable,*

$$\begin{aligned}\int_{C[a,b]} & f(X_1(x),X_2(x),\dots,X_n(x))\,d\omega_\varphi(x)\\ &\stackrel{*}{=} (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(u_1,u_2,\dots,u_n)\exp\{-\frac{1}{2}\sum_{j=1}^n u_j\}\,d\prod_{i=1}^n m_L(u_1,u_2,\dots,u_n)\end{aligned}$$

where $\stackrel{*}{=}$ means that if one side exists then both sides exist and the two values are equal.

Let $\{e_k\mid k=1,2,\dots\}$ be a complete orthonormal set in $L^2([a,b],m_L)$ such that each e_k is of bounded variation. For f in $L^2([a,b],m_L)$ and x in $C[a,b]$, let

$$\int_a^b f(s)\,\widehat{dx}(s) = \lim_{n\to\infty}\int_a^b\Big[\sum_{k=1}^n\int_a^b f(u)e_k(u)\,dm_L(u)\,e_k(v)\Big]dx(v)$$

if the limit exists.

$\int_a^b f(s)\widehat{dx}(s)$ is called the Paley-Wiener-Zygmund integral of f according to x. By the routine method in the theory of Wiener space, we can prove that the integral $\int_a^b f(s)\widehat{dx}(s)$ is independent on the orthonormal set $\{e_k \mid k = 1,2,\dots\}$ and the Paley-Wiener-Zygmund integral exists ω_φ-a.e. $x \in C[a,b]$.

Remark. In 1980, Cameron and Storvick introduced the definitions and some related theories of the spaces S, S' and S'' of Wiener functionals. If we replace $(C_0[a,b], m_w)$ by $(C[a,b], \omega_\varphi)$ in their paper, we can prove various results on $(C[a,b], \omega_\varphi)$ which are similar to Cameron and Storvick's results in [7].

§ 4. The Generalized Fernique's Theorem for Analogue of Wiener Measure Space

From Lemma 2.5, we have the following lemma.

Lemma 4.1.

$$m_\varphi(\{x \in C \mid \sup_{0\le s\le 1} |x(s) - x(0)| \ge K\}) \le \frac{1}{K}\sqrt{\frac{2}{\pi}}\exp\{-\frac{K^2}{2}\} \tag{4.1}$$

for a positive real number K.

In this section, we investigate the existence of the integral $\int_C \exp\{\alpha(\sup_{0\le s\le 1} |x(s)|)^p\}\, dm_\varphi(x)$ for two positive real numbers α, p.

Theorem 4.2. *For* $0 < p < 2$, $\int_C \exp\{\alpha(\sup_{0\le s\le 1} |x(s) - x(0)|)^p\}\, dm_\varphi(x)$ *is finite for all positive real number* α. *If* $p = 2$ *then* $\int_C \exp\{\alpha \sup_{0\le s\le 1} |x(s) - x(0)|^p\}\, dm_\varphi(x)$ *is finite for* $0 < \alpha < \frac{1}{2}$.

Theorem 4.3. *If* $0 < p < 1$ *and* $\int_{\mathbb{R}} \exp\{2\alpha|u|^p\}\, d\varphi(u)$ *is finite for some positive real number* α, *then* $\int_C \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\}\, dm_\varphi(x)$ *is finite.*

Theorem 4.4. *If* $1 \le p < 2$ *and* $\int_{\mathbb{R}} \exp\{2^p\alpha|u|^p\} d\varphi(u)$ *is finite, then*

$$\int_C \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\}\, dm_\varphi(x)$$

is finite.

Theorem 4.5. *If* $\alpha < \frac{1}{2}$ *and* $\int_{\mathbb{R}} \exp\{4\alpha|u|^2\}\,d\varphi(u)$ *is finite then*

$$\int_C \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^2\}\,dm_\varphi(x)$$

is finite.

Remark. If $p > 2$ and $\alpha > 0$ then by Theorem 2.1,

$$\begin{aligned}
(4.2)\qquad \int_C \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\}\,dm_\varphi(x) &\ge \int_C \exp\{\alpha|x(1)|^p\}\,dm_\varphi(x)\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\{\alpha|u_1|^p - \frac{1}{2}(u_1-u_0)^2\}\,dm_L(u_1)\,d\varphi(u_0)\\
&\ge \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{|u_1|\le 1} \exp\{\alpha|u_1|^p - \frac{1}{2}(u_1-u_0)^2\}\,dm_L(u_1)\,d\varphi(u_0)\\
&\quad + \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{|u_1|\ge 1} \exp\{\alpha|u_1|^2 - \frac{1}{2}(u_1-u_0)^2\}\,dm_L(u_1)\,d\varphi(u_0)\\
&= +\infty.
\end{aligned}$$

Remark. Suppose $\varphi = \delta_0$, that is, (C, m_φ) is the concrete Wiener measure space. Then by the theorems above, $\int_C \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\}\,dm_\varphi(x)$ is finite for $0 < p < 2$ and all real number α and $\int_C \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^2\}\,dm_\varphi(x)$ is finite for $\alpha < \frac{1}{2}$. Moreover, $\int_C \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\}\,dm_\varphi(x) = +\infty$ for $p > 2$ and $\alpha > 0$.

§ 5. An Integration Formula for Analogue of Wiener Measure

In this section, we investigate the integral of functionals such as $F(x) = \left(\int_0^t x(s)^2\,dm_L(s)\right)^n$ and $G(x) = \exp\{\lambda \int_0^t x(s)^2\,dm_L(s)\}$ and we give some corollaries, follows from our results.

Lemma 5.1. *Let* $0 = s_0 < s_1 < s_2 < \cdots < s_n = t$. *Suppose* u_0^{2n} *is* φ*-integrable. Then*

$$(5.1)\qquad \int_{C[0,t]} \prod_{j=1}^n x(s_j)^2\,d\omega_\varphi(x)$$

$$= \sum_{k,n}{}' \{ \frac{\prod_{l=2}^n \{(l - \sum_{j=n+2-l}^n k_j)(2l - 2\sum_{j=n+2-l}^n k_j - 1)\}}{(n - \sum_{j=1}^n k_j)!(2n - 2\sum_{j=1}^n k_j - 1)!! \prod_{j=1}^n k_j!} \prod_{j=1}^n (s_j - s_{j-1})^{k_j}\} \int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^n k_j}\,d\varphi(u_0).$$

Theorem 5.2. *Let* $F(x)=\left(\int_0^t x(s)^2 dm_L(s)\right)^n$ *on* $C[0,t]$ *where n is a natural number. Suppose* u_0^{2n} *is* φ*-integrable. Then*

$$(5.2)\quad \int_{C[0,t]} F(x)\,d\omega_\varphi(x)$$

$$= n!\sum_{k,n}{}' \frac{\prod_{l=2}^{n}\{(l-\sum_{j=n+2-l}^{n}k_j)(2l-2\sum_{j=n+2-l}^{n}k_j-1)\}\,t^{n+\sum_{j=1}^{n}k_j}}{(n+\sum_{j=1}^{n}k_j)!(n-\sum_{j=1}^{n}k_j)!(2n-2\sum_{j=1}^{n}k_j-1)!!\prod_{j=1}^{n}(2k_j-1)!!}\int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n}k_j}\,d\varphi(u_0).$$

In Theorem 5.2, by putting $t=1$ and $\varphi=\delta_0$, the Dirac measure at the origin $0\in\mathbb{R}$, ω_φ is the concrete Wiener measure on $C_0[0,t]$, that is, $\omega_\varphi=m_w$, $\int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n}k_j}\,d\varphi(u_0)=0$ if $n\neq\sum_{j=1}^{n}k_j$ and $\int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n}k_j}\,d\varphi(u_0)=1$ if $n=\sum_{j=1}^{n}k_j$. So, we have the following corollary.

Corollary 5.3. *Let* $F(x)=\left(\int_0^1 x(s)^2 dm_L(s)\right)^n$ *on* $C_0[0,1]$*. Then*

$$(5.3)\quad \int_{C[0,1]} F(x)\,dm_w(x)=\frac{1}{2^n(2n-1)!!}\sum_{k,n}{}'\frac{\prod_{l=2}^{n}\{(l-\sum_{j=n+2-l}^{n}k_j)(2l-2\sum_{j=n+2-l}^{n}k_j-1)\}}{\prod_{j=1}^{n}(2k_j-1)!!}$$

Theorem 5.4. *Suppose* $\lambda t<\frac{1}{2}$ *and* $\exp\{u^{2n}\}$ *is* φ*-integrable on* $\mathbb{R}$ *for all natural number n. Let* $G(x)=\exp\{\lambda\int_0^t x(s)^2 dm_L(s)\}$ *on* $C[0,t]$*. Then* $G(x)$ *is* ω_φ*-integrable and*

$$(5.4)\quad \int_{C[0,t]} G(x)\,d\omega_\varphi(x)$$

$$=\varphi(\mathbb{R})+\sum_{n=1}^{\infty}\lambda^n\sum_{k,n}{}'\frac{\prod_{l=2}^{n}\{(l-\sum_{j=n+2-l}^{n}k_j)(2l-2\sum_{j=n+2-l}^{n}k_j-1)\}\,t^{n+\sum_{j=1}^{n}k_j}}{(n+\sum_{j=1}^{n}k_j)!(n-\sum_{j=1}^{n}k_j)!(2n-2\sum_{j=1}^{n}k_j-1)!!\prod_{j=1}^{n}(2k_j-1)!!}$$

$$\times\int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n}k_j}\,d\varphi(u_0).$$

In Theorem 3.1 of section 4, by putting $h(u)=0$ on $[0,t]$, if F is $\omega_{\varphi_{-\alpha}}$-integrable then $F(x+\alpha)$ is $\omega_{\varphi_{-\alpha}}$-integrable and

$$\int_{C[0,t]} F(x)\,d\omega_{\varphi_{-\alpha}}(x)=\int_{C[0,t]} F(x+\alpha)\,d\omega_\varphi(x).$$

Using this, we can prove the following corollary.

Corollary 5.5. *Suppose* $\lambda t < \frac{1}{2}$ *and* $\exp\{u^{2n}\}$ *is* $\varphi_{-\alpha}$*-integrable where* α *is a real number. Let* $G(x) = \exp\{\lambda \int_0^t (x(s)+\alpha)^2 dm_L(s)\}$ *on* $C[0,t]$*. Then* $G(x)$ *is* ω_φ*-integrable and*

$$(5.5)\quad \int_{C[0,t]} G(x)\, d\omega_\varphi(x)$$
$$= \varphi(\mathbb{R}) + \sum_{n=1}^{\infty} \lambda^n {\sum_{k,n}}' \frac{\prod_{l=2}^{n}\{(l-\sum_{j=n+2-l}^{n} k_j)(2l-2\sum_{j=n+2-l}^{n} k_j - 1)\}\, t^{n+\sum_{j=1}^{n} k_j}}{(n+\sum_{j=1}^{n} k_j)!\,(n-\sum_{j=1}^{n} k_j)!\,(2n-2\sum_{j=1}^{n} k_j - 1)!!\,\prod_{j=1}^{n}(2k_j-1)!!}$$
$$\times \int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n} k_j}\, d\varphi_{-\alpha}(u_0)$$

In Theorem5.4, putting $t = 1$ and $\varphi = \delta_0$, we have the following corollary by [4].

Corollary 5.6. *For any positive real number* λ*,*

$$\int_{C_0[0,1]} \exp\{-\lambda \int_0^1 x(s)^2 dm_L(s)\}\, dm_w(x)$$
$$= 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!2^n(2n-1)!!} {\sum_{k,n}}' \frac{\prod_{l=2}^{n}\{(l-\sum_{j=n+2-l}^{n} k_j)(2l-2\sum_{j=n+2-l}^{n} k_j - 1)\}}{\prod_{j=1}^{n}(2k_j-1)!!} = (\cosh\sqrt{2\lambda})^{-\frac{1}{2}}.$$

§ 6. Probabilities of Analogue of Wiener Paths Crossing Continuously Differentiable Curves

In this section, we give the analogue of Wiener measure m_φ of $\{x \in C[0,T] \mid x(0) < f(0)$ and $x(s_0) \geq f(s_0)$ for some $s_0 \in [0,T]\}$ by use of integral equation techniques. This result is a generalization of Park and Paranjape's 1974 result [31].

Let $T > 0$ be given and m_w the standard Wiener measure on the space $C_0[0,T]$ of all continuous functions x with $x(0) = 0$. From [45] and [46], we can found the following equations: for $b \geq 0$,

$$(6.1)\quad m_w(\{x \in C_0[0,T] \mid \sup_{0\leq t\leq T} x(t) \geq b\}) = 2\int_{b/\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

and

$$(6.2)\quad m_w(\{x \in C_0[0,T] \mid \sup_{0\leq t\leq T} (x(t) - at) \geq b\})$$
$$= \int_{(aT+b)/\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + e^{-2ab} \int_{-\infty}^{(aT-b)/\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$$

In 1974, Park and Paranjape proved the following theorem [31].

Theorem 6.1. *Let $f(t)$ be continuous on $[0,T]$, differentiable in $(0,T)$, and satisfy $|f'(t)| \leq \frac{C}{t^p}$ $(0 < p < \frac{1}{2})$ for some constant C. Then for $b \geq -f(0)$,*

$$(6.3) \quad m_w(\{x \in C_0[0,T] \mid \sup_{0\leq t\leq T} (x(t) - f(t)) \geq b\})$$
$$= 2\int_{(f(T)+b)/\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du - 4\int_0^T M(T,t)\Big[\int_{(f(T)+b)/\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du\Big] dt$$
$$+ \sum_{n=1}^{\infty} 4^n \int_0^T K_n(T,t)\Big[2\int_{(f(t)+b)/\sqrt{t}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$
$$- 4\int_0^t M(t,s)\int_{(f(s)+b)/\sqrt{s}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du ds\Big] dt,$$

where

$$M(t,s) = \begin{cases} \dfrac{\partial}{\partial s}\displaystyle\int_{-\infty}^{(f(t)-f(s))/\sqrt{t-s}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du & (0 \leq s < t \leq T), \\ 0 & (0 \leq t < s \leq T), \end{cases}$$

$$K_1(T,t) = \int_t^T M(T,s)M(s,t)ds,$$

and

$$K_{n+1}(T,t) = \int_t^T K_n(T,s)K_1(s,t)ds.$$

The main purpose of this section is to find the analogue of Wiener measure m_φ of $\{x \in C[0,T] \mid \sup_{0\leq t\leq T} (x(t) - f(t)) \geq 0\}$ for continuously differentiable function f on $[0,T]$, which is a generalization of Theorem 6.1.

Throughout in this section, $\int_a^b f(u)du$ means the Henstock integral of f.

Let f: $[0,T] \to \mathbb{R}$ be continuously differentiable and $f(s) = 0$ if $s \leq 0$. For $t \in [0,T]$, the limit $\lim_{s\to t^-} \frac{f(t) - f(s)}{\sqrt{t-s}}$ exists and equals to 0.

For $x \in C[0,T]$, let $\tau(x)$ be the first hitting time of the curve f from below by x, that is, $x(\tau(x)) = f(\tau(x))$. If x never reaches the curve f, let $\tau(x) = +\infty$.

For $t \in [0,T]$, let

$$(6.4) \qquad A_t = \{x \in C[0,T] \mid x(0) < f(0) \text{ and for some } s_0 \in [0,t],\ x(s_0) \geq f(s_0)\}.$$

Let $G\colon \mathbb{R} \to \mathbb{R}$ be a function with

$$G(t) = \begin{cases} 0 & (t < 0), \\ m_\varphi(A_t) & (0 \le t \le T), \\ m_\varphi(A_T) & (T < t). \end{cases}$$

Lemma 6.2. *G is increasing and continuous with $G(0) = 0$.*

Lemma 6.3. *If $0 \le s < t \le T$ then $\tau(x) = s$ and $x(t) - x(s)$ are independent.*

The following theorem is one of main theorems in these notes.

Theorem 6.4. *For $0 < t \le T$, $G(t)$ satisfies the following Volterra's integral equation of the second kind*

$$(6.5)\quad G(t) = 2\int_{-\infty}^{f(0)} \Big[\int_{f(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(u_1 - u_0)^2)}{2t}\} du_1\Big] d\varphi(u_0) - 2\int_0^t G(s) M(t,s) ds$$

where

$$M(t,s) = \begin{cases} \dfrac{\partial}{\partial s} \displaystyle\int_{-\infty}^{(f(t)-f(s))/\sqrt{t-s}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du & (0 \le s < t \le T), \\ 0 & (0 \le t \le s \le T). \end{cases}$$

The equality (6.5) and the change of order of integration gives

$$\begin{aligned}(6.6)\qquad G(t) = {} & 2\int_{-\infty}^{f(0)} \Big[\int_{f(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(u_1 - u_0)^2)}{2t}\} du_1\Big] d\varphi(u_0) \\ & -4\int_0^t \Big[\int_{-\infty}^{f(0)} \Big[\int_{f(s)}^{+\infty} \frac{1}{\sqrt{2\pi s}} \exp\{-\frac{(u_1 - u_0)^2)}{2s}\} du_1\Big] d\varphi(u_0)\Big] M(t,s) ds \\ & +4\int_0^t \Big[\int_z^t M(s,z) M(t,s) ds\Big] G(z) dz,\end{aligned}$$

if $M(s,z)M(t,s)G(z)$ is integrable on $\{(s,z) \mid 0 \le z < s \le t\}$.
By [47], we obtain the main theorem in these notes.

Theorem 6.5. *If $\int_z^t M(s,z) M(t,s) ds$ is square integrable on $\{(z,t) \mid 0 \le z < t \le T\}$ then the equation (6.5) has one and essentially only one solution in the class L^2. This solution is given by the formula*

(6.7)

$$G(t) = 2\int_{-\infty}^{f(0)} \Big[\int_{f(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(u_1 - u_0)^2)}{2t}\} du_1\Big] d\varphi(u_0)$$

$$+\sum_{n=1}^{\infty}(-1)^n 2^{n+1}\int_0^t\Big[\Big[\int_{-\infty}^{f(0)}\Big[\int_{f(s)}^{+\infty}\frac{1}{\sqrt{2\pi s}}\exp\{-\frac{(u_1-u_0)^2)}{2s}\}du_1\Big]d\varphi(u_0)\Big]H_n(t,s)ds,$$

where $H_1(t,s)=M(t,s)$ *and* $H_{n+1}(t,s)=\int_s^t H_n(t,z)H_1(z,s)dz.$

Remark. If $\varphi=\delta_0$ then the equation (6.3) and the equation (6.7) are exactly same.

Remark. Let $\varphi=\delta_0$ and $f(t)=b$ a constant function with $b\geq 0$. The $M(t,s)=0$ for $0\leq s<t\leq T$, so we have the equation (6.1), that is,

$$G(t)=2\int_b^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{u^2}{2t}\}du.$$

§ 7. The Relationship Between Conditional Expectation and Bartle Integral with Respect to a Vector Measure V_φ

In this section, we will show that the Bartle integral with respect to V_φ can be written as the iterated integrals with respect to complex-valued measure. From this, we recognize the relation between the Bartle integral and the conditional expectation on $(C[a,b],\omega_\varphi)$ [41].

Let φ be a probability measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Let n be a non-negative integer. Let X be a $\mathbb{R}^{n+1}$-valued measurable function on $(C[a,b],\mathcal{B}(C[a,b]),\omega_\varphi)$. We write P_X for a measure on $(\mathbb{R}^{n+1},\mathcal{B}(\mathbb{R}^{n+1}))$ determined by X, that is, $P_X(E)=\omega_\varphi(X^{-1}(E))$ for $E\in\mathcal{B}(\mathbb{R}^{n+1})$.

For $\varphi\in\mathcal{M}(\mathbb{R})$ and $B\in\mathcal{B}(C[a,b])$, let $[V_\varphi(B)](E)=\omega_\varphi(B\cap X^{-1}(E))$. Then V_φ is a measure-valued measure on $(C[a,b],\mathcal{B}(C[a,b]))$ in the total variation norm sense.

Theorem 7.1. *Let φ be a probability measure on* $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ *and f bounded measurable on* $(C[a,b],\mathcal{B}(C[a,b]))$. *Then*

$$\big[(\mathrm{Ba})-\int_{C[a,b]}f(x)dV_\varphi(x)\big](E)=\int_E E(f|X)(\xi)dP_X(\xi)$$

for $E\in\mathcal{B}(\mathbb{R}^{n+1})$.

For a non-negative finite real valued measure in $\mathcal{M}(\mathbb{R})$, let φ^N be a normalized measure of φ, that is, $\varphi^N(E)=\dfrac{\varphi(E)}{|\varphi|(\mathbb{R})}$ for E in $\mathcal{B}(\mathbb{R})$ if φ is a non-zero measure and φ^N is a zero measure if φ is a zero measure. For φ in $\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi=\sum_{j=1}^{4}\alpha_j\varphi_j$, $\omega_\varphi=\sum_{j=1}^{4}\alpha_j\omega_{\varphi_j}$ and for $j=1,2,3,4$, $\omega_{\varphi_j}=|\varphi_j|(\mathbb{R})\varphi_j^N$. Hence, for $\varphi\in\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi=\sum_{j=1}^{4}\alpha_j\varphi_j$, for $B\in\mathcal{B}(C[a,b])$ and for $E\in\mathcal{B}(\mathbb{R})$,

$$\big[V_\varphi(B)\big](E)=\big[\sum_{j=1}^{4}\alpha_j|\varphi_j|(\mathbb{R})V_{\varphi_j^N}(B)\big](E),$$

so we have

$$V_\varphi = \sum_{j=1}^{4} \alpha_j |\varphi_j|(\mathbb{R}) V_{\varphi_j^N}.$$

Theorem 7.2. *Let $\varphi \in \mathcal{M}(\mathbb{R})$. For a bounded measurable function f on $(C[a,b], \mathcal{B}(C[a,b]))$ and $X(x) = x(b)$,*

$$\left[(\mathrm{Ba}) - \int_{C[a,b]} f(x)\, dV_\varphi(x)\right](E) = \frac{1}{2\pi} \int_E \int_{\mathbb{R}} e^{-i\xi u} \int_{C[a,b]} e^{iux(t)} f(x)\, d\omega_\varphi(x)\, dm_L(u)\, dm_L(\xi)$$

for $E \in \mathcal{B}(\mathbb{R})$.

Remark. By putting $\varphi = \delta_0$, $\omega_\varphi = \omega$ and $X(x) = x(b)$, the classical Wiener measure and

$$\left[(\mathrm{Ba}) - \int_{C[a,b]} f(x)\, dV_\varphi(x)\right](E) = \int_{X-1(E)} f(x)\, d\omega(x).$$

Here f is a bounded measurable function and $E \in \mathcal{B}(\mathbb{R})$.

Theorem 7.3 (The Wiener Integration Formula for V_φ). *Suppose for $k = 1, 2, \dots, n$, i_k is a nonnegative integer such that $m = n + \sum_{j=1}^{n} i_j + 1$ and $a \equiv t_0 \equiv t_{0,0} < t_{0,1} < t_{0,2} < \cdots < t_{0,i_1} < t_1 \equiv t_{0,i_1+1} \equiv t_{1,0} < t_{1,1} < t_{1,2} < \cdots < t_{n-1,i_n} < t_n \equiv t_{n-1,i_n+1} \equiv b$ and for $j = 1, 2, \dots, n$. Let $X(x) = (x(t_0), x(t_1), \dots, x(t_n))$. If $f\colon \mathbb{B}^m \to \mathbb{R}$ is a Borel measurable function then the following equality holds*:

$$\begin{aligned}(7.1)\qquad & \left[(\mathrm{Ba}) - \int_{C[a,b]} f(y(t_{0,0}), y(t_{0,1}), \dots, y(t_{n-1,i_n+1}))\, dV_\varphi^{J_{\vec{t}}}(y)\right](E) \\ & \overset{*}{=} \int_{\mathbb{R}} \Big[\int_{\mathbb{R}^{m-1}} f(u_{0,0}, u_{0,1}, \dots, u_{n-1,i_n+1}) W_{m+1} \prod_{g=0}^{n} \chi_{E^{[g]}}(u_{g,0}) \\ & \qquad \cdot d\Big(\prod_{i=1}^{m-1} \omega\Big)(u_{0,0}, u_{0,1}, \dots, u_{n-1,i_n+1})\Big] dm_L(u_{0,0}),\end{aligned}$$

where $E^{[g]}$ is the g^{th}-section of E.

§ 8. The Simple Formula for Conditional Expectation on Analogue of Wiener Measure Space

In this section, we prove the simple formula for conditional expectation on analogue of Wiener measure. Throughout in this section, let $a = t_0 < t_1 < \cdots < t_n = b$ be given, let

$$[y](s) = \sum_{j=1}^{n} \chi_{[t_{j-1}, t_j)}(s)[y(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}}(y(t_j) - y(t_{j-1}))] + y(b)\xi_{\{b\}}(s)$$

for $y \in C[a,b]$ and

$$[u](s) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j)}(s)[u_{j-1} + \frac{s-t_{j-1}}{t_j - t_{j-1}}(u_j - u_{j-1})] + u_n \xi_{\{b\}}(s)$$

for $(u_0, u_1, \ldots, u_n) \in \mathbb{R}^{n+1}$.

By [38], we have following theorem from the direct calculations $E(\exp\{i\lambda_1 X + i\lambda_2 Y\}) = E(\exp\{i\lambda_1 X\})E(\exp\{i\lambda_2 Y\})$ and $E(\exp\{i\lambda_1 X + i\lambda_3 Z\}) = E(\exp\{i\lambda_1 X\})E(\exp\{i\lambda_3 Z\})$.

Theorem 8.1. *Let φ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $a = t_0 < t_1 < \cdots < s_1 < t_{j-1} < s_2 < t_j < s < \cdots < t_n = b$ and X, Y and Z three functions from $C[a,b]$ into $\mathbb{R}$ with $X(y) = y(s) - [y](s)$, $Y(y) = y(s_1)$ and $Z(y) = y(s_2)$, respectively. Then X and Y are stochastically independent and X and Z are stochastically independent.*

In 2008, Professor D. H. Cho [9] proved the next theorem by the quite different and long method on the analogue of Wiener space over paths in $\mathbb{B}$ compare with our proof in [38].

Theorem 8.2 (The Simple Formula for Conditional Expectation). *Let φ be a Borel probability measure on $\mathbb{R}$. Let $J_{\vec{t}}$: $C[a,b] \to \mathbb{R}^{n+1}$ be the function with $J_{\vec{t}}(y) = (y(t_0), y(t_1), \ldots, y(t_n))$. Let F be m_φ-integrable on $C[a,b]$. Then for $E \in \mathcal{B}(\mathbb{R}^{n+1})$,*

$$[(\text{Ba}) - \int_{C(\mathbb{B})} F(y) dV^{\varphi}_{J_{\vec{t}}}(y)](E) = \int_E E^{\varphi}(F|J_{\vec{t}}) dP^{\varphi}_{J_{\vec{t}}}(\vec{u}), \tag{8.1}$$

that is,

$$E^{\varphi}(F|J_{\vec{t}}) = E(F(y - [y] + [\vec{u}])).$$

We know that for any bounded measurable function F on $C[a,b]$ and for any probability measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, there is a conditional expectation $E^{\varphi}(F|J_{\vec{t}})$. What happen if the probability measure φ change?

Theorem 8.3 (The Uniqueness Theorem for Giving Distributions). *For a bounded measurable function F on $C[a,b]$, there is a unique conditional expectation $E(F|J_{\vec{t}})$, independent of the selection of the distribution φ such that*

$$[(\text{Ba}) - \int_{C[a,b]} F(x) dV^{\varphi}_{J_{\vec{t}}}(x)](E) = \int_E E(F|J_{\vec{t}})(\vec{u}) dP^{\varphi}_{J_{\vec{t}}}(\vec{u})$$

for any $E \in \mathcal{B}(\mathbb{R}^{n+1})$ and for any Borel probability measure φ on $\mathbb{R}$.

Remark. In Theorem 8.3, if we take $\varphi = \delta_0$ then u_0 does not appear in the representation of $E(F|J_{\vec{t}})$ because $u_0 = 0$.

§ 9. A Measure-Valued Feynman-Kac Formula

Cameron and Storvick [6] introduced an operator-valued function space integral in 1968. Johnson and Lapidus [18] established the existence theorem of the operator-valued function space integral as an operator from $L^2(\mathbb{R}^N)$ to itself for certain functionals involving some Borel measures, and in 1987, Lapidus [23] proved that the integral satisfies the Schrödinger wave equation. In 1992, Chang and the first author [8] established the existence theorem of the operator-valued function space integral as an operator from L^p to $L^{p'}$ $(1 < p < 2)$ for certain functionals involving some Borel measures. The first author proved that the integral satisfies a Volterra-Stieljes integral equation in [36] In this section, we will achieve the measure-valued Feynman-Kac formula for the integral with respect to a measure-valued measure of suitable functional. Throughout in this section and the next sections, we assume $X(x) = x(b)$ and $V_\varphi^X = V_\varphi$ ([39]).

Theorem 9.1. *Let $\varphi \in \mathcal{M}(\mathbb{R})$, η a complex-valued Borel measure on $[a,b]$ and $\theta \in L_{\varphi;\infty,1;\eta}$. Then*

$$|\theta(s,x(s))| \le \|\theta(s,\cdot)\|_{\varphi;\infty}$$

for $|\eta| \times \omega_{|\varphi|}$-a.e. $(s,x) \in [a,b] \times C[a,b]$.

Throughout this section let $\eta = \mu + \nu$ be a complex-valued Borel measure on $[a,b]$ such that μ is the continuous part of η and $\nu = \sum\limits_{p=0}^{n} c_p \delta_{\tau_p}$ where $a = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = b$ and c_p $(p = 0, 1, \ldots, n)$ are complex numbers, $\varphi \in \mathcal{M}(\mathbb{R})$ and $\theta \in L_{\varphi;\infty,1;\eta}$. For non-negative integers q and $j_1, \ldots, j_n$ with $q = j_1 + j_2 + \cdots + j_n$, let

$$\begin{aligned}\triangle_{q;j_1,j_2,\ldots,j_n} = \{(s_{1,1}, s_{1,2}, \ldots, s_{1,j_1}, s_{2,1}, \ldots, s_{n-1,j_{n-1}}, s_{n,1}, \ldots, s_{n,j_n}) \mid \tau_0 = a < s_{1,1} < \\ \cdots < s_{1,j_1} < \tau_1 < s_{2,1} < \cdots < \tau_{n-1} < s_{n,1} < \cdots < s_{n,j_n} < \tau_n = b\}.\end{aligned}$$

For convenience, we set $M_{\theta(s,\cdot)} \equiv M_{\theta(s)}$ for $a \le s \le b$ and $\tau_0 = s_{0,0}$, $\tau_n = t = s_{n,j_n+1}$ and $\tau_k = s_{k+1,0} = s_{k,j_k+1}$ for $k = 1, 2, \ldots, n-1$. For non-negative integers $m, q_0, \ldots, q_{n+1}, j_1, \ldots, j_n$ with $m = q_0 + q_1 + \cdots + q_{n+1}$ and $q_{n+1} = j_1 + j_2 + \cdots + j_n$, let $K(m,n,q,j)$: $\triangle_{q_{n+1};j_1,j_2,\ldots,j_n} \times C[a,b] \to \mathbb{C}$ be a function defined by

$$K(m,n,q,j)((s_{1,1},\ldots,s_{n,j_n}),x) = [\prod_{i=0}^{n} \theta(\tau_i, x(\tau_i))^{q_i}][\prod_{i=1}^{n}\prod_{j=1}^{j_i} \theta(s_{i,j}, x(s_{i,j}))]$$

and $D(m,n,q,j)$: $\triangle_{q_{n+1};j_1,j_2,\ldots,j_n} \to \mathbb{R}$ a function defined by

$$D(m,n,q,j)(s_{1,1},\ldots,s_{n,j_n}) = [\prod_{i=0}^{n} \|\theta(\tau_i,\cdot)\|_{\varphi;\infty}^{q_i}][\prod_{i=1}^{n}\prod_{j=1}^{j_i} \|\theta(s_{i,j},\cdot)\|_{\varphi;\infty}].$$

Lemma 9.2. (1) $|K(m,n,q,j)| \le D(m,n,q,j)$ *in* $|\mu| \times \omega_{|\varphi|}$*-a.e.*
(2) *It follows that*

$$|\int_{\triangle_{q_{n+1};j_1,\dots,j_n}} D(m,n,q,j)(s_{1,1},\dots,s_{n,j_n})\, d(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)(s_{1,1},\dots,s_{n,j_n})|$$
$$\le \frac{1}{q_{n+1}!}(\prod_{i=0}^{n}\|\theta(\tau_i,\cdot)\|_{\varphi;\infty}^{q_i})(\|\theta\|_{\varphi;\infty,1;\mu})^{q_{n+1}}.$$

(3) $D(m,n,q,j)$ *is* $(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)\times V_\varphi$*-Bartle integrable on* $\triangle_{q_{n+1};j_1,\dots,j_n}\times C[0,t]$.

Lemma 9.3. $\theta(s,x(s))$ *is* $\mu\times V_\varphi$*-Bartle integrable on* $[0,t]\times C[0,t]$.

Theorem 9.4. (1) $K(m,n,q,j)$ *is* $(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)\times V_\varphi$*-Bartle integrable.*

(2) *For* $\prod_{i=1}^{n}\prod_{j=1}^{j_i}|\mu|$*-a.e.* $(s_{1,1},\dots,s_{n,j_n})$, $K(m,n,q,j)((s_{1,1},\dots,s_{n,j_n}),\cdot)$ *is* V_φ*-Bartle integrable.*

(3) (Ba)$-\int_{C[0,t]} K(m,n,q,j)((s_{1,1},\dots,s_{n,j_n}),x)\,dV_\varphi(x)$ *is* $\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu$*-Bochner integrable.*

The proof of the following theorem is patterned to some extent on earlier work by Johnson and Lapidus in [18] but the present setting requires a number of new concepts and results in the previous parts of this section.

Theorem 9.5 (A Measure-Valued Feynman-Kac Formula). $\exp\{\int_{[a,b]}\theta(s,x(s))d\eta(s)\}$ *is* V_φ*-Bartle integrable on* $C[a,b]$ *and for* $E\in\mathcal{B}(\mathbb{R})$,

$$[(\mathrm{Ba})-\int_{C[a,b]}\exp\{\int_{[a,b]}\theta(s,x(s))\,d\eta(s)\}\,dV_\varphi(x)](E)$$
$$=\sum_{m=0}^{\infty}\sum_{q_0+\dots+q_{n+1}=m}\frac{\prod_{p=0}^{n}c_p^{q_p}}{\prod_{p=0}^{n}q_p!}\sum_{j_1+\dots+j_n=q_{n+1}}$$
$$\int_{\triangle_{q_{n+1};j_1,\dots,j_n}}[(L_n\circ L_{n-1}\circ\dots\circ L_1)(T(s_{1,1},\varphi,\theta(0,\cdot)^{q_0}))](E)\,d(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)(s_{1,1},\dots,s_{n,j_n}).$$

Moreover,

$$|(\mathrm{Ba})-\int_{C[a,b]}\exp\{\int_{[a,b]}\theta(s,x(s))\,d\eta(s)\}\,dV_\varphi(x)|(\mathbb{R})\le 4|\varphi|(\mathbb{R})[\exp\{\|\theta\|_{\varphi;\infty,1;\eta}\}].$$

Here, for $k=2,3,\dots,n$,

$$L_k = M_{\theta(\tau_k)^{q_k}}\circ S_{\tau_k-s_{k,j_k}}\circ M_{\theta(s_{k,j_k})}\circ S_{s_{k,j_k}-s_{k,j_k-1}}\circ\dots\circ M_{\theta(s_{k,1})}\circ S_{s_{k,1}-s_{k,0}}$$

and

$$L_1 = M_{\theta(\tau_1)^{q_1}} \circ S_{\tau_1 - s_{1,j_1}} \circ M_{\theta(s_{1,j_1})} \circ S_{s_{1,j_1} - s_{1,j_1-1}} \circ \cdots \circ M_{\theta(s_{1,1})} .$$

From Theorem 9.5, directly we deduce the following corollaries.

Corollary 9.6. *In Theorem 9.5, we assume that $\eta = \mu$, an arbitrary continuous measure on $[a,b]$. Then for E in $\mathcal{B}(\mathbb{R})$*

$$\begin{aligned}
&[(\mathrm{Ba})\text{–}\int_{C[a,b]} \exp\{\int_{[a,b]} \theta(s,x(s))\, d\eta(s)\}\, dV_\varphi(x)](E) \\
&= \sum_{m=0}^{\infty} \int_{\triangle_m} [(S_{t-s_m} \circ M_{\theta(s_m)} \circ \cdots \circ S_{s_2-s_1} \circ M_{\theta(s_1)})(T(s_1,\varphi,\theta^0 \equiv 1))](E)\, d(\prod_{i=1}^{m} \mu)(s_1,s_2,\ldots,s_m),
\end{aligned}$$

where $\triangle_m = \{(s_1,s_2,\ldots,s_m) \in [0,t]^m \mid 0 < s_1 < s_2 < \cdots < s_m < t\}$.

Corollary 9.7. *In Theorem 9.5, we assume that $\eta = \nu = \sum_{p=0}^{n} c_p \delta_{\tau_p}$, a discrete measure on $[a,b]$ with finite support. Then for $E \in \mathcal{B}(\mathbb{R})$,*

$$\begin{aligned}
&[(\mathrm{Ba})\text{–}\int_{C[a,b]} \exp\{\int_{[a,b]} \theta(s,x(s))\, d\eta(s)\}\, dV_\varphi(x)](E) \\
&= \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_n=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \\
&\quad [(M_{\theta(\tau_n)^{q_n}} \circ S_{\tau_n-\tau_{n-1}} \circ \cdots \circ S_{\tau_2-\tau_1} \circ M_{\theta(\tau_1)^{q_1}})(T(\tau_1,\varphi,\theta(0,\cdot)^{q_0}))](E).
\end{aligned}$$

Corollary 9.8. *In Theorem 9.5, we assume that $c_n = 0$. Then for E in $\mathcal{B}(\mathbb{R})$,*

$$\begin{aligned}
&[(\mathrm{Ba})\text{–}\int_{C[a,b]} \exp\{\int_{[a,b]} \theta(s,x(s))\, d\eta(s)\}\, dV_\varphi(x)](E) \\
&= \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_n=m} \frac{\prod_{p=0}^{n-1} c_p^{q_p}}{\prod_{p=0}^{n-1} q_p!} \sum_{j_1+\cdots+j_n=q_n} \int_{\triangle_{q_n;j_1,\ldots,j_n}} \\
&\quad [((S_{t-s_{n,j_n}} \circ M_{\theta(s_{n,j_n})} \circ \cdots \circ S_{s_{n,1}-\tau_{n-1}}) \circ L_{n-1} \circ \cdots \circ L_1) \\
&\quad (T(s_{1,1},\varphi,\theta(0,\cdot)^{q_0}))](E)\, d(\prod_{i=1}^{n} \prod_{j=1}^{j_i} \mu)(s_{1,1},\ldots,s_{n,j_n}).
\end{aligned}$$

§ 10. A Volterra Integral Equation for the Measure-Valued Feynman-Kac Formula

In this section, we prove that the equality in Theorem 9.5, satisfies a suitable Volterra integral equation.

Throughout this section, let $a = 0 = \tau_0 < \tau_1 < \cdots < \tau_n = t < \tilde{t} = b$ and let η be a Borel measure on $[0, \tilde{t}\,]$ such that $\eta = \mu + \nu$ where μ is the continuous part of η and $\nu = \sum_{p=0}^{n} c_p \delta_{\tau_p}$; further let $\theta \in L^{\tilde{t}}_{\varphi;\infty,1;\eta}$. Let

$$u(t') = (\mathrm{Ba})\text{-}\int_{C[0,t']} \exp\{\int_{[0,t']} \theta(s, x(s))\, d\eta(s)\}\, dV_\varphi(x)$$

for $t < t' \le \tilde{t}$.

The following theorem is the counterpart for the measure-valued measure V_φ of the integral equation for the Feynman-Kac formula with Lebesgue-Stieljes measure, obtained by Lapidus in [23, 24, 25] and for the Feynman-Kac formula with an operator-valued measure, obtained by Kluvanek in [20].

Theorem 10.1 (The Measure-Valued Feynman-Kac Formula). *For $t < t' \le \tilde{t}$, $u(t')$ satisfies a Volterra integral equation, that is,*

$$u(t') = S_{t'-t}(u(t)) + (\mathrm{Bo})\text{-}\int_{(t,t']} (S_{t'-s} \circ M_{\theta(s)}) u(s)\, d\mu(s)\,.$$

Corollary 10.2. *Under the assumptions in Corollary* 9.6, *for* $0 < t' \le \tilde{t}$, *$u(t')$ satisfies a Volterra integral equation, that is,*

$$u(t') = \; S_{t'}(\varphi) + (\mathrm{Bo})\text{-}\int_{(0,t']} (S_{t'-s} \circ M_{\theta(s)})(u(s))\, d\mu(s)\,.$$

Corollary 10.3. *Under the assumptions in Corollary* 9.7, *for* $0 < t' \le \tilde{t}$,

$$u(t') = \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_n=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!}$$
$$[S_{t'-t} \circ M_{\theta(\tau_n)^{q_n}} \circ S_{\tau_n - \tau_{n-1}} \circ \cdots \circ S_{\tau_2 - \tau_1} \circ M_{\theta(\tau_1)^{q_1}}](T(\tau_1, \varphi, \theta(0,\cdot)^{q_0})),$$
$$u(t') = S_{t'-t}(u(t)),$$

and

$$(\mathrm{Bo})\text{-}\int_{(t,t']} (S_{t'-s} \circ M_{\theta(s)})(u(s))\, d\mu(s) = 0, \quad \textit{a zero operator.}$$

§ 11. The Dobrakov integral on the analogue of Wiener space

In this section, we will treat the theory of Dobrakov integral over $C[a,b]$. For $B \in \mathcal{B}(C[a,b])$, let $V(B)\colon \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$ with $[V(B)](\varphi) = V_\varphi(B)$. Then $V(B)$ is a bounded linear operator on $\mathcal{M}(\mathbb{R})$. From [37], we can check that the following facts.

Lemma 11.1. *For $u_0 \in \mathbb{R}$, let $P_{u_0} = \{x \in C[a,b] | x(a) = u_0\}$. Then $\hat{V}(P_{u_0}) = 1$.*

Theorem 11.2. *Let F be a subset of $\mathbb{R}$ and let $P(F) = \{x \in C[a,b] | x(0)$ belongs to $F\}$. Then F is finite if and only if $\hat{V}(P(F))$ is finite if and only if $P(F)$ is integrable.*

Using Theorem 1.3, we have the following theorem.

Theorem 11.3. *V is an operator-valued measure countably additive in the strong operator topology but is not an operator-valued measure countably additive in the uniform operator topology.*

Theorem 11.4. *$\mathcal{M}(\mathbb{R})$ is not a weakly complete Banach space i.e., there is a subspace of $\mathcal{M}(\mathbb{R})$ which is isomorphic to the space c_0.*

Theorem 11.5. *If F is a finite subset of $\mathbb{R}$, then V is continuous on $P_0(F)$.*

Lemma 11.6. *Let the semivariation $\hat{m}$ be continuous on an integrable set, let A be an integrable set and let f be a bounded strongly function. Then the function $f \cdot \chi_A$ integrable.*

Theorem 11.7 (The Wiener Integral Formula for Operator-Valued Measure). *Let $F = \{u_1, u_2, \dots, u_n\}$ and let $a = t_0 < t_1 < \cdots < t_n = b$ be given. Suppose H is a function from $P(F)$ into $\mathcal{M}(\mathbb{R})$ such that $H(x) = \delta_{x(a)} h_{x(a)}(x(t_1), x(t_2), \dots, x(t_n))$ and h_{u_k} $(k = 1, 2, \dots, n)$ are bounded measurable functions on $\mathbb{R}^n$. Then the Dobrakov integral $\int_{P(F)} H(x)dV(x)$ exists and the following equality holds.*

$$
\begin{aligned}
&[\int_{P(F)} H(x)dV(x)](E) \\
&= \frac{1}{\prod_{i=1}^{n} \sqrt{2\pi(t_i - t_{i-1})}} \sum_{k=1}^{n} \int_E [\int_{\mathbb{R}^{n-1}} h_{u_k}(v_1, v_2, \dots, v_n) \\
&\exp\{-\frac{1}{2} \sum_{i=2}^{n} \frac{(v_i - v_{i-1})^2}{t_i - t_{i-1}}\} \exp\{-\frac{1}{2} \frac{(v_1 - u_k)^2}{t_1}\} dm_L(v_1) \cdots dm_L(v_{n-1})] dm_L(v_n)
\end{aligned}
$$

for all Borel subset E of $\mathbb{R}$.

Example 11.8. Let $t = 2$ and let $F = \{3, 5\}$. Let $H(x) = \delta_{x(0)} \exp\{-x(1)^2\}$. Then for $E \in \mathcal{B}(\mathbb{R})$,

$$
[\int_{P(E)} H(x)dV(X)](E) = \frac{e^{-3}}{2\sqrt{2\pi}} \int_E \exp\{-\frac{3}{8}(u_2 - 1)^2\} du_2 + \frac{e^{-\frac{25}{3}}}{2\sqrt{2\pi}} \int_E \exp\{-\frac{3}{8}(u_2 - \frac{5}{3})^2\} du_2.
$$

§ 12. The Operational Calculus for a Measure-Valued Dyson Series

In this section, we investigate Feynman's operational calculus for a measure-valued Dyson series [35]. Throughout this section, let t_1 and t_2 be two real numbers with $0 < t_1 < t_2$ and let

$\varphi \in \mathcal{M}(\mathbb{R})$. Let $X^{a,b}\colon C[a,b] \to \mathbb{R}$ be a function with $X^{a,b}(y) = y(b)$ for $a < b$. For $E \in \mathcal{B}(\mathbb{R})$ and for $B \in \mathcal{B}(C[a,b])$, $V_\varphi^{a,b}(B) = \omega_\varphi(B \cap X^{a,b}(E))$.

Theorem 12.1. *Let* $(s_0, s_1, s_2, \ldots, s_{m+n}) \in \mathbb{R}^{m+n+1}$ *with* $0 < s_0 < s_1 < \cdots < s_m = t_1 < s_{m+1} < \cdots < s_{m+n} = t_2$. *Let* f_1 *and* f_2 *be two complex-valued Borel measurable functions on* $\mathbb{R}^{m+1}$ *and* $\mathbb{R}^n$, *respectively such that*

$$f_1(u_0, u_1, \ldots, u_m) W(m+1; (s_0, s_1, \ldots, s_m); u_0, u_1, \ldots, u_m)$$

is $|\varphi| \times \prod_{j=1}^{m} m_L$*-integrable on* $\mathbb{R}^{m+1}$ *and*

$$f_1(u_0, u_1, \ldots, u_m) f_2(u_{m+1}, \ldots, u_{m+n}) W(m+n+1; (s_0, s_1, \ldots, s_{m+n+1}); u_0, u_1, \ldots, u_{m+n})$$

is $|\varphi| \times \prod_{j=1}^{m+n} m_L$*-integrable on* $\mathbb{R}^{m+n+1}$. *Then*

$$F_1(x) = f_1(x(s_0), x(s_1), \ldots, x(s_m))$$

is V_φ^{0,t_1}*-Bartle integrable on* $C[0,t_1]$,

$$F(x) = f_1(x(s_0), x(s_1), \ldots, x(s_m)) f_2(x(s_{m+1}), \ldots, x(s_{m+n}))$$

is V_φ^{0,t_2}*-Bartle integrable on* $C[0,t_2]$ *and*

$$F_2(x) = f_2(x(s_{m+1}), \ldots, x(s_{m+n}))$$

is $V_{\tilde{\varphi}}^{t_1,t_2}$*-Bartle integrable on* $C[t_1,t_2]$, *where*

$$\tilde{\varphi}(E) = [(\mathrm{Ba})\text{-}\int_{C[0,t_1]} F_1(x)\, dV_\varphi^{0,t_1}(x)](E)$$

for $E \in \mathcal{B}(\mathbb{R})$. *Moreover,*

$$(\mathrm{Ba})\text{-}\int_{C[0,t_2]} F(x)\, dV_\varphi^{0,t_2}(x) = (\mathrm{Ba})\text{-}\int_{C[t_1,t_2]} F_2(x)\, dV_{\tilde{\varphi}}^{t_1,t_2}(x).$$

Remark. Let $\varphi \in \mathcal{M}(\mathbb{R})$, let $P_1 : C[0,t_2] \to C[0,t_1]$ be a function with $[P_1(x)](s) = x(s)$ for $0 \le s \le t_1$ and let $P_2\colon C[0,t_2] \to C[t_1,t_2]$ be a function with $[P_2(x)](s) = x(s)$ for $t_1 \le s \le t_2$. Then by Theorem 12.1, $V_\varphi^{0,t_2}(I) = V_{\tilde{\varphi}}^{t_1,t_2}(P_2(I))$ for $I \in \mathcal{I}$ where $\tilde{\varphi} = V_\varphi^{0,t_1}(P_1(I))$. But it is not true always that $V_\varphi^{0,t_2}(B) = V_{\tilde{\varphi}}^{t_1,t_2}(P_2(B))$ for $B \in \mathcal{B}(C[0,t_2])$ where $\tilde{\varphi} = V_\varphi^{0,t_1}(P_1(B))$. Because, putting $\varphi = \delta_0$, $t_1 = 1$, $t_2 = 2$ and $B = \{x \in C[0,t_2] |$ either $x(1) \ge 0$ and $x(2) \ge 0$ or $x(1) < 0$ and $x(2) < 0$ holds$\}$, $B \in \mathcal{B}(C[0,2])$, $[V_\varphi^{0,2}(B)](\mathbb{R}) = \frac{1}{2}$, $V_\varphi^{0,1}(P_1(B)) = V_\varphi^{0,1}(C[0,1]) = S_1(\delta_0)$, the standard normal distribution, and

$$[V_{S_1(\delta_0)}^{1,2}(P_2(B))](\mathbb{R}) = [S_1 \circ S_1(\delta_0)](\mathbb{R}) = [S_2(\delta_0)](\mathbb{R}) = 1.$$

Here, we establish the operational calculus for a measure-valued Dyson series, the following theorem in this note.

Theorem 12.2. *Under the assumptions in the above theorem, let* $g(z) = \exp(z)$ *and* $\eta = \mu + \nu$ *a complex-valued Borel measure on* $[0,t_2]$ *such that* μ *is a continuous part of* η *and* $\nu = \sum_{p=0}^{m+n} c_p\delta_{\tau_p}$ *where* $0 = \tau_0 < \tau_1 < \cdots < \tau_m = t_1 < \tau_{m+1} < \cdots < \tau_{m+n} = t_2$ *and* c_p $(p = 0,1,\ldots,m+n)$ *are complex numbers. Then* $\exp(\int_{[0,t_1]}\theta(s,x(s))d\eta(s))$, $\exp(\int_{(t_1,t_2]}\theta(s,x(s))d\eta(s))$ *and* $\exp(\int_{[0,t_2]}\theta(s,x(s))d\eta(s))$ *are all* V_φ^{0,t_1}-, $V_{\tilde{\varphi}}^{t_1,t_2}$- *and* V_φ^{0,t_2}-*Bartle integrable of* x *on* $C[0,t_1]$, $C[t_1,t_2]$ *and* $C[0,t_2]$, *respectively, where*

$$\tilde{\varphi} = (\mathrm{Ba})\text{–}\int_{C[0,t_1]}\exp(\int_{[0,t_1]}\theta(s,x(s))d\eta(s))dV_\varphi^{0,t_1}(x).$$

Moreover,

$$\begin{aligned}&(\mathrm{Ba})\text{–}\int_{C[0,t_2]}\exp(\int_{[0,t_2]}\theta(s,x(s))d\eta(s))dV_\varphi^{0,t_2}(x)\\ &= (\mathrm{Ba})\text{–}\int_{C[t_1,t_2]}\exp(\int_{(t_1,t_2]}\theta(s,x(s))d\eta(s))dV_{\tilde{\varphi}}^{t_1,t_2}(x).\end{aligned}$$

Remark. (a) Let θ be a constant function on $[0,t_2]\times\mathbb{R}$, say $\theta(s,u) = c$, let η be the Lebesgue measure on $[0,t_2]$ and let $\varphi = \delta_0$. Then

$$\int_{C[0,t_1]}\exp(\int_{[0,t_1]}\theta(s,x(s))d\eta(s))dV_\varphi^{0,t_1}(x) = \exp(ct_1)S_{t_1}(\delta_0) \equiv \tilde{\varphi}$$

and

$$\begin{aligned}&\int_{C[t_1,t_2]}\exp(\int_{[t_1,t_2]}\theta(s,x(s))d\eta(s))dV_{\tilde{\varphi}}^{t_1,t_2}(x)\\ &= \int_{C[0,t_2]}\exp(\int_{[0,t_2]}\theta(s,x(s))d\eta(s))dV_\varphi^{0,t_2}(x),\end{aligned}$$

so, a formula, given in Theorem 12.2, holds.
(b) Taking $g(z) = z$ in (a)

$$\int_{C[0,t_1]}g(\int_{[0,t_1]}\theta(s,x(s))d\eta(s))dV_\varphi^{0,t_1}(x) = ct_1S_{t_1}(\delta_0) \equiv \psi$$

and

$$\int_{C[t_1,t_2]}g(\int_{[t_1,t_2]}\theta(s,x(s))d\eta(s))dV_\psi^{t_1,t_2}(x) = c^2t_1(t_2-t_1)S_{t_2}(\delta_0).$$

Since $\int_{C[0,t_2]}g(\int_{[0,t_2]}\theta(s,x(s))d\eta(s))dV_\varphi^{0,t_2}(x) = ct_2S_{t_2}(\delta_0)$ a formula, given in Theorem 12.2, doesn't hold for the general function g.

§ 13. Fourier-Feynman Transform on Analogue of Wiener Space

In this section, we will develop the theories of fourier-Feynman transform on analogue of Wiener space. First of all, we will establish the existence theorem for our transform. Moreover, we will find the properties of it. In developing our theories, the rotation theorem is key role, so we assume that a measure φ has the Radon-Nikodym derivative with respect to the Lebesgue measure having a form $\frac{d\varphi}{dm_L}(x) = Ae^{-ax^2}$ where A, α are two positive real numbers.

Lemma 13.1. *For a non-zero complex number λ with $Re\,\lambda \geq 0$ and for $f \in L^2(\mathbb{R}^{n+1})$, define*

$$g(v_0, v_1, \ldots, v_n) = \left(\frac{\lambda}{2\pi}\right)^{\frac{n+1}{2}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(u_0, u_1, \cdots, u_n) e^{-\frac{\lambda}{2}\sum_{j=0}^{n}(u_j - v_j)^2}\, dm_L(u_0) \cdots dm_L(u_n),$$

then $g \in L^2(\mathbb{R}^{n+1})$ and $\|g\|_2 \leq \|f\|_2$.

Lemma 13.2. *For $1 \geq p \geq 2$ and all non-zero real number q, if $F_1 \overset{an}{\approx} F_2$, then the existence of $T_q^{(p)}(F_1)$ assure the existence of $T_q^{(p)}(F_2)$, and $T_q^{(p)}(F_1) \overset{an}{\approx} T_q^{(p)}(F_2)$.*

Theorem 13.3. *For a partition $a = t_0 < t_1 < \cdots < t_n = b$, let $F(x) = f(x(t_0), x(t_1), \ldots, x(t_n))$ be in $\mathcal{A}_n$. Then for each $y \in C[a,b]$ and for all complex numbers λ with* $\mathrm{Re}\,\lambda > 0$, $\int_{C[a,b]}^{\text{an anw}\lambda} F(x+y)d\omega_\varphi(x)$ *exists. Moreover,*

$$(13.1)\quad \begin{aligned} &\int_{C[a,b]}^{\text{an anw}\lambda} F(x+y)d\omega_\varphi(x) \\ &= A\lambda^{\frac{n+1}{2}}\left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(v_0, v_1, \ldots, v_n) \\ &\quad e^{-\frac{\lambda}{2}\sum_{j=1}^{n}\frac{[(v_j - v_{j-1}) - (y(t_j) - y(t_{j-1}))]^2}{t_j - t_{j-1}}} e^{-a\lambda(v_0 - y(t_0))^2} dm_L(v_n) \cdots dm_L(v_1) dm_L(v_0). \end{aligned}$$

Moreover, let $h(y(t_0), y(t_1), \ldots, y(t_n); \lambda)$ be the right side in (13.1)*, then*

$$\|h(y(t_0), y(t_1), \ldots, y(t_n); \lambda)\|_2 \leq A\sqrt{\frac{\pi}{a}}\|f\|_2$$

and $h(w_0, w_1, \ldots, w_n; \lambda)$ is analytic of λ.

Theorem 13.4. *Let $f \in L^2(\mathbb{R}^{n+1})$ and let q be a non-zero real number. For a partition $a = t_0 < t_1 < \cdots < t_n = b$, define*

$$(13.2)\quad \begin{aligned} &g(v_0, v_1, \ldots, v_n) \\ &= \frac{A}{\sqrt{2\pi}}(-iq)^{\frac{n+1}{2}}\left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_0, u_1, \ldots, u_n) \\ &\quad e^{\frac{iq}{2}\sum_{j=1}^{n}\frac{[(u_j - u_{j-1}) - (v_j - v_{j-1})]^2}{t_j - t_{j-1}}} e^{iqa(u_0 - v_0)^2}\, dm_L(u_n) \cdots dm_L(u_1) dm_L(u_0). \end{aligned}$$

Then g is in $L^2(\mathbb{R}^{n+1})$, *and for all* $u_0, u_1, \ldots, u_n$

$$f(u_0,u_1,\ldots,u_n) \tag{13.3}$$
$$= \frac{\sqrt{2}a}{A\sqrt{\pi}}(iq)^{\frac{n+1}{2}}[\prod_{j=1}^{n} 2\pi(t_j-t_{j-1})]^{-\frac{1}{2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(v_0,v_1,\ldots,v_n)$$
$$e^{-\frac{iq}{2}\sum_{j=1}^{n}\frac{[(u_j-u_{j-1})-(v_j-v_{j-1})]^2}{t_j-t_{j-1}}} e^{-iqa(u_0-v_0)^2}\, dm_L(v_n)\cdots dm_L(v_1)dm_L(v_0),$$

and

$$\|f\|_2 = \frac{4a^2}{A}\|g\|_2. \tag{13.4}$$

Theorem 13.5. *For a nonzero real number q and a partition* $a = t_0 < t_1 < \cdots < t_n = b$, *if* $F(x) = f(x(t_0),\ldots,x(t_n)) \in \mathcal{A}_n$, *then* $G \equiv T_q(F)$ *exists and* $G(y) \approx g(y(t_0),\ldots,y(t_n)) \in \mathcal{A}_n$ *where*

$$g(w_0,w_1,\ldots,w_n) \tag{13.5}$$
$$= \frac{A}{\sqrt{2\pi}}(-iq)^{\frac{n+1}{2}}[\prod_{j=1}^{n} 2\pi(t_j-t_{j-1})]^{-\frac{1}{2}} \lim_{B\to\infty}\int_{D_B}\cdots\int_{D_B} f(v_0,v_1,\ldots,v_n)$$
$$e^{\frac{iq}{2}\sum_{j=1}^{n}\frac{[(v_j-v_{j-1})-(w_j-w_{j-1})]^2}{t_j-t_{j-1}}} e^{iqa(v_0-w_0)^2}\, dm_L(v_n)\cdots dm_L(v_1)dm_L(v_0).$$

Moreover,

$$\|T_q(F)\|_2 \le 4\sqrt{a^3}\|g\|_2. \tag{13.6}$$

References

[1] Bearman J. E., Rotation in the product of two Wiener spaces, *Proc. Amer. Math. Soc.* **3** (1952), 129–137.

[2] Billingsley, P., *Convergence of Probability Measures*, John Wiley & Sons, New York, 1968.

[3] Burrill, C. W., *Measure, Integration and Probability*, McGraw-Hill, New York, 1972.

[4] Chiang, T. S., Chow, Y. and Lee, Y. J., A formula for $E_W \exp\{-2^{-1}a^2\|x+y\|_2^2\}$, *Proc. Amer. Math. Soc.* **100** (1987), 721–724.

[5] Cameron, R. H. and Martin, W. T., Transformations of Wiener integrals under translations, *Ann. Math.* **45** (1944), 386–396.

[6] Cameron, R. H. and Storvick, D. A., An operator-valued function space integral and a related integral equation, *J. Math. Mech.* **18** (1968), 517–552.

[7] ———, Some Banach algebras of analytic Feynman integrable functionals, Lecture Notes **798**, *Springer-Verlarg* (1980), 18–67.

[8] Chang, K. S. and Ryu, K. S., Analytic operator-valued function space integrals as an $\mathcal{L}(L_p, L_{p'})$ theory, *Trans. Amer. Math. Soc.* **330** (1992), 697–709.

[9] Cho, D. H., A simple formula for an analogue of conditional Wiener integrals and its applications, *Trans. Amer. Math. Soc.* **360** (2008), 3795–3811.

[10] Cohn, D. L., *Measure Theory*, Birkhauser, Boston, 1980.

[11] Diestel, J. and Uhl, J. J., *Vector measures*, Mathematical Survey, No 15, A. M. S., 1977.

[12] Dobrakov, I., On integration in Banach space, *I, Czech. Math. L.*, **20** (1970), 511–536.

[13] Dunford, N. and Schwartz, J. T., *Linear Operators, Part* I, *General Theory*, Pure and Applied Mathematics **7**, Wiley Interscience, New York, 1958.

[14] C. George, *Exercises in Integration*, Springer-Verlag New York, 1984.

[15] Halmos, P. R., *Measure Theory*, Springer-Verlag, New York, 1950.

[16] Hewitt, E. and Stromberg, K., *Real and Abstract Analysis*, Springer-Verlag, New York, 1965.

[17] Im, M. K., The transformation theorem on analogue of Wiener space, *J. Kore Soc. Math. Educ. Ser. B: Pure Appl. Math.* **14** (2007), 317–333.

[18] Johnson, G. W. and Lapidus, M. L., Generalized Dyson series, generalized Feynman diagrams, the Feynman integral and Feynman's operational calculus, *Memoirs A. M. S.* **62** (1986), 1–78.

[19] ______, *The Feynman integral and Feynman's operational calculus*, Oxford Mathematical Monographs, Oxford Univ. Press, 2000.

[20] Kluvanek, I., Operator valued measures and perturbations of semi-groups, *Arch. Rational Mech. Anal.* **81** (1983), 161–180.

[21] Kluvanek, I. and Knowles, G., *Vector Measures and Control Systems*, Math. Studies **20**, North-Holland, 1975.

[22] Kuo, H. H., *Gaussian Measures in Banach Spaces*, Lecture Notes in Math. **463**, Springer, 1975.

[23] Lapidus, M. L., The Feynman-Kac formula with a Lebesgue-Stieltjes measure and Feynman's operational calculus, *Stud. Appl. Math.* **76** (1987), 93–132.

[24] ______, Strong product integration of measures and the Feynman-Kac formula with a Lebesgue-Stieltjes measure, *Circ. Math. Palermo (2) Suppl.*, **17** (1987), 271–312.

[25] ______, The Feynman-Kac formula with a Lebesgue-Stieltjes measure: an integral equation in the general case, *Integral Equations and Operator Theory*, **12** (1989), 163–210.

[26] Larsen, R., *Functional Analysis*, Marcel Dekker, INC, New York, 1973.

[27] Lewis, D. R., Integration with respect to vector measure, *Pacific J. Math.*, **33** (1970), 157–165.

[28] Novinger, W. P., Mean convergence in L^p space, *Proc. Amer. Math. Soc.*, **34** (1972), 627–628.

[29] Okikiolu, G. G., *Aspect of the theory of bounded linear operators in L_p space*, Academic Press, London, 1971.

[30] Parthasarathy, K. R., *Probability measures on metric spaces*, Academic Press, New York, 1967.

[31] Park, C. and Paranjape, S. R., Probabilities of Wiener paths crossing differentiable curves, *Pacific J. Math.*, **53** (1974), 579–583.

[32] Park, C. and Skoug, D. L., A simple formula for conditional Wiener integrals with applications, *Pacific J. Math.*, **135** (1988), 381–394.

[33] Reed M. and Simon B., *Functional Analysis* I, Academic Press, New York, 1972.

[34] Rudin, W., *Real and Complex Analysis*, 3nd. ed., Mcgraw-Hill, New York, 1987.

[35] Ryu, K. S., The operational calculus for a measure-valued Dyson series, *J. Korean Math. Soc.*, **43** (2006), 703–715.

[36] ______, Volterra-Stieltjes integral equation and the operator-valued function space integral as an $\mathcal{L}(L_p, L_{p'})$ theory, *Bull. Korean Math. Soc.* **36** (1999), 171–181.

[37] ______, The Dobrakov integral over paths, *J. Chungcheong Math. Soc.* **19** (2006), 61–68.

[38] ______, The simple formula of conditional expectation on analogue of Wiener measure, *Honam Math. J.* **30** (2008), 723–732.

[39] Ryu, K. S. and Im, M. K., A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula, *Trans. Amer. Math. Soc.*, **354** (2002), 4921–4951.

[40] ______, An analogue of Wiener measure and its applications, *J. Korean Math. Soc.*, **39** (2002),

801–819.

[41] ———, The measure-valued Dyson series and its stability theorem, *J. Korean Math. Soc.*, **43** (2006), 461–489.

[42] Ryu, K. S., Im, M. K. and Choi, K. S., Survey of the theories for analogue of Wiener measure space, *Inter. Infor. Sci.*, **15** (2009), 319–337.

[43] Ryu, K. S. and Shim, S. H., The rotation theorem on analogue of Wiener measure, *Honam Math. J.*, **29** (2007), 577–588.

[44] S. Saks, *Theory of the integral*, Hafner, 1936.

[45] Shepp, L. A., Radon-Nikodym derivatives of Gaussian measures, *Ann. Math. Stat.*, **37** (1966), 321–354.

[46] Tucker, H. G., *A graduate course in probability*, Academic press, New York, 1967.

[47] Tricomi, F. G., *Integral Equations*, Interscience Publishers, Inc., 1957.

[48] Wiener, N., Differential space, *J. Math. Phys.*, **2** (1923), 131–174.

[49] Yamasaki, Y., Measures on infinite dimensional space, *World Scientific in Pure Math.* **15** (1985).

[50] Yeh, J., *Stochastic processes and the Wiener integral*, Marcel Deckker, New York, 1973.

[51] ———, Inversion of conditional expectations, *Pacific J. Math.*, **52** (1974), 631–640.

[52] Yosida, K., *Functional Analysis, 4th Edition*, Springer-Verlag Berlin, 1974.

[53] Yoo, I., Convolution and the Fourier-Wiener transform on abstract Wiener space, *Rocky Mountain J. Math.*, **25** (1995), 1577–1587.