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Residual vanishing of concentration arising in the mean field equations

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Abstract

In this short report, we study the Sawada-Suzuki equation. In the positive case, we prove the property called *Residual vanishing* which means that a blow-up solution sequence (more precisely, its subsequence) converges to a finite sum of Dirac's measures in the sense of measure.

1 Introduction

In this report, we consider the Sawada-Suzuki equation ([6]):

$$\begin{cases} -\Delta v_n = \lambda_n \int_I \alpha \left(\frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\alpha) & \text{in } \Omega \\ \int_\Omega v_n = 0, \end{cases} \tag{1.1}$$

where (λ_n, v_n) is a solution sequence to (1.1), λ_n a non-negative number sequence tending to some non-negative number λ_0, $I = [-1, 1]$, $\Omega = (\Omega, g)$ a two dimensional orientable compact Riemannian manifold, and $\mathcal{P}(d\alpha)$ a Borel probability measure on I. According to the result of [4], the following alternative holds:

(i) (*Compactness*) $\limsup_{n\to\infty} \|v_n\|_\infty < +\infty$, namely, there exist $v \in \mathcal{E}$ and a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that $v_{n_k} \to v$ in $\mathcal{E}$ as $k \to \infty$, where

$$\mathcal{E} = \left\{ v \in H^1(\Omega) \mid \int_\Omega v = 0 \right\}.$$

(ii) (*Concentration*) $\limsup_{n\to\infty} \|v_n\|_\infty = +\infty$, namely, the set $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$ is a non-empty and finite set, and there exists $0 \le s_\pm \in L^1(\Omega)$ such that

$$\nu_{\pm,n} := \lambda_n \int_{I_\pm} \frac{\alpha e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) dx \stackrel{*}{\rightharpoonup} \nu_\pm = s_\pm dx + \sum_{x_0 \in \mathcal{S}_\pm} m(x_0) \delta_{x_0}(dx) \tag{1.2}$$

in $\mathcal{M}(\Omega)$ with $m(x_0) \ge 4\pi$ for all $x_0 \in \mathcal{S}_\pm$, where $I_+ = (0, 1]$, $I_- = [-1, 0)$, δ_x is the Dirac measure supported at x, $\mathcal{M}(\Omega) = C(\Omega)^*$ and

$$\mathcal{S}_\pm = \{x_0 \in \Omega \mid \text{there exists } \{x_n\} \subset \Omega \text{ such that } x_n \to x_0 \text{ and } v_n(x_n) \to \pm\infty\}. \tag{1.3}$$

It is natural to ask whether $s_{\pm}$ is zero or not in (1.2). If this is the case, we call this property *residual vanishing* in this report. In the positive case, we obtain

Proposition 1. *If (ii) above holds and $I = I_+$, then $s = s_+ = 0$.*

Remark 1. *We note that $\mathcal{S} = \mathcal{S}_+$ in the case $I = I_+$, see [4] for details. The proof of this fact is based on the boundedness from below of the Green function associated to $-\Delta$ on Ω, i.e.,*

$$\begin{cases} -\Delta_x G(x,y) = \delta_y - \frac{1}{|\Omega|} & \text{in } \Omega \\ \int_\Omega G(x,y)dx = 0, & \forall y \in \Omega, \end{cases}$$

see [1].

Remark 2. *Residual vanishing also holds in the case $I = I_-$.*

Remark 3. *It is open whether residual vanishing is true or not in the general case. On the contrary, the problem is not solved even in the simple case $\mathcal{P}(d\alpha) = \frac{1}{2}(\delta_{-1} + \delta_1)$ treated in [5].*

It is not difficult to show *residual vanishing* in the case $\mathcal{P}(d\alpha) = \delta_p$ for $p \in I$ by a direct application of the result (Theorem 3) of [2]. Just to be safe, we show it here, assuming $p = 1$ for simplicity, i.e.,

$$-\Delta v_n = \lambda_n \left(\frac{e^{v_n}}{\int_\Omega e^{v_n}} - \frac{1}{|\Omega|} \right).$$

Fix $x_0 \in \mathcal{S}$. If it fails then it holds that

$$\liminf_{n\to\infty} \int_\Omega e^{v_n} < +\infty.$$

We introduce

$$z_n = v_n - \log \int_\Omega e^{v_n}$$

and obtain

$$-\Delta z_n = \lambda_n e^{z_n} - \frac{\lambda_n}{|\Omega|} \quad \text{in } \Omega.$$

It follows from the assumption of contradiction that $z_n \to +\infty$ (for some subsequence still denoted by the same notation). Since λ_n is uniformly bounded and $-\lambda_n/|\Omega|$ can be regarded as a simple perturbed term, we can safely apply the result of [2] to the equation of z_n to find that $z_n \to -\infty$ in $B(x_0, r_0) \setminus \{x_0\}$ for $0 < r_0 \ll 1$, where $B(x,r)$ denotes a disk centerd at x with radius r for $x \in \mathbf{R}^2$ and $r > 0$, in particular, B_r in the case $x = 0$. On the other hand, z_n is bounded below in $B(x_0, r_0) \setminus \{x_0\}$ since $\mathcal{S} = \mathcal{S}_+ \neq \emptyset$, a contradiction.

Still, it seems to be difficult to directly apply the result of [2] to the general positive case. To overcome this difficulty, we introduce the key transformation, see (2.3) below, and then develop a blowup analysis.

This report consists of three sections. We prove Proposition 1 in Section 2, and several lemmas stated there are shown in Section 3.

2 Proof of Proposition 1

In this section, we write I and $\mathcal{S}$ by I_+ and $\mathcal{S}_+$, respectively, in order to stress that we treat the positive case.

To prove the proposition, we have only to show

$$\mathcal{P}(\{\alpha \in I_+ \mid \liminf_{n\to\infty} \int_\Omega e^{\alpha v_n} = +\infty\}) = \mathcal{P}(I_+). \tag{2.1}$$

To confirm this, we fix $\omega \subset\subset \Omega \setminus \mathcal{S}_+$. Then, it holds that

$$\begin{aligned} 0 &\leq \int_\omega s_+ dx = \lim_{n\to\infty} \int_\omega \nu_{+,n} = \lim_{n\to} \lambda_n \int_\omega \int_{I_+} \left(\frac{\alpha e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} - \frac{1}{|\Omega|} \mathcal{P}(d\alpha) \right) \\ &\leq (\lambda_0 + 1) C(\omega) \lim_{n\to\infty} \int_{I_+} \frac{\mathcal{P}(d\alpha)}{\int_\Omega e^{\alpha v_n}} = 0 \end{aligned}$$

because $\lambda_n \to \lambda_0$ and v_n is uniformly bounded in ω. Hence, we obtain $s = 0$ in ω by $0 \leq s_{+,n} \in L^1(\Omega)$. Since $\omega \subset\subset \Omega \setminus \mathcal{S}_+$ is arbitrary, the proposition holds if (2.1) is true.

Now, we suppose that (2.1) is false. Then, there exists a number α_* such that

$$0 < \alpha_* := \sup\{\alpha \in I_+ \mid \liminf_{n\to\infty} \int_\Omega e^{\alpha v_n} < +\infty\} \quad \text{and} \quad \mathcal{P}((0, \alpha_*]) > 0. \tag{2.2}$$

Fix $x_0 \in \mathcal{S}_+$ and take $r_0 > 0$ satisfying $\overline{B(x_0, r_0)} \cap \mathcal{S}_+ = \{x_0\}$. It is possible to take such an r_0 because $\mathcal{S}$ is a finite set. We may assume $x_0 = 0$ by a translation. Then, there exist $x_n \in B_{r_0}$ and $\alpha_n \in \mathbf{R}$ such that

$$\begin{gathered} x_n \to 0 \quad v_n(x_n) = \max_{\overline{B_{3r_0}}} v_n \to +\infty, \\ e^{\alpha_n v_n(x_n)} = \int_{I_+} \frac{\alpha e^{\alpha v_n(x_n)}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha). \end{gathered} \tag{2.3}$$

For this α_n, we obtain the following lemmas shown in next section.

Lemma 1. *There exists $C_1 > 0$, independent of n, such that*

$$\int_{I_+} \frac{\alpha e^{(\alpha - \alpha_n) v_n(x)}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) \leq C_1$$

for all $x \in \overline{B_{2r_0}}$.

Lemma 2. *We have*

$$\alpha_n \to \alpha_0 \in [\alpha_*, 1],$$

passing to a subsequence.

Here, we develop a blow-up argument. Set

$$\begin{cases} w_n(x) = \alpha_n v_n(x_n) - L, \\ \tilde{w}_n(x) = w_n(\sigma_n x + x_n) + 2\log\sigma_n, \\ \sigma_n = e^{-w_n(x_n)/2} \ (\to 0 \text{ by Lemma 2}), \end{cases}$$

where $L \gg 1$ will be determined later on. The function $\tilde{w}_n = \tilde{w}_n(x)$ is a solution to

$$\begin{cases} -\Delta\tilde{w}_n = \alpha_n \tilde{V}_n(x)e^{\tilde{w}_n} - \sigma_n^2 \frac{\alpha_n\lambda_n}{|\Omega|}\int_{I_+}\alpha\mathcal{P}(d\alpha) \quad \text{in } B_{r_0/\sigma_n} \\ \tilde{w}_n \le \tilde{w}_n(0) = 0 \quad \text{in } B_{r_0/\sigma_n} \\ \int_{B_{r_0/\sigma_n}} \tilde{V}_n e^{\tilde{w}_n} \le m(0), \end{cases} \tag{2.4}$$

where

$$\tilde{V}_n(x) = e^L \cdot \lambda_n \int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(\sigma_n x + x_n)}}{\int_\Omega e^{\alpha v_n}}\mathcal{P}(d\alpha).$$

Lemma 3. *There exist* $\tilde{w} \in C^2(\mathbf{R}^2)$ *and* $0 < \tilde{V} \in C^2(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$ *such that*

$$\tilde{w}_n \to \tilde{w}, \quad \tilde{V}_n \to \tilde{V} \qquad \text{in } \mathbf{R}^2$$

and

$$\begin{cases} -\Delta\tilde{w} = \alpha_0\tilde{V}(x)e^{\tilde{w}} \quad \text{in } \mathbf{R}^2 \\ \tilde{w} \le \tilde{w}(0) = 0 \quad \text{in } \mathbf{R}^2 \\ \int_{\mathbf{R}^2}\tilde{V}e^{\tilde{w}} \le m(0). \end{cases} \tag{2.5}$$

Lemma 3 is also shown in next section.

For a solution $\tilde{w}$ to (2.5), we set

$$\tilde{\phi}(x) = \frac{\alpha_0}{2\pi}\int_{\mathbf{R}^2}\tilde{V}(y)e^{\tilde{w}(y)}\log\frac{|x-y|}{1+|y|}dy, \tag{2.6}$$

complying [3]. Noting that

$$\tilde{V}e^{\tilde{w}} \in L^1 \cap L^\infty(\mathbf{R}^2), \tag{2.7}$$

we find that the function $\tilde{\phi}$ set by (2.6) is well-defined in $\mathbf{R}^2$, and can show the following lemma because the proof of Lemma 1.1 of [3] is applicable to our case, see also Remark below.

Lemma 4. *There exists $C_2 > 0$, independent of L, such that*

$$\tilde{w}(x) \geq -\beta \log(1+|x|) - C_2 \tag{2.8}$$

for $x \in \mathbf{R}^2$, where

$$\beta = \frac{\alpha_0}{2\pi} \int_{\mathbf{R}^2} \tilde{V} e^{\tilde{w}}. \tag{2.9}$$

Remark 4. *In Lemma 1.1 of [3], the integrability condition $\int_{\mathbf{R}^2} e^{\tilde{w}} dx < +\infty$ is assumed to show the estimates from above and below for solutions and the estimate from below for β. However, it is not required if one only needs the estimate from below (2.8).*

Proof of Proposition 1: Fix $R \gg 1$. It follows from Lemmas 3-4 that

$$v_n(x) \geq v_n(x_n) - \frac{\beta}{\alpha_n} \log\left(1 + \left|\frac{x-x_n}{\sigma_n}\right|\right) - \frac{C_2}{\alpha_n} + \varepsilon_n$$

for all $x \in B(x_n, \sigma_n R)$, where ε_n is a quantity converging to 0 as $n \to \infty$. This ε_n may be changed in the following but keeps the property that $\varepsilon_n \to 0$. We obtain

$$\begin{aligned} \int_{B(x_n,\sigma_n)} e^{\alpha v_n} &\geq e^{\alpha v_n(x_n) - \alpha C_2/\alpha_n - 1} \int_{B(x_n,\sigma_n R)} \left(1 + \left|\frac{x-x_n}{\sigma_n}\right|\right)^{-\alpha\beta/\alpha_n} dx \\ &= e^{(\alpha-\alpha_n)v_n(x_n)} \cdot e^{L - \alpha C_2/\alpha_n - 1} \int_{B_R} (1+|x|)^{-\alpha\beta/\alpha_n} dx \end{aligned} \tag{2.10}$$

for all $\alpha \in I_+$. Thus, (2.3) and (2.10) yield

$$\begin{aligned} 1 &= \int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(x_n)}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) \\ &\leq \varepsilon_n + \int_{[\alpha_n,1]} \frac{\int_{B(x_n,\sigma_n)} e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \cdot \frac{\alpha}{e^{L-\alpha C_2/\alpha_n - 1} \int_{B_R} (1+|x|)^{-\alpha\beta/\alpha_n} dx} \mathcal{P}(d\alpha) \\ &\leq \varepsilon_n + \frac{1}{e^{L - C_2/\alpha_n - 1} \int_{B_R} (1+|x|)^{-\beta/\alpha_n} dx}. \end{aligned} \tag{2.11}$$

Since $\beta/\alpha_n \leq (\alpha_0/\alpha_n) \cdot (m(0)/2\pi)$ by (2.9) and Lemma 3, inequality (2.11) implies

$$1 \leq \varepsilon_n + \frac{1}{e^{L-C_2/\alpha_n - 1} \int_{B_R} (1+|x|)^{-\frac{\alpha_0}{\alpha_n} \cdot \frac{m(0)}{2\pi}} dx},$$

or

$$1 \leq \frac{e^{1+C_2/\alpha_0} - L}{\int_{B_R} (1+|x|)^{-\frac{m(0)}{2\pi}} dx},$$

which is a contradiction if L is sufficiently large. The proof is complete. $\square$

3 Proof of Lemmas 1-3

As having announced in the previous sections, we show Lemmas 1-3 in this section. We again consider the positive case (i.e., $\mathcal{S} = \mathcal{S}_+$ and $I = I_+$) in what follows.

Proof of Lemma 1: Since $\mathcal{S} = \mathcal{S}_+$, there exists $C_3 > 0$, independent of n, such that $v_n > -C_3$ in Ω. We use (2.3) and Jensen's inequality to calculate

$$\begin{aligned}
&\int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(x)}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) \\
&\leq \int_{I'_{+,n}} \frac{\alpha e^{-(\alpha_n-\alpha)v_n(x)}}{\int_\Omega e^{\alpha v_n}} + \int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(x_n)}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) \\
&\leq \frac{\alpha_n \mathcal{P}(I'_{+,n}) e^{\alpha_n C_3}}{|\Omega|} + 1 \leq \frac{e^{C_3}}{|\Omega|} + 1
\end{aligned}$$

for all $x \in \overline{B_{2r_0}}$ and n, where

$$I'_{+,n} = \begin{cases} (0, \alpha_n) & \text{if } \alpha_n > 0 \\ \emptyset & \text{if } \alpha_n \leq 0. \end{cases}$$

The lemma is completely shown. □

Proof of Lemma 2: Put $\alpha_0 = \lim_{n\to} \alpha_n$.
Assume that $\alpha_0 > 1$. Then, there exists $\delta > 0$ such that

$$e^{(1+\delta)v_n(x_n)} \leq e^{\alpha_n v_n(x_n)},$$

that is, by Jensen's inequality,

$$e^{\frac{\delta}{2}v_n(x_n)} \leq \int_{I_+} \frac{\alpha e^{(\alpha-1-\delta/2)v_n(x_n)}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) \leq e^{-\frac{\delta}{2}v_n(x_n)} |\Omega|^{-1}$$

for $n \gg 1$, which is a contradiction because $v_n(x_n) \to +\infty$.

Next, assume that $\alpha_0 \leq 0$. In the case that $\mathcal{P}((0, \alpha_*)) > 0$, there exists $0 < \varepsilon \ll 1$ such that $\mathcal{P}([\varepsilon, \alpha_* - \varepsilon]) > 0$, and therefore

$$\begin{aligned}
1 &= \int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(x_n)}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) \\
&\geq \int_{[\varepsilon, \alpha_*-\varepsilon]} \frac{\alpha e^{(\alpha-\varepsilon/2)v_n(x_n)}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) \\
&\geq c(\varepsilon) e^{\frac{\varepsilon}{2}v_n(x_n)} \mathcal{P}([\varepsilon, \alpha_* - \varepsilon]) \to +\infty
\end{aligned}$$

as $n \to \infty$, a contradiction. In the case that $\mathcal{P}(\{\alpha_*\}) = \mathcal{P}((0,\alpha_*]) > 0$, it holds that $\liminf_{n\to} \int_\Omega e^{\alpha_* v_n} < +\infty$, and hence

$$\begin{aligned} 1 &= \int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(x_n)}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) \\ &\geq \alpha_* e^{(\alpha_*-\alpha_n)v_n(x_n)} \left(\int_\Omega e^{\alpha_* v_n} \right)^{-1} \mathcal{P}(\{\alpha_*\}) \to +\infty \end{aligned}$$

as $n \to \infty$, a contradiction.

We have shown that $\alpha_0 \in (0,1]$. It is left to show that $\alpha_0 \geq \alpha_*$. To prove this, we finally assume that $\alpha_0 \in (0,\alpha_*)$. Consider

$$\varphi_n = \alpha_n v_n - \log \int_\Omega e^{\alpha_n v_n}.$$

Passing to a subsequence, we have

$$\varphi_n(x_n) \to +\infty. \tag{3.1}$$

The function $\varphi_n = \varphi_n(x)$ satisfies

$$\begin{cases} -\Delta\varphi_n = K_n(x)e^{\varphi_n} - \frac{\alpha_n\lambda_n}{|\Omega|} \int_{I_+} \alpha\mathcal{P}(d\alpha) & \text{in } B_{2r_0} \\ \int_\Omega e^{\varphi_n} = 1, \end{cases} \tag{3.2}$$

where

$$K_n(x) = \alpha_n\lambda_n \left(\int_\Omega e^{\alpha_n v_n} \right) \int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(x)}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha).$$

Lemma 1 and the boundedness $\liminf_{n\to\infty} \int_\Omega e^{\alpha_n v_n} < +\infty$ show that there exists $C_4 > 0$, independent of n, such that

$$0 \leq K_n \leq C_4 \quad \text{in } B_{2r_0}. \tag{3.3}$$

Consequently, (3.1)-(3.3) assure that

$$\varphi_n \to -\infty \quad \text{locally uniformly in } B_{2r_0}\backslash\{0\} \tag{3.4}$$

by virtue of the result of [2]. However, (3.4) is false since $\mathcal{S} = \mathcal{S}_+$ and $\liminf_{n\to\infty} \int_\Omega e^{\alpha_n v_n} < +\infty$. □

Proof of Lemma 3: It follows from Lemma 2 that

$$0 \leq \tilde{V}_n \leq e^{L(\lambda_0+1)}C_1 \quad \text{in } B_{r_0/\sigma_n}$$

for $n \gg 1$. We also have

$$0 \leq e^{\tilde{w}_n} \leq 1 \quad \text{in } B_{r_0/\sigma_n}$$

for all n, and

$$\sigma_n^2 \frac{\alpha_n \lambda_n}{|\Omega|} \int_{I_+} \alpha \mathcal{P}(d\alpha) \to 0$$

as $n \to \infty$. Combining these properties with $\tilde{w}_n(0) = 0$, we can safely apply the result of [2] to find that, for every $R > 0$, there exists $C_5(R) > 0$ such that

$$\tilde{w}_n \geq -C_5(R) \quad \text{in } B_R \tag{3.5}$$

for $n \gg 1$. Thus, the elliptic regularity and a diagonal arugument show that there exists $\tilde{w} \in C^{1+\alpha}(\mathbf{R}^2)$, $\alpha \in (0,1)$, such that

$$\tilde{w}_n \to \tilde{w} \quad \text{in } C^{1+\alpha}_{loc}(\mathbf{R}^2). \tag{3.6}$$

Noting the definitions of $\tilde{V}_n$ and $\tilde{w}_n$, we see that there exists $\tilde{V} \in C^{1+\alpha}(\mathbf{R}^2)$, $\alpha \in (0,1)$, such that

$$\tilde{V}_n \to \tilde{V} \quad \text{in } C^{1+\alpha}_{loc}(\mathbf{R}^2). \tag{3.7}$$

We again use the elliptic regularity, together with (3.6)-(3.7), and conclude the relation (2.5) and $\tilde{w}, \tilde{V} \in C^2(\mathbf{R}^2)$.

It is clear that $\tilde{V} \in L^\infty(\mathbf{R}^2)$ by Lemma 1, and therefore, we must show that $\int_{\mathbf{R}^2} \tilde{V} e^{\tilde{w}} \leq m(0)$ and that $\tilde{V} > 0$ in $\mathbf{R}^2$.

For every $R > 0$ and $0 < r \ll 1$,

$$\begin{aligned}
\int_{B_R} \tilde{V} e^{\tilde{w}} &\leq \liminf_{n\to\infty} \int_{B_R} \tilde{V}_n e^{\tilde{w}_n} \leq \liminf_{n\to\infty} \int_{B_{r/\sigma_n}} \tilde{V}_n e^{\tilde{w}_n} \\
&= \liminf_{n\to\infty} \int_{B(x_n,r)} \nu_{+,n} \leq m(0) + \int_{B_{2r}} \nu_+
\end{aligned}$$

by the Fatou lemma, the definitions of w_n, $\tilde{w}_n$, σ_n and $\tilde{V}_n$, and (1.2). Letting $R \uparrow +\infty$ and $r \downarrow 0$, we obtain $\int_{\mathbf{R}^2} \tilde{V} e^{\tilde{w}} \leq m(0)$.

Finally, we use the definitions of w_n, $\tilde{w}_n$, σ_n and $\tilde{V}_n$, (3.5), $\tilde{w}_n \leq 0$ and (1.2) to obtain $C_6(R) > 0$, independent of $n \gg 1$, such that

$$\begin{aligned}
\tilde{V}_n(x) &= e^L \lambda_n \int_{I_+} \frac{\alpha e^{\frac{\alpha-\alpha_n}{\alpha_n}(\tilde{w}_n(x)+\alpha_n v_n(x_n))}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) \\
&\geq e^{L-C_6(R)} \lambda_n \int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(x_n)}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) = e^{L-C_6(R)} \lambda_n
\end{aligned}$$

for all $x \in B_R$ and $n \gg 1$, and for every $R > 0$, which means $\tilde{V} > 0$ in $\mathbf{R}^2$ because $\lambda_n \to \lambda_0 > 0$ by $\mathcal{S} = \mathcal{S}_+ \neq \emptyset$. □

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