数理解析研究所講究録
第1798巻 2012年 73-84

Variational formulation of nonlinear hydrodynamic stability

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I. INTRODUCTION

Nonlinear stability of hydrodynamic equilibria has been an important issue for gaining a better understanding of phenomena such as pattern formation and transition to turbulence. In 1944, Landau [1] deduced a heuristic scenario about development of a weakly unstable mode towards nonlinear regime, in which the nonlinear self-interaction of the dominant mode generates second harmonics and distorts the mean fields. The evolution of the dominant mode's amplitude A is possibly governed by the Landau equation, $d|A|^2/dt = 2\gamma|A|^2 - l|A|^4$, where $\gamma > 0$ is the linear growth rate and l is a constant that determines whether the linear growth will be saturated or not [2]. Although this simple model is not valid for all situations, such reduction to a low degree-of-freedom system is very informative. However, the derivation of this greatly reduced equation (so-called the amplitude equation) from the hydrodynamic equations requires a lot of algebra and approximations. While the heuristic method have been successful in many fundamental hydrodynamic problems, it often faces technical difficulties such as "indeterminable mean field" (the equations for the nonlinearly-generated mean fields are underdetermined and require a further assumption to fix them).

The goal of this paper is to propose a systematic approach for deriving the amplitude equations (which describe the local bifurcations of equilibria) by means of the variational principle, namely, Hamilton's principle. In comparison to the conventional approach, the use of the variational principle shortens the detailed calculation of the wave-wave interactions and elucidates the properties peculiar to dynamical systems such as the energy conservation. Moreover, we will show that the determination of the wave-induced mean fields can be carried out immediately owing to the Lagrangian viewpoint of fluid. This fact was exemplified by our recent work on elliptically-strained rotating flow [3]. This paper tries to present more general understanding that can comprehend various hydrodynamic problems.

II. CONVENTIONAL APPROACH TO AMPLITUDE EQUATION

In this section, we review how the amplitude equation is derived conventionally from the hydrodynamic equations of the *Eulerian* description. While the use of the Eulerian description of fluid looks simple and straightforward at first viewing, we will indicate that some inevitable problems occur within the Eulerian framework.

Let us symbolically denote the hydrodynamic equations for Eulerian variables by $\partial_t u = X(u)$. For example, the variables are a set of velocity, density and pressure fields, say $u = (\boldsymbol{v}, \rho, p)^T$, for the case of compressible fluids. By assuming the perturbation expansion ($\epsilon \ll 1$):

$$u = u_e + \epsilon u^{(1)} + \frac{\epsilon^2}{2} u^{(2)} + \frac{\epsilon^3}{3!} u^{(3)} + \dots \tag{1}$$

around an equilibrium state, u_e s.t. $X(u_e) = 0$, the nonlinear stability of u_e can be analyzed as follows. First, to leading order, we solve the linearized problem,

$$O(\epsilon): \quad \partial_t u^{(1)} = DX_e(u^{(1)}), \tag{2}$$

where DX_e denotes the linearized operator at u_e. The linear solution predicts an occurrence of an unstable eigenmode, $\hat{u}_\Theta^{(1)} \propto e^{i\Theta}$, which depends on a certain phase $\Theta = -(\omega + i\gamma)t + k_y y + k_z z$ ($\gamma > 0$ corresponds to instability, and the wavenumbers, k_y and k_z, are well-defined if the equilibrium u_e is uniform in the y, z directions). In order to formulate nonlinear evolution of this unstable eigenmode with a sufficiently small γ (i.e., in the neighborhood of the bifurcation point), Landau resorted what is called the multiple scale analysis. Namely, the solution is assumed to take the form of

$$u^{(1)} = A(T)\hat{u}_\Theta^{(1)}(t) + \text{c.c.}, \quad (T = \epsilon^2 t) \tag{3}$$

where $A(T)$ represents a slowly varying part of the amplitude and will be determined later by the higher-order analysis. In the second order, the equation for $u^{(2)}$ takes a form of

$$O(\epsilon^2): \quad \partial_t u^{(2)} - DX_e(u^{(2)}) = D^2 X_e(u^{(1)}, u^{(1)}). \tag{4}$$

By substituting (3) into the right hand side of this equation, the inhomogeneous solution can be written as

$$u^{(2)} = 2|A|^2 \hat{u}_0^{(2)} + (A^2 \hat{u}_{2\Theta}^{(2)} + \text{c.c.}), \tag{5}$$

where $\hat{u}_0^{(2)}$ is understood as the wave-induced mean fields and $\hat{u}_{2\Theta}^{(2)} (\propto e^{2\Theta})$ as the second harmonics. In the third order, the equation for $u^{(3)}$ is derived as

$$O(\epsilon^3): \quad \partial_t u^{(3)} - DX_e(u^{(3)}) = 3D^2X_e(u^{(1)}, u^{(2)}) + D^3X_e(u^{(1)}, u^{(1)}, u^{(1)}) - 6\partial_T u^{(1)}, \tag{6}$$

where the last term comes from the assumption of slowly varying amplitude. When (3) and (5) are substituted into the right hand side, one can find that the right hand side includes the fundamental harmonics $\propto e^{i\Theta}$ and it would force the inhomogeneous solution $u^{(3)}$ to diverge in proportion to $te^{i\Theta}$. Such a failure of perturbation expansion can be avoided by imposing the solvability condition. Since the homogeneous part of the equation has the eigenmode $\hat{u}_{\Theta}^{(1)}$ as a solution, one also needs to solve the eigenmode $\hat{v}_{\overline{\Theta}}^{(1)} \propto e^{i\overline{\Theta}}$ of the adjoint operator $(DX_e)^*$. The solvability condition is then $\langle \hat{v}_{\overline{\Theta}}^{(1)}$, R.H.S of (6) $\rangle = 0$, which ends up with the equation of $A(T)$

$$\partial_T A = -lA|A|^2, \tag{7}$$

where l is the constant conjectured by Landau. In fact, the analytical evaluation of this constant l is laboring especially when there are many field variables in u.

Obviously, a lot of assumptions are required to justify the above procedure depending on the problems. We expect that the mathematical justification should be given in the context of the center manifold theory, whereas the hydrodynamic problem is still challenging because of the infinite dimensionality. However, even if we leave aside the detailed analysis, we can notice that there is a crucial defect in the above Landau's procedure.

The problem is that the equation for the 2nd-order mean field $\hat{u}_0^{(2)}$ is *underdetermined* in many cases, i.e., further reasonable constraint is required to fix $\hat{u}_0^{(2)}$. The reason for this arbitrariness of $\hat{u}_0^{(2)}$ is intuitively understood as follows.

Let us consider a fluid in the slab geometry, such as the parallel shear flow (see FIG. 1). If one assumes the uniformity of the equilibrium state u_e along the y, z directions, it is well-known that the equilibrium condition $X(u_e) = 0$ does not so much

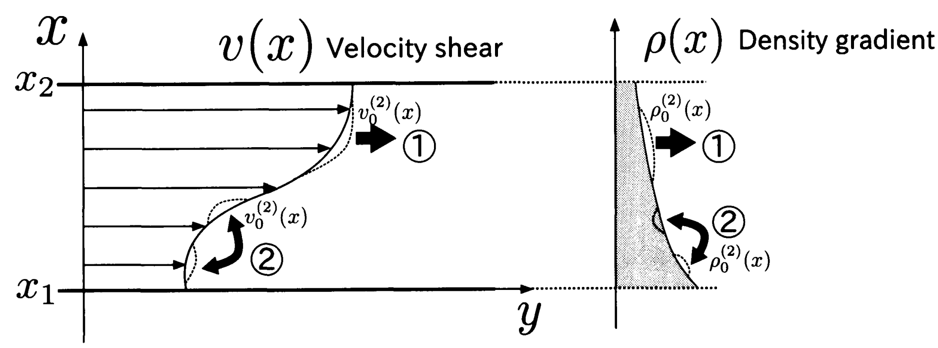

FIG. 1: Example of equilibrium state in slab geometry and modifications by some secondary fields

restrict the velocity and density profiles of equilibrium fields along the x direction; in fact, they can be almost arbitrary. It follows that, when one studies stability of a certain equilibrium state u_e, there exist infinite number of neighboring equilibrium states whose profiles are slightly different from u_e. Therefore, the equation for the 2nd-order mean fields $\hat{u}_0^{(2)}$ becomes an underdetermined problem, that is, almost arbitrary $\hat{u}_0^{(2)}$ is allowed to produce a modified equilibrium solution.

However, this arbitrariness should be removed in a legitimate manner. We can find it from the following physical reasons. For instance, suppose that the modification $\hat{u}_0^{(2)}$ causes some excesses of velocity and density (as shown by ① of FIG. 1). Such changes in mean fields clearly violate the momentum and mass conservation laws, and hence they cannot occur as a consequence of the fluid motion induced by the unstable mode. Instead, let us consider another $\hat{u}_0^{(2)}$ that causes some interchanges of momentum and mass (as shown by ② of FIG. 1). These changes are now possible if and only if the corresponding motion (with $k_y \neq 0$) of the fluid elements is accompanied. Therefore, we can infer that the changes in the mean fields should be somehow determined in such a way that they are generated by the actual fluid motion preserving the conservation laws. To ascertain this conjecture, we naturally necessitate the Lagrangian viewpoint of fluid dynamics.

III. LAGRANGIAN FOR NONLINEAR DISPLACEMENT OF FLUID

As the starting point, we employ the Newcomb's Lagrangian theory [5], which is originally devised as the variational principle for ideal magnetohydrodynamic

(MHD) equations;

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla p + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \tag{8}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{9}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{10}$$

$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s = 0 \tag{11}$$

($\boldsymbol{v}$: velocity, $\boldsymbol{B}$: magnetic field, ρ: density, s: specific entropy, $p(\rho, s)$: pressure). We remark that this variational principle is directly applicable to inviscid compressible fluids by neglecting the magnetic field and moreover to inviscid incompressible fluids (i.e., the Euler equations) by restricting the fluid motion to be incompressible. Let us denote by $\boldsymbol{x}(t)$ the position of an infinitesimal fluid element at time t. Newcomb's Lagrangian is defined as a functional of the fluid flow map $\varphi_t : \boldsymbol{x}(0) \in D \to \boldsymbol{x}(t) \in D$, which is mathematically a one-parameter ($t \in \mathbb{R}$) diffeomorphism group on the fluid domain $D \subset \mathbb{R}^3$. The (Eulerian) velocity field is given by $\boldsymbol{v}(\boldsymbol{x}(t), t) = d\boldsymbol{x}/dt(t)$. Then, the three conservation laws (9)-(11) are formally solved as $\varphi_t^*\beta_t = \beta_0, \ \ \varphi_t^*\varrho_t = \varrho_0, \ \ \varphi_t^* s_t = s_0$, where φ_t^* denotes the pull-back of n-forms and we use the following identifications,

$$\boldsymbol{B} \ \Leftrightarrow \ \text{2-form: } \beta_t = J(\boldsymbol{x})[B^1(\boldsymbol{x}, t)dx^2 \wedge dx^3 + B^2(\boldsymbol{x}, t)dx^3 \wedge dx^1 + B^3(\boldsymbol{x}, t)dx^1 \wedge dx^2], \tag{12}$$

$$\rho \ \Leftrightarrow \ \text{3-form: } \varrho_t = \rho(\boldsymbol{x}, t)J(\boldsymbol{x})dx^1 \wedge dx^2 \wedge dx^3, \tag{13}$$

$$s \ \Leftrightarrow \ \text{0-form: } s_t = s(\boldsymbol{x}, t), \tag{14}$$

where $J(\boldsymbol{x})$ is the Jacobian of the general coordinates $\boldsymbol{x} = (x^1, x^2, x^3)$. In this way, the Eulerian field variables $u = (\boldsymbol{v}, \boldsymbol{B}, \rho, s)^T$ are related to the fluid motion φ_t.

The Lagrangian [5] is given by

$$L(u) = \int_D \left[\frac{\rho}{2}|\boldsymbol{v}|^2 - \frac{1}{2}|\boldsymbol{B}|^2 - \rho U(\rho, s)\right] d^3x, \tag{15}$$

where $U(\rho, s)$ denotes the internal energy per unit volume. Note that one has to take a variation of this Lagrangian with respect to the fluid motion, $\boldsymbol{x}(t) \mapsto$

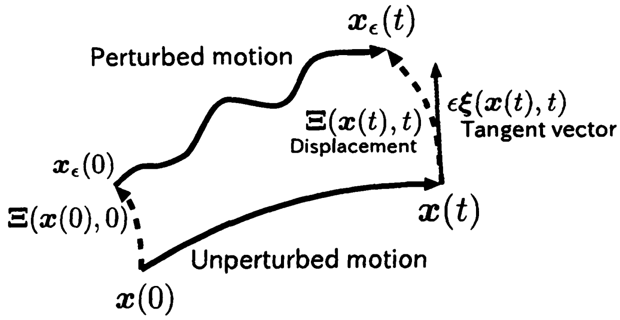

FIG. 2: Unperturbed and perturbed orbits of a fluid element

$\boldsymbol{x}(t)+\epsilon\boldsymbol{\xi}(\boldsymbol{x}(t),t)$. The associated variation of u is found to be $u \mapsto u+\epsilon\delta_{\boldsymbol{\xi}} u$, where

$$\delta_{\boldsymbol{\xi}} u \stackrel{\text{def}}{=} \begin{pmatrix} \partial_t \boldsymbol{\xi} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v} \\ \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) \\ -\nabla\cdot(\rho\boldsymbol{\xi}) \\ -\boldsymbol{\xi}\cdot\nabla s \end{pmatrix}. \tag{16}$$

This operator $\delta_{\boldsymbol{\xi}}$ acts on each components of u as the Lie derivative along $\boldsymbol{\xi}$. Then, as shown by Newcomb, Hamilton's principle indeed reproduces the equation of motion (8);

$$0 = \delta_{\boldsymbol{\xi}} \int L(u)dt = \int\int_D \left[\rho\frac{D\boldsymbol{v}}{Dt} + \nabla p - (\nabla\times\boldsymbol{B})\times\boldsymbol{B}\right]\cdot\boldsymbol{\xi} d^3x dt \quad \text{for } \forall\boldsymbol{\xi}$$

where the variation $\boldsymbol{\xi}$ should be tangential to the boundary of the domain D.

Now, let us formulate the variational principle for nonlinear perturbation based on the above Newcomb's theory. From now on, we regard $\boldsymbol{x}(t) = \varphi_t(\boldsymbol{x}(0))$ as an unperturbed fluid motion and denote a perturbed motion by $\boldsymbol{x}_\epsilon(t) = \boldsymbol{x}(t) + \Xi(\boldsymbol{x}(t),t)$, where $\Xi(\boldsymbol{x},t)$ is the displacement of the orbit (see FIG. 2). Since $\Xi(\boldsymbol{x},t)$ is not a vector field, it is convenient to express this perturbation by an exponential map,

$$\boldsymbol{x}_\epsilon = e^{\epsilon\boldsymbol{\xi}\cdot\nabla}\boldsymbol{x} = \boldsymbol{x} + \epsilon\boldsymbol{\xi} + \frac{\epsilon^2}{2}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} + \frac{\epsilon^3}{6}\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}) + \dots \tag{17}$$

where $\epsilon \ll 1$ is a small amplitude parameter and $\epsilon = 0$ corresponds to the unperturbed state. The relation between the nonlinear displacement Ξ and the vector field $\boldsymbol{\xi}$ is, of course,

$$\Xi = \epsilon\boldsymbol{\xi} + \frac{\epsilon^2}{2}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} + \frac{\epsilon^3}{6}\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}) + O(\epsilon^4). \tag{18}$$

The corresponding variation of the Eulerian variables u is represented by the Lie series,

$$u_\epsilon = u + \epsilon\delta_{\boldsymbol{\xi}} u + \frac{\epsilon^2}{2}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}} u + \frac{\epsilon^3}{6}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}} u + \dots. \tag{19}$$

Here, we present a general formula which transforms the Lie-series expansion (with respect to $\boldsymbol{\xi}$) into the expansion with respect to nonlinear displacement Ξ.

$$\begin{aligned}\text{Formula:}\quad e^{\epsilon\delta_{\boldsymbol{\xi}}} =& 1 + \epsilon\delta_{\boldsymbol{\xi}} + \frac{\epsilon^2}{2}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}} + \frac{\epsilon^3}{6}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}} + O(\epsilon^4)\\ =& 1 + \delta_\Xi + \frac{1}{2}\delta^2_{\Xi,\Xi} + \frac{1}{6}\delta^3_{\Xi,\Xi,\Xi} + O(\epsilon^4).\end{aligned} \tag{20}$$

where

$$\delta^2_{\boldsymbol{\eta},\boldsymbol{\xi}} \stackrel{\text{def}}{=} \delta_{\boldsymbol{\eta}}\delta_{\boldsymbol{\xi}} - \delta_{\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}}, \tag{21}$$

$$\delta^3_{\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\xi}} \stackrel{\text{def}}{=} \delta_{\boldsymbol{\zeta}}\delta^2_{\boldsymbol{\eta},\boldsymbol{\xi}} - \delta^2_{\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta},\boldsymbol{\xi}} - \delta^2_{\boldsymbol{\eta},\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi}}, \tag{22}$$

$$\delta^n_{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\dots,\boldsymbol{\xi}_n} \stackrel{\text{def}}{=} \delta_{\boldsymbol{\xi}_1}\delta^{n-1}_{\boldsymbol{\xi}_2,\dots,\boldsymbol{\xi}_n} - \sum_{j=2}^{n}\delta^{n-1}_{\boldsymbol{\xi}_2,\dots,\boldsymbol{\xi}_1\cdot\nabla\boldsymbol{\xi}_j\dots,\boldsymbol{\xi}_n}. \tag{23}$$

We can prove that $\delta^n_{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\dots,\boldsymbol{\xi}_n}$, $n = 2, 3, \dots$, are *symmetric* with respect to any permutation of subscript vector fields. (Proof: Use the Jacobi identity, $\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\eta}} - \delta_{\boldsymbol{\eta}}\delta_{\boldsymbol{\xi}} = \delta_{\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta}-\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}}$ for all $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.) In this formula, note that Ξ is approximately regarded as a vector field using the relation (18).

The perturbation expansion of the Lagrangian is then written as

$$\begin{aligned}L(u_\epsilon) =& L + \epsilon\delta_{\boldsymbol{\xi}}L + \frac{\epsilon^2}{2}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}}L + \frac{\epsilon^3}{6}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}}L + \dots\\ =& L + \delta_\Xi L + \frac{1}{2}\delta^2_{\Xi,\Xi}L + \frac{1}{3!}\delta^3_{\Xi,\Xi,\Xi}L + \dots.\end{aligned} \tag{24}$$

Let us assume that this expansion is carried out around an given equilibrium state u_e. Then, the first-order term $\delta_\Xi L$ vanishes for any Ξ, since the unperturbed state is already an extremum of the Lagrangian. The Lagrangian for the nonlinear displacement Ξ is therefore obtained in the form of

$$L[\Xi] = \frac{1}{2}\delta^2_{\Xi,\Xi}L + \frac{1}{3!}\delta^3_{\Xi,\Xi,\Xi}L + \frac{1}{4!}\delta^4_{\Xi,\Xi,\Xi,\Xi}L + \dots \tag{25}$$

$$= \int \frac{\rho}{2}\left|\frac{D\Xi}{Dt}\right|^2 d^3x - \frac{W^{(2)}(\Xi,\Xi)}{2} - \frac{W^{(3)}(\Xi,\Xi,\Xi)}{3!} - \frac{W^{(4)}(\Xi,\Xi,\Xi,\Xi)}{4!} - \dots \tag{26}$$

where $D/Dt = \partial/\partial t + \boldsymbol{v} \cdot \nabla$ (ρ and $\boldsymbol{v}$ are, respectively, equilibrium density and velocity) [12]. We refer to $W^{(n)}$ as nth-order potential energy, which can be systematically derived by operating $\delta^n_{\Xi,\ldots,\Xi}$ on Newcomb's Lagrangian (15). In the limit of infinitesimally small Ξ, one can neglect $W^{(n)}$, $n \geq 3$, and (26) reproduces the quadratic Lagrangian derived by Dewar [6]. By introducing a linear operator $\mathcal{F}$ by $W^{(2)} = -\int \Xi \cdot \mathcal{F}\Xi d^3x$ (which turns out to be symmetric), the Euler-Lagrange equation corresponds to the well-known linearized equation of motion $\rho D^2\Xi/Dt^2 = \mathcal{F}\Xi$ [7]. Existence of this variational formulation of the linearized systems has been noted by many pioneering works [8–10] and utilized extensively in linear stability analysis. However, the generalization to nonlinear (or finite amplitude) displacement has not been so straightforward since Ξ is not a vector field, but as a diffeomorphism map. Pfirsch & Sudan [11] derived the third-order potential energy $W^{(3)}$ in the absence of equilibrium flow ($\boldsymbol{v} = 0$) and showed a few special examples of equilibria at which $W^{(3)}$ vanishes for marginal modes. Our method presented here can obtain a more complete form of $W^{(3)}$ with cubic symmetry [12], including flow ($\boldsymbol{v} \neq 0$), and moreover shows that the application of the Lie series expansion is a key technique for finding symmetric forms of higher-order potential energies $W^{(n)}$.

IV. AMPLITUDE EQUATIONS FOR WAVE-WAVE INTERACTIONS

Once the Lagrangian for the displacement Ξ is formulated as in (26), we can derive the amplitude equations in a systematic way. Since the Euler-Lagrange equation is written as

$$\rho\frac{D^2\Xi}{Dt^2} = \mathcal{F}\Xi + \frac{1}{2}\mathcal{F}^{(2)}(\Xi,\Xi) + \frac{1}{3!}\mathcal{F}^{(3)}(\Xi,\Xi,\Xi) + \ldots, \tag{27}$$

nonlinear forces $\mathcal{F}^{(n)}(\ldots)$, $n \geq 2$, defined by $W^{(n)}(\ldots) = -\int \Xi \cdot \mathcal{F}^{(n)}(\ldots)d^3x$, are responsible for nonlinear wave-wave interactions.

If there exist three waves satisfying the resonance conditions for frequencies, $\omega_a + \omega_b + \omega_c = 0$, and wave numbers, $k_a + k_b + k_c = 0$, this triad will be coupled via the quadratic force $\mathcal{F}^{(2)}$. Let us express such resonant three eigenmodes by

$$\Xi = \sum_{j=a,b,c} A_j(\epsilon t)\hat{\boldsymbol{\xi}}_j e^{-i\omega_j t} + \text{c.c.}, \tag{28}$$

where $\hat{\boldsymbol{\xi}}_j$ is the eigenfunction of the linearized system (corresponding to the eigenvalue ω_j) and $A_j(\epsilon t)$ is the slowly varying amplitude. By substituting this expression into the Lagrangian (26), the application of the averaged Lagrangian method [13] gives

$$L[\Xi] = -\sum_{j=a,b,c} i\mu_j \frac{dA_j^*}{dt} A_j - \left(W_{a,b,c}^{(3)} A_a^* A_b A_c + \text{c.c.} \right) + O(\epsilon^4) \tag{29}$$

where

$$\mu_j = 2 \int \left[\hat{\boldsymbol{\xi}}_j^* \cdot \rho(\omega_j + i\boldsymbol{v} \cdot \nabla) \hat{\boldsymbol{\xi}}_j \right] d^3x \tag{30}$$

$$W_{a,b,c}^{(3)} = W^{(3)}(\hat{\boldsymbol{\xi}}_a^*, \hat{\boldsymbol{\xi}}_b, \hat{\boldsymbol{\xi}}_c) \tag{31}$$

The variation with respect to A_j yields the amplitude equations for resonant three modes,

$$\mu_a \frac{dA_a}{dt} = -iW_{a,b,c}^{(3)} A_b A_c, \quad \mu_b \frac{dA_b^*}{dt} = iW_{a,b,c}^{(3)} A_a^* A_c, \quad \mu_c \frac{dA_c^*}{dt} = iW_{a,b,c}^{(3)} A_a^* A_b. \tag{32}$$

Note that $N_j = \mu_j |A_j|^2$ corresponds to the wave action of each mode, and they satisfy the energy conservation, $\omega_a N_a + \omega N_b + \omega_c N_c = \text{const}$. The coupling coefficient $W_{a,b,c}^{(3)}$ is measured by the 3rd-order potential energy $W^{(3)}$, where the cubic symmetry of $W^{(3)}$ is directly related to the Hamiltonian property of the amplitude equations. Hence, one needs neither to invoke the solvability condition nor to spend any effort on proving the energy conservation among the resonant triad.

If the three-wave resonance is absent or ineffective, the four-wave resonance via the cubic force $\mathcal{F}^{(3)}$ becomes dominant as the next nonlinearity. In particular, a linearly unstable mode will be subject to the self-interaction via $\mathcal{F}^{(3)}$ as well as the non-resonant interaction with the second harmonics via $\mathcal{F}^{(2)}$, which results in saturation or acceleration of the growth as is the case for Landau's picture. The derivation of the amplitude equation is as follows. First, we are supposed to solve the eigenvalue problem

$$\mathcal{E}(\omega + i\gamma)\hat{\boldsymbol{\xi}}_\Theta \stackrel{\text{def.}}{=} [\rho(-i\omega + \gamma + \boldsymbol{v} \cdot \nabla)^2 - \mathcal{F}]\hat{\boldsymbol{\xi}}_\Theta = 0 \tag{33}$$

and identify an unstable eigenmode $\hat{\boldsymbol{\xi}}_\Theta \propto e^{i\Theta}$ including a phase $\Theta = -\omega t + k_y y + k_z z$. Then, we seek the solution in the form of

$$\Xi = \Xi^{(1)} + \tfrac{1}{2}\Xi^{(2)} \quad \text{with} \quad \Xi^{(1)} = A(\epsilon t)(\hat{\boldsymbol{\xi}}_\Theta + \text{c.c.})$$

and substitute this into the Lagrangian (26). The equation for the 2nd-order displacement $\Xi^{(2)}$ is given by

$$\left(\rho\frac{D^2}{Dt^2} - \mathcal{F}\right)\Xi^{(2)} = \mathcal{F}^{(2)}(\Xi^{(1)}, \Xi^{(1)}). \tag{34}$$

The inhomogeneous solution would be written in the form of

$$\Xi^{(2)} = 2|A|^2\hat{\boldsymbol{\xi}}_0^{(2)} + A^2(\hat{\boldsymbol{\xi}}_{2\Theta}^{(2)} + \text{c.c.}) \tag{35}$$

where $\hat{\boldsymbol{\xi}}_0^{(2)} = \mathcal{E}^{-1}(0)\mathcal{F}^{(2)}(\hat{\boldsymbol{\xi}}_\Theta, \hat{\boldsymbol{\xi}}_\Theta^*)$ is the mean displacement and $\hat{\boldsymbol{\xi}}_{2\Theta}^{(2)} = \mathcal{E}^{-1}(2\omega)\mathcal{F}^{(2)}(\hat{\boldsymbol{\xi}}_\Theta, \hat{\boldsymbol{\xi}}_\Theta) \propto e^{2i\Theta}$ is the second harmonics. Actually, we cannot solve $\hat{\boldsymbol{\xi}}_0^{(2)}$ uniquely because $\mathcal{E}(0)$ is not invertible. But, we will see later that this indeterminacy does not matter for the final result. Consequently, the Lagrangian (averaged over the phase Θ) is reduced to

$$L[\Xi] = I\left|\frac{dA}{dt}\right|^2 + \gamma^2 I|A|^2 - W\frac{|A|^4}{4} \tag{36}$$

where $I = \int \rho|\hat{\boldsymbol{\xi}}_\Theta|^2 d^3x$ and

$$W = W^{(3)}(\hat{\boldsymbol{\xi}}_\Theta, \hat{\boldsymbol{\xi}}_\Theta^*, \hat{\boldsymbol{\xi}}_0^{(2)}) + \mathrm{Re}W^{(3)}(\hat{\boldsymbol{\xi}}_\Theta, \hat{\boldsymbol{\xi}}_\Theta, \hat{\boldsymbol{\xi}}_{2\Theta}^{(2)*}) + W^{(4)}(\hat{\boldsymbol{\xi}}_\Theta, \hat{\boldsymbol{\xi}}_\Theta, \hat{\boldsymbol{\xi}}_\Theta^*, \hat{\boldsymbol{\xi}}_\Theta^*). \tag{37}$$

The Euler-Lagrange equation is therefore

$$\frac{d^2A}{dt^2} = \gamma^2 A - \frac{W}{2I}A|A|^2. \tag{38}$$

The resultant constant $W \in \mathbb{R}$ indicates whether the linear instability $A \propto e^{\gamma t}$ will be decelerated ($W > 0$) or accelerated ($W < 0$) due to the nonlinearity. Again, there is no need to invoke the solvability condition, since we do not have to solve the adjoint problem owing to the symmetric property of the expanded Lagrangian (26).

Moreover, in contrast to the conventional Eulerian approach, this Lagrangian approach can avoid the problem of the ambiguity of 2nd-order mean fields. Although the 2nd-order mean displacement $\hat{\boldsymbol{\xi}}_0^{(2)}$ is still ambiguous (or underdetermined) even in the above analysis, it turns out to cause no change in the resultant amplitude equation.

To show this fact, it is essential to recognize a special class of vector fields defined by $\{\boldsymbol{\eta} : \partial_t\boldsymbol{\eta} = 0, \delta_{\boldsymbol{\eta}} u = 0\}$. Such a $\boldsymbol{\eta}$ generates a symmetry group of the equilibrium state u, that is, a displacement along $\boldsymbol{\eta}$ does not change the equilibrium fields. For the case of the slab equilibrium shown in FIG. 1, such the vector fields are found to be $\boldsymbol{\eta} = (0, \eta_y(x), \eta_z(x))$ with arbitrary functions $\eta_y(x), \eta_z(x)$. Since $\delta_{\boldsymbol{\eta}} u = 0$, one can easily prove that the vector fields $\{\boldsymbol{\eta}\}$ span the eigenspace of the zero eigenvalue $\mathcal{E}(0)\boldsymbol{\eta} = 0$, and hence the 2nd-order mean displacement $\hat{\boldsymbol{\xi}}_0^{(2)}$ is underdetermined ($\hat{\boldsymbol{\xi}}_0^{(2)} + \boldsymbol{\eta}$ is also a solution). On the other hand, according to (19), the 2nd-order Eulerian mean field is given by

$$u_0^{(2)} = \delta_{\hat{\boldsymbol{\xi}}_0^{(2)}} u + \delta^2_{\hat{\boldsymbol{\xi}}_\Theta, \hat{\boldsymbol{\xi}}_\Theta^*} u \tag{39}$$

and the indeterminacy of $\hat{\boldsymbol{\xi}}_0^{(2)}$ will be eliminated by the property $\delta_{\boldsymbol{\eta}} u = 0$. Therefore, $u_0^{(2)}$ is determined uniquely so as to deserve the mean field induced by the actual fluid displacement. Since the Lagrangian $L(u)$ is defined for u, one can always pay no attention to displacements along $\{\boldsymbol{\eta}\}$ that cause no change in u. This fact further simplifies our analysis. For example, we remark that, if the stability of the slab equilibrium of incompressible fluid is concerned, $\{\boldsymbol{\eta}\}$ spans the whole functional space of $\hat{\boldsymbol{\xi}}_0^{(2)}$ and hence there is no longer need to solve $\hat{\boldsymbol{\xi}}_0^{(2)}$ in this case.

V. SUMMARY

In this paper, we have revisited the conventional approach to nonlinear hydrodynamic stability, tracing back to Landau's idea. By reviewing its general procedure of analysis, we pointed out that there is a lack of strategy for determining the secondary mean fields induced by the linearly unstable mode. This ambiguity of the mean fields originates from the existence of a lot of neighboring equilibrium solutions, but most of them are not actually accessible via the fluid motion that preserves the conservation laws of mass and momentum. To determine the secondary mean fields in a unified manner, we naturally necessitate the Lagrangian description of fluid.

By focusing on the nonlinear displacement Ξ of fluid elements, we have formulated the variational principle and derived the equation of motion for Ξ. The perturbation analysis with respect to Ξ can restrict our attention to only the perturbed states that are accessible from the unperturbed state via some fluid motion. Therefore, once Ξ is solved perturbatively, the secondary mean fields is determined automatically without any additional effort.

With this variational principle for Ξ, we can derive the amplitude equations more efficiently and accurately than the conventional method. For example, number of variables is reduced in many cases, e.g., $(\boldsymbol{v}, \boldsymbol{B}, \rho, s) \mapsto \Xi$, because the conservation laws (9)-(11) are formally solved and built-in as constraints on perturbations. We have also shown that there is no need to solve the adjoint eigenfunction for the purpose of imposing the solvability condition, because the linearized system (or the 2nd-order Lagrangian) is already symmetric with respect to the variable Ξ. It should be emphasized that the amplitude equations are derived from the averaged Lagrangian and hence explicitly possess a Hamiltonian structure. This fact is advantageous since even the verification of the energy conservation is often laboring in the conventional approach.

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