Kinematic variational principle for vortical structure of Euler flows and beyond

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Abstract

Coherent vortices, being durable for some time, are often observed nature. These vortices may be modeled as steady solutions of the Euler equations for an inviscid incompressible fluid. A steady incompressible Euler flow is characterized as an extremal of the total kinetic energy with respect to perturbations constrained to an isovortical sheet (coadjoint orbits). Kelvin (1878) argued that an elongated vortex of circular cross-section is realized as a maximum energy state, and recently his conjecture has been numerically and mathematically confirmed. This guarantees its stability if the disturbances are restricted two-dimensional isovortical ones. An isovortical perturbation preserves vortex-line topology and is expressible most efficiently by the Lagrangian variables. We show how topological ideas work in the variational formulation for characterizing a steady solution of the Euler equation.

This is generalized to a steadily moving vortical flow. According to Kelvin-Benjamin's principle, a steady distribution of vorticity, relative to a moving frame, is realized as the state that maximizes the total kinetic energy, under the constraint of constant hydrodynamic impulse, with respect to variations preserving the vorticity-field topology. Combined with an asymptotic solution of the Euler equations for a family of vortex rings, we can skip the detailed solution for the flow field to obtain the translation velocity of a vortex ring valid to third order in a small parameter, the ratio of the core radius to the ring radius.

1   Introduction

Coherent structures formed in natural and practical flows are commonly characterized by dominance of vorticity in the cores over in the surrounding region. A century and a half ago, the field of vortex dynamics started with a single piece of paper written by Helmholtz. In his seminal paper [1], Helmholtz proved a distinguishing feature of the vorticity that vortex lines are frozen into the fluid. In the same paper, he studied motion of vortex rings. By an elaboration from the Euler equations, now being widely known through Lamb's textbook [2], Helmholtz had reached an identity for traveling speed of a thin axisymmetric vortex ring, steadily translating in an inviscid incompressible fluid of infinite extent. Helmholtz-Lamb's method is recapitulated in a recent article [3]. On those days, vortex rings were hot as possible entities of atoms embedded in the ether. The implication of Helmholtz' laws, invariance in time of the circulation and linkages of vortex lines, led Kelvin to this belief. The idea of the vortex atoms was pursued by J. J. Thomson [4].

By a deep insight into the formation of a columnar vortex along the central line of a rotating tank filled with water, Kelvin [5] envisioned that a columnar vortex should be a state of the maximum of the kinetic energy, with respect to perturbations that maintain the circulation. An almost century passed before Kelvin's variational principle was mathematically formulated and proved. Arnol'd [6] proved that a steady solution of the Euler equations of an incompressible fluid is an extremal of the total kinetic energy with respect to the kinematically accessible perturbations. We mean by *kinematically accessible perturbations* the perturbation flow field for which the perturbed vorticity is frozen into the perturbed flow field. The kinematically accessible perturbations may be alternatively said to be the *isovortical perturbations* or occasionally the *rearrangements*. For planar flows, numerical method for relaxing the flow field to the state of the maximum energy was developed by Vallis *et al.* [7, 8], and Kelvin's envision that an isolated vortex is a maximum-energy state was numerically demonstrated. Mathematical proof for steady isolated vortex as the maximum-energy states has been given, by inventing rearrangement inequalities, in various setting (see, for example, [9, 10, 11]).

The main theme of this conference is a statistical approach to collection of vortices, being pioneered by Onsager [12], by which characteristics of coherent structures are well described. Energetics of an isolated vortex may provide us with a complimentary view to formation of coherent structures, though the relation of the energetics with statistical behavior is yet to be clarified. This article gives a rough sketch of how to use the energetics to find the steady vortices and travelling speed of a steadily moving vortex ring.

In §2, we give an outline of the relaxation method to detect a steady flow corresponding to the state of energy maximum. Kelvin's variational principle can be extended to make allowance for motion by adding a constraint of constant impulse [13, 14, 15]; a stationary configuration of vorticity in an inviscid incompressible fluid, in a steadily moving frame, is realizable as an extremal of energy on an isovortical sheet under the constraint of constant impulse. The rest of article is concerned with the variational principle for motion of vortex rings. Kelvin-Benjamin's variational principle is adapted to find the traveling speed of steady vortices [15, 16, 17, 18]. This variational principle is applied to motion of vortex rings.

Take the density of fluid to be $\rho_f = 1$ and define the hydrodynamic impulse by

$$P = \frac{1}{2} \iiint x \times \omega dV. \tag{1}$$

The translation velocity U of a vortex ring is then calculable through the variation

$$\delta H - U \cdot \delta P = 0, \tag{2}$$

under the constraint that, for any smooth Lagrangian displacement of fluid particles, the vorticity is frozen into the fluid. Section 4 touches upon this principle, which is the theme of ref. [15]. Intriguingly, the same principle encompasses motion of a vortex ring ruled by the cubic nonlinear Schrödinger equation, which serves as a model for superfluid liquid helium and a Bose-Einstein condensate, at zero temperature [19].

The relaxation scheme toward the maximum-energy state by monotonically increasing energy relies upon the existence of an upper bound of the kinetic energy, given an initial configuration. This upper bound is provided by the Casimir invariants as generalizing the enstrophy

[7, 8, 14]. For a barotropic fluid, all the topological invariants, like the Casimir invariants and the helicity, are variants of the cross helicity [20]. Section 3 gives a brief account of this unified view of the topological invariants. After stating Kelvin-Benjamin's variational principle in §4, we show in §5 how the kinematic variational principle is used to calculate the travelling speed of a vortex ring. The variational principle enables us to derive a higher-order correction, in the ratio of core and ring-radii, a small parameter, to the travelling speed of an inviscid vortex ring. At high Reynolds numbers, the viscosity plays only a secondary role of selecting the vorticity profile in the core. With incorporating this profile, the inviscid formula is applicable to vortex rings in the regime of high Reynolds numbers [15]. Our formula significantly improves Saffman's first-order formula [21, 22], and fits well with the result of numerical simulation at a moderate Reynolds number.

The last section (§6) discusses Onsager's statistical theory of point vortices [12], along with its difference from and similarity with Kelvin's variational principle.

2 Steady vortical flows of the Euler equations

A steady solution of the Euler equations is known to be a state of energy maximum with respect to kinematically accessible perturbations or a state of an energy extremizing rearrangement for which the Casimir invariants are fixed [6]. Here, we give a proof of this theorem in three dimensions to gain an insight into the variational structure.

Under the assumption that the fluid is incompressible, we can introduce the vector potential A for the velocity field u ($u = \nabla \times A$). We assume that the vorticity $\omega = \nabla \times u$ is localized in some finite region in such a way that the velocity decreases sufficiently rapidly. These assumptions admit a representation of the total kinetic energy H of the fluid, filling an unbounded space, as

$$H = \frac{1}{2} \iiint u^2 \mathrm{d}V = \frac{1}{2} \iiint \omega \cdot A \mathrm{d}V, \tag{3}$$

where the density of fluid is set to be unity. We confine ourselves to steady motion $\bar{u}$, which obeys

$$\nabla \times (\bar{u} \times \omega) = 0. \tag{4}$$

Consequently, there exists a globally defined spatial function $h(x)$ such that

$$\bar{u} \times \omega = \nabla h. \tag{5}$$

Suppose that fluid particles undergo an infinitesimal displacement $\delta\xi$ while preserving the volume of an arbitrary fluid element:

$$x \to \tilde{x} = x + \delta\xi(x); \quad \nabla \cdot \delta\xi = 0. \tag{6}$$

We impose the condition that the flux of vorticity through an arbitrary material surface be unchanged throughout the process of the displacement. Its local representation is [7, 14]

$$\delta\omega = \nabla \times (\delta\xi \times \omega). \tag{7}$$

Using the definition $\bar{u} = \nabla \times \bar{A}$ of the vector potential, we can deduce, from (5) and (7),

$$\bar{A} \cdot \delta\omega = -\nabla \cdot \left\{ h\delta\xi + \bar{A} \times (\delta\xi \times \omega) \right\}. \tag{8}$$

The variation δH of the kinetic energy, subjected to the variation of fluid-particle positions (6), is calculated as

$$\delta H = \iiint A \cdot \delta\omega \, dV = -\iint \left\{ h\delta\xi + \bar{A} \times (\delta\xi \times \omega) \right\} \cdot n \, dA. \tag{9}$$

The surface integral is taken over the closed surface receding to infinity that bounds the whole region. It vanishes under the assumption that the vorticity $|\omega|$ decays sufficiently rapidly with distance $|x|$, say exponentially in $|x|$. Under the same assumption, h approaches a constant h_∞ at large distances $|x|$, and the first term of the surface integral vanishes, with the aid of the Gauss theorem, owing to (6). Consequently, the proof of $\delta H = 0$ has been completed.

Relying on this property, a numerical algorithm for seeking an energy extremizing rearrangement was developed [7, 8]. This numerical algorithm belongs to the category of the simulated annealing (SA). A state of energy minimum is also a steady Euler flow. As a matter of fact, this is a state of zero kinetic energy and, in the numerical construction, takes the form of a region of high and low vorticity interpenetrating each other with ever creating finer structures and is therefore named the Kelvin sponge [8].

Given an initial distribution $\omega_0(x)$ of vorticity, the vorticity kinematically accessible to $\omega_0(x)$ is generated by

$$\frac{\partial\omega}{\partial t} = \nabla \times (u \times \omega), \tag{10}$$

with convection velocity u taken to be arbitrary smooth vector field. Introduce the vector potential $v(x)$ for $\omega(x)$ defined by $\omega(x) = \nabla \times v$. The advection equation that monotonically changes the energy while keeping the Casimir invariants is constructed by choosing the advection velocity in (10) u, for example, as

$$u = v + \alpha\frac{\partial v}{\partial t}, \tag{11}$$

for some constant α. The kinetic energy E of fluid contained in the domain $\mathscr{D}$ indeed monotonically changes

$$\frac{d}{dt}\frac{1}{2}\int_\mathscr{D} v^2 dV = -\alpha\int_\mathscr{D} \left(\frac{\partial v}{\partial t}\right)^2 dV. \tag{12}$$

A choice of $\alpha < 0$ realizes the monotonic increase of the energy. The existence of a state of energy maximum is guaranteed if the energy id bounded above. For planar flows in a two-dimensional domain $\mathscr{A}$, an upper bound is brought from the enstrophy

$$\Omega = \int_\mathscr{A} \omega^2 dA, \tag{13}$$

a Casimir invariant, combined with the Poincaré inequality

$$E \leq kA\Omega = \text{const.} \tag{14}$$

where A is the area of $\mathscr{A}$ and $k(>)$ is a constant depending on the domain shape. With judicious choice of u in relation to v, Vallis *et al.* convincingly demonstrated that an isolated steady vortex corresponds to the state of the energy maximum, kinematically accessible from the initial vorticity distribution ω_0.

Recently, Flierl and Morrison [24] proposed a new numerical algorithm, the 'Dirac simulated annealing', for finding states of energy maxima or minima, of the two-dimensional Euler flow in an infinite space, while preserving not only the Casimir invariants but also desired integrals. The latter are called the Dirac constraints. Even number of constraint functionals are chosen in such a way that the antisymmetric matrix whose ij-th entry consists of the Poisson bracket of the i-th and the j-th constraints be nonsingular. With use of the Dirac constraints, a generalized Dirac bracket is defined by extending the original noncanonical Poisson bracket. The Dirac bracket makes not only the original Casimir invariants but also the newly introduced constraints (=the Dirac constraints) Casimir invariants. With the angular impulse and the strain moment chosen as the Dirac constraints, for instance, the constancy of these two moments maintains the compactness of the core during the evolution under the Dirac simulated annealing (DSA-) dynamics for some time. For later times, tines develop from the points on the core boundary corresponding to the major axis which break the numerical computation, and thereafter the system begins to axisymmetrize.

Long-living vortices are commonly observed in nature and also in planetary flows as represented by the Jupiter red spot. A view that this is a statistically (quasi-) equilibrium state is promising and is well-received as discussed in several other papers in this volume. As a possible alternative view, a long-living vortex may be simply modeled by steady solutions of the Euler equations for an inviscid fluid. Nycander [9] and Emamizadeh [10] considered an isolated vortex embedded in a simple shear flow, with the vorticity of the embedded vortex and the background shear being both positive. Previously, they proved the existence of a stable vortex, as the energy maximizer. The maximizer of the kinetic energy is attained in the class of the doubly Steiner symmetric rearrangements (DSS); given a measurable and non-negative function f, the DSS of f is symmetric with respect to the both axes $x = 0$ and $y = 0$, a decreasing function of x for $x > 0$ and fixed y, and a decreasing function of y for $y > 0$ and fixed x.

Are isolated vortices in a planar flow all the maximum states of the kinetic energy on the corresponding isovortical sheets? This is not necessarily true. Saffman and Szeto [25] calculated numerically a family of steadily co-rotating vortex pairs and found two branches of solutions in the plane of the angular impulse J and the excess energy E. The upper branch is assumed to be the maximum state of the excess energy and thus is likely to give stable solutions. In this keeping, the simultaneous turning point in E and J is identified with the point for a change of stability. Dristchel [26] pointed out that this is not the case; the loss of stability occurs at a point along the upper branch, and moreover, a new solution branch bifurcates from this point.

Luzzatto-Fegiz and Williamson [27] put their basis on the variational principles established by Arnol'd [6] and generalized by Benjamin [13]. For a steady flow in a steadily rotating frame, the constancy of the angular impulse should be imposed as an additional constraint. For a steady flow in a steadily translating frame, the constancy of the linear impulse should be imposed as an additional constraint (see §3). They showed for planar motion of vortex patches, vortices of uniform vorticity that this principle is faithfully effected in the velocity-impulse plane, rather

than the energy-impulse plane. The gain or loss of the stability of a solution branch has link with a turning point of the impulse in the velocity-impulse plane. Moreover, they resolved the issue of overlooking bifurcation branches in the previous treatment, by introducing the 'IVI diagram'. By placing weak point vortices (=imperfections) at stagnation points, bifurcated branches associated with a solutions of lower symmetry are captured. This method not only detects supposedly all the branches, but also automatically determines stability or instability.

Is an isolated vortex corresponding to the energy maximizer stable? This may be, if limited to planar flows, in many cases true as the solution is realized as an isolated point on a given isovortical sheet. But realistic flows are three dimensional, for which the answer should be generically no. If account is taken of three-dimensional waves on an isolated vortices, in most cases, there are an infinite number of negative-energy waves as well as an infinite number of positive-energy waves [28]. In a three-dimensional setting, stable configurations are exceptions, and instability of a vortex is unavoidable.

3 Unified view of topological invariants

The helicity is a topological invariant of an ideal fluid in three dimensions [29]. Two-dimensional ideal flows admit an integral of any function of vorticity as topological invariants called the Casimir invariant. This is extended to axisymmetric flows [14]. In this section, we briefly remark that all the topological invariants are variants of the cross-helicity. This unified view is gained from the fact that Noether's theorem associated with the particle relabeling symmetry does not discriminate between two and three dimensions [20].

We start from the vorticity equations (10) for a barotropic fluid filling a domain $\mathscr{D}$. Since we are concerned with the kinematics of ideal barotropic flows, the advection velocity u may be an arbitrary smooth vector field so that the vorticity ω may be unrelated to $\nabla \times u$. We take compressibility into account, and the fluid density ρ_f obeys the equation of continuity $D\rho_f/Dt + \rho_f \nabla \cdot u = 0$. Here $D/Dt = \partial/\partial t + u \cdot \nabla$ is the Lagrangian derivative. The law of mass conservation holds true without reference to the detailed form of velocity field u, and therefore pertains to the kinematics.

Suppose that $\mathscr{D}$ is simply connected. Impose the following boundary condition on ω:

$$\omega \cdot n = 0 \quad \text{on } \partial\mathscr{D}, \tag{15}$$

or in case the domain $\mathscr{D}$ is unbounded,

$$|\omega| \to 0 \quad \text{sufficiently rapidly as } |x| \to \infty. \tag{16}$$

Then for a given solenoidal vector field $\omega(x,t)$, there exists a vector potential $v(x,t)$ defined, over $\mathscr{D}$, by $\omega = \nabla \times v$. The vector potential is determined only up to the gauge transformation. The evolution equation of v, obtained by taking the uncurl of (10), is named the Euler-Poincaré equations [30], and, when specialized as $v = u$, is made coincident with the Euler equations.

Let us introduce another solenoidal vector field $B(x,t)$ which is frozen into the fluid. The equation of B takes the same form as (10), and the boundary condition to be imposed is the

same as (15) or (16). The cross helicity

$$\mathcal{H}[\omega, B] = \int_{\mathcal{D}} v \cdot B \, \mathrm{d}V \tag{17}$$

is invariant even if the advection velocity field u is different from v [20]. The helicity is a special case of (17) of taking $B = \omega$ and $u = v$.

For two-dimensional flows on the xy-plane with velocity provided by $u(x,t) = (u_x(x,y,t), u_y(x,y,t), 0)$, there is a family of integral invariants for planar flows in a domain $\mathcal{A}$, namely integrals of arbitrary function of $\omega = \partial u_y/\partial x - \partial u_x/\partial y$. For a compressible barotropic fluid, it is superseded by

$$Q = \int_{\mathcal{A}} \omega f\left(\frac{\omega}{\rho_f}\right) \mathrm{d}A, \tag{18}$$

where f is an arbitrary function. This integral is termed the generalized enstrophy [7]. Invariance of (18) is a direct consequence of the restriction of (10) to two dimensions,

$$\frac{D}{Dt} f\left(\frac{\omega}{\rho_f}\right) = 0, \tag{19}$$

and the conservation law of the vorticity flux or Kelvin's circulation theorem. Introducing $F = \nabla \times f e_z$, (19) is converted into

$$\frac{\partial F}{\partial t} = \nabla \times (u \times F). \tag{20}$$

A topological invariant is manufactured by replacing B by F in (17) with the volume integral of unit length in z over the domain $\mathcal{A}$. This integral is reduced, after a partial integration, to (18), except for a boundary term. The latter vanishes, in a typical case that $f(\omega/\rho)$ approaches zero sufficiently rapidly as the boundary $\partial\mathcal{A}$ recedes to infinity,

4 Kelvin-Benjamin's variational principle

By adding constraint of constant impulse (the linear momenta) or angular impulse (the angular momentum), a stationary vortical flow in a moving frame is realizable as an extremal of the kinetic energy.

The variational principle described in §2 is augmented by the term associated with the constraint of constant impulse. We confine ourselves to steady motion, with constant speed U, of a region with vorticity and assume that the flow is stationary in a frame moving with U. It is expedient to partition the velocity u as $u = \bar{u} + U$. Correspondingly, the vector potential $A(x)$ is augmented as $A = \bar{A} - x \times U/2$. The variation (9) of the kinetic energy, subjected to the variation of fluid-particle positions (6), is augmented as

$$\delta H = \iiint A \cdot \delta\omega \, \mathrm{d}V = U \cdot \left(\frac{1}{2}\iiint x \times \delta\omega \, \mathrm{d}V\right)$$
$$- \iint \left\{ h\delta\xi + \bar{A} \times (\delta\xi \times \omega) \right\} \cdot n \, \mathrm{d}A. \tag{21}$$

The surface integrals vanish as before, and we are left only with the volume integral in (9). The variation of the hydrodynamic impulse (1) is

$$\delta P = \frac{1}{2} \iiint x \times \delta \omega \, dV. \tag{22}$$

With this form, (21) is reckoned upon as the variational principle (2) for the translation speed U of the vortex region.

In the sequel, we restrict this theorem to motion of a steadily moving axisymmetric vortex ring. An iso-vortical sheet is of infinite dimension. A family of solutions of the Euler equations includes a few parameters. By imposing certain relations among these parameters, we can maintain the solutions on a single iso-vortical sheet, and the restricted family of the solutions constitutes a finite dimensional set on the sheet. Thus the traveling speed of a vortex ring may be calculable through (2). Fraenkel-Saffman's formula is obtained in this framework [15], though excluded from the list of [17]. This principle is extensible to higher orders, whereby the $O(\varepsilon^3)$ corrections are produced [15]. This is a topic of the following section.

5   High-Reynolds-number vortex ring

The inner solution for steady motion of a vortex ring, or quasi-steady motion in the presence of viscosity, is found by solving the Euler or the Navier-Stokes equations, subject to the matching condition, in powers of the small parameter ε, the ratio of the core- to the ring-radii [32]. To work out the inner solution, we introduce the relative velocity $\tilde{u}$ in the meridional plane by $u = \tilde{u} + (\dot{R}, \dot{Z})$. Here a dot stands for differentiation with respect to time. Let us nondimensionalize the inner variables. We introduce, in the core cross-section, local polar coordinates (r, θ) around the core center. The radial coordinate is normalized by the core radius $\varepsilon R_0 (= \sigma)$ and the local velocity (u, v), relative to the moving frame, by the maximum velocity $\Gamma/(\varepsilon R_0)$. The normalization parameter for the ring speed $(\dot{R}(t), \dot{Z}(t))$, the slow dynamics, should be Γ/R_0. The suitable dimensionless inner variables are thus defined as

$$r^* = r/\varepsilon R_0, \quad t^* = t \Big/ \frac{R_0}{\Gamma}, \quad \psi^* = \frac{\psi}{\Gamma R_0}, \quad \zeta^* = \zeta \Big/ \frac{\Gamma}{R_0^2 \varepsilon^2}, \quad \tilde{u}^* = \tilde{u} \Big/ \frac{\Gamma}{R_0 \varepsilon}, \quad (\dot{R}^*, \dot{Z}^*) = (\dot{R}, \dot{Z}) \Big/ \frac{\Gamma}{R_0}. \tag{23}$$

The difference in normalization between the last two of (23) should be kept in mind. Correspondingly to (23), the kinetic energy H and the hydrodynamic impulse P are normalized as $H^* = H/\Gamma^2 R_0$, $P_z^* = P_z/\Gamma R_0^2$. Hereinafter we drop the superscript $*$ for dimensionless variables. Dimensionless form of the radial position R of the core center is $R = 1 + \varepsilon^2 R^{(2)} + O(\varepsilon^3)$. We can maintain the first term to be unity by adjusting disposable parameters, bearing with the origin of coordinates, in the first-order field [32]. The second-order correction $\varepsilon^2 R^{(2)}$ is tied with the viscous expansion.

A glance at the Euler or the Navier-Stokes equations shows that the dependence, on θ, of the solution in a power series in ε is

$$\psi = \psi^{(0)}(r) + \varepsilon \psi_{11}^{(1)}(r) \cos \theta + \varepsilon^2 \left[\psi_0^{(2)}(r) + \psi_{21}^{(2)}(r) \cos 2\theta \right] + O(\varepsilon^3), \tag{24}$$

$$\zeta = \zeta^{(0)}(r) + \varepsilon \zeta_{11}^{(1)}(r) \cos \theta + \varepsilon^2 \left[\zeta_0^{(2)}(r) + \zeta_{21}^{(2)}(r) \cos 2\theta \right] + O(\varepsilon^3). \tag{25}$$

Upon substitution from (24) and (25), we obtain a representation, to $O(\varepsilon^2)$ in dimensionless form, $H = H^{(0)} + \varepsilon^2 H^{(2)}$ and $P_z = P^{(0)} + \varepsilon^2 P^{(2)}$ of the kinetic energy and the z component of the hydrodynamic impulse (1), as

$$H^{(0)} = -2\pi^2 \int_0^\infty r\zeta^{(0)}\psi^{(0)}\mathrm{d}r, \quad H^{(2)} = -2\pi^2 \int_0^\infty r\left(\frac{1}{2}\zeta_{11}^{(1)}\psi_{11}^{(1)} + \zeta^{(0)}\psi_0^{(2)} + \zeta_0^{(2)}\psi^{(0)}\right)\mathrm{d}r, \quad (26)$$

$$P^{(0)} = \pi, \quad P^{(2)} = \pi\left(2R^{(2)} - 4\pi d^{(1)}\right), \tag{27}$$

where $d^{(1)} = d_1/(\Gamma\sigma^2)$ is the dimensionless strength of dipole.

Evaluation of (26) and (27) is relatively easy as these do not include the quadrupole field $\psi_{21}^{(2)}$ and $\zeta_{21}^{(2)}$. Given $\zeta^{(0)}$ to $O(\varepsilon^0)$, the azimuthal velocity to $O(\varepsilon^0)$ satisfies $v^{(0)} = -\partial\psi^{(0)}/\partial r$, and the Stokes streamfunction complying with the matching condition is, to $O(\varepsilon^0)$,

$$\psi^{(0)} = -\int_0^r v^{(0)}(r')\mathrm{d}r' + \lim_{r\to\infty}\left\{\int_0^r v^{(0)}(r')\mathrm{d}r' - \frac{1}{2\pi}\left[\log\left(\frac{8}{\varepsilon r}\right) - 2\right]\right\}. \tag{28}$$

Without viscosity, the vorticity profile $\zeta^{(0)}$ may be taken to be arbitrary, but viscosity plays the role of selecting its functional form [33]. It is expedient to handle the streamfunction $\tilde{\psi}$ for the flow relative to the coordinates moving with the same speed $\dot{Z}$ as the vortex ring along the z-direction, namely, $\psi = -\dot{Z}\rho^2/2 + \tilde{\psi}$. The first-order solution comprises a dipole field. Denoting the dipole coefficient of the streamfunction for the flow, relative to the moving frame, to be $\tilde{\psi}_{11}^{(1)} = \psi_{11}^{(1)} + r\dot{Z}^{(0)}$, the coefficient function $\tilde{\psi}_{11}^{(1)}$ is given by

$$\tilde{\psi}_{11}^{(1)} = -v^{(0)}\left\{\frac{r^2}{2} + \int_0^r \frac{\mathrm{d}r'}{r'[v^{(0)}(r')]^2}\int_0^{r'} r''\left[v^{(0)}(r'')\right]^2\mathrm{d}r''\right\} + c_{11}^{(1)}v^{(0)}, \tag{29}$$

where $c_{11}^{(1)}$ is a disposable parameter tied with choice of the origin $r = 0$ of the local coordinates. The vorticity is found from $\zeta_{11}^{(1)} = a\tilde{\psi}_{11}^{(1)} + r\zeta^{(0)}$ with $a(r,t) = -1/v^{(0)}(\partial\zeta^{(0)}/\partial r)$. The Fourier coefficient $\tilde{\psi}_0^{(2)}(r)$ of the monopole component of $O(\varepsilon^2)$, relative to the moving coordinate frame, defined by $\tilde{\psi}_0^{(2)} = \psi_0^{(2)} + \dot{Z}^{(0)}r^2/4$ is written in terms of $v^{(0)}$, $\tilde{\psi}_{11}^{(1)}$ and $\zeta_0^{(2)}$. The $O(\varepsilon^2)$ monopole component $\zeta_0^{(2)}$ of vorticity obeys a heat-conduction equation with source terms [32].

The leading-order term $H^{(0)}$ of energy is evaluated with ease, by introducing (28) into (26), which is expressed, in dimensional variables, as

$$H_0/\Gamma^2 = \frac{1}{2}R_0\left\{\log\left(\frac{8R_0}{\sigma}\right) + A - 2\right\}, \tag{30}$$

where $H_0 = \Gamma^2 R_0 H^{(0)}$ and A is given by

$$A = \lim_{r\to\infty}\left\{\frac{4\pi^2}{\Gamma^2}\int_0^r r'v_0(r')^2\mathrm{d}r' - \log\left(\frac{r}{\sigma}\right)\right\}. \tag{31}$$

We pose, as a natural profile of local velocity field featuring a vortex ring,

$$v_0(r) = -\frac{\Gamma}{2\pi r}f\left(\frac{r}{\sigma}\right), \quad \zeta_0 = \frac{\Gamma}{2\pi r}\frac{d}{dr}f\left(\frac{r}{\sigma}\right); \quad f(\xi) = O(\xi^2) \text{ as } \xi \to 0, \quad f(\xi) \to 1 \text{ as } \xi \to \infty,$$

$$(32)$$

where f is an arbitrary function, though subjected to the above boundary conditions. The parameter σ introduces the scale for the core thickness. Suppose that the fluid particles occupying a toroidal region of radius r around the center circle of radius R is mapped to another toroidal region of radius $\hat{r}$ around the center circle of radius $\hat{R}$. To maintain these flow fields on an iso-vortical sheet, it is necessary for the local circulation along any material loop to remain unchanged [6, 14, 20]. Preservation of material volume enforces $2\pi^2 r^2 R = 2\pi^2 \hat{r}^2 \hat{R}$, $2\pi^2 \sigma^2 R = 2\pi^2 \hat{\sigma}^2 \hat{R}$, from which follows $r/\sigma = \hat{r}/\hat{\sigma}$. Consequently, the local circulation around the circle of radius r, $\Gamma(r) = 2\pi \int_0^r \zeta_0(r')r'dr' = \Gamma f(r/\sigma)$, is made invariant: $\Gamma(r) = \Gamma(\hat{r})$. Under an infinitesimal perturbation of $R \to \hat{R} = R + \delta R$, $\sigma \to \hat{\sigma} = \sigma + \delta\sigma$, with $R = R_0 + R_2$, (5) demands that, at each order, $\sigma^2 R_0 = $ const. and $\sigma^2 R_2 = $ const., and therefore that $2\delta\sigma/\sigma = -\delta R_0/R_0 = -\delta R_2/R_2$. We can show that, under this perturbation, $\hat{A} = A + O\left((\delta R)^2\right)$. In view of these constraints, the variation of (30) with respect to an iso-vortical perturbation becomes

$$\delta H_0 = \frac{\Gamma^2}{2}\left[\log\left(\frac{8R_0}{\sigma}\right) + A - \frac{1}{2}\right]\delta R_0.$$

$$(33)$$

The variation of the leading term of impulse $P_0 = \Gamma\pi R_0^2$ is $\delta P_0 = 2\pi\Gamma R_0 \delta R_0$, and application of (2) retrieves Fraenkel-Saffman's formula [21, 31]:

$$U_0 = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8R_0}{\sigma}\right) + A - \frac{1}{2}\right\}.$$

$$(34)$$

The third-order correction U_2 to the translation speed of the vortex ring requires evaluation of $H^{(2)}$. For an inviscid vortex ring in steady motion, $R_2 = R_0\varepsilon^2 R^{(2)} \equiv 0$ without loss of generality, and, after some manipulations, we arrive at

$$U_2 = \frac{1}{R_0^3}\left\{\frac{d_1}{2}\left[\log\left(\frac{8R_0}{\sigma}\right) - 2\right] - \pi\Gamma B + \frac{\pi}{2\Gamma}\int_0^\infty r^4 \zeta_0 v_0 dr\right\},$$

$$(35)$$

where $v_0 = \Gamma v^{(0)}/\sigma$ and $\zeta_0 = \Gamma\zeta^{(0)}/\sigma^2$ are dimensional variables, and

$$B = \lim_{r\to\infty}\left\{\frac{1}{\Gamma^2}\int_0^r r'v_0\tilde{\psi}_{11}^{(1)}dr' + \frac{r^2}{16\pi^2}\left[\log\left(\frac{r}{\sigma}\right) + A\right] + \frac{d_1}{2\pi\Gamma}\log\left(\frac{r}{\sigma}\right)\right\}.$$

$$(36)$$

This is an extension, to $O(\varepsilon^3)$, of Fraenkel-Saffman's formula (34).

Even if viscosity is switched on, the higher-order asymptotics U_2 is not invalidated at a large Reynolds number. Taking, as the initial condition, a circular line vortex of radius R_0,

$$\zeta(\rho, z, 0) = \Gamma\delta(\rho - R_0)\delta(z - Z) \quad \text{at } t = 0,$$
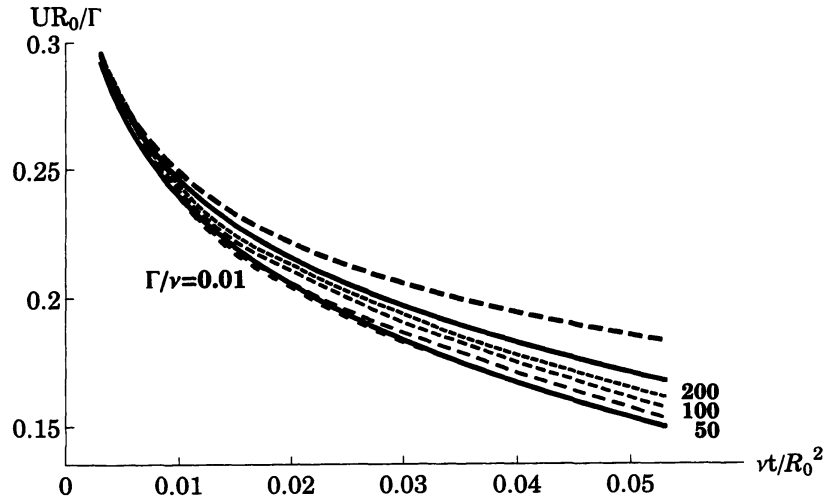
$$(37)$$

Figure 1: Variation of speed of a viscous vortex ring with time. The upper and lower solid lines are the high- (39) and low-Reynolds number asymptotics [34], respectively, while the thick dashed line is the Saffman's formula (34). The dashed lines are the values read off from the graph of numerical simulations [23].

the leading-order vorticity ζ_0 is given by

$$\zeta_0 = \frac{\Gamma}{4\pi v t} e^{-r^2/4vt}, \tag{38}$$

where v is the kinematic viscosity and t is the time measured from the instant at which the core is infinitely thin [21, 33], and the inhomogeneous heat-conduction equation governing $\zeta_0^{(2)}$ becomes tractable, with an introduction of similarity variables. The parameters $c_{11}^{(1)}$ in (29) and R_2, both being functions of t, play a common role of specifying the radial position of the ring at $O(\varepsilon^2)$ relative to R_0. This redundancy is removed, for instance, by taking $c_{11}^{(1)} \equiv 0$. Thus we are led to an extension of Saffman's formula (34) in the form

$$U \approx \frac{\Gamma}{4\pi R_0} \left\{ \log\left(\frac{4R_0}{\sqrt{vt}}\right) - 0.55796576 - 3.6715912\frac{vt}{R_0^2} \right\}. \tag{39}$$

Fig. 1 displays the comparison of the asymptotic formula (39) with a direct numerical simulation of the axisymmetric Navier-Stokes equations [23]. The normalized speed UR_0/Γ of the ring is drawn as a function of normalized time vt/R_0^2 for its small values. The upper thick solid line is our formula (39), and the thick broken line is the first-order truncation (34). The dashed lines are the results of the numerical simulations, attached with the circulation Reynolds number Γ/v, ranging from 0.01 to 200. Augmented only with a single correction term, (35) appears to furnish a close upper bound on the translation speed. Notably, the large-Reynolds-number asymptotic formula (39) compares fairly well with the numerical result of even moderate and small Reynolds numbers.

The variational principle (2), which comprises only the total energy H and the impulse P_z, dispenses with $\psi_{21}^{(2)}$ and $\zeta_{21}^{(2)}$. A further simplification is achieved by relying on the variational principle with kinematic constraints.

6 Discussions

Onsager [12] developed a method of statistical mechanics for ensembles of point vortices in a bounded domain on a plane. For small values of the total energy, low-energy configurations are favored. Opposite-signed vortices have tendency to couple together, or vortices have tendency to stay close to the boundary to couple with their image vortices. For large values of the total energy, the temperature becomes negative. This implies that high-energy configurations are favored. Same-signed vortices are clustered, which may be responsible for the formation of large-scale coherent vortices in two-dimensional turbulence. In this theory, the boundedness of the domain, high energy and statistics are key ingredients for a large-scale coherent structure. On the other hand, for Kelvin's variational principle, a large-scale structure with high vorticity of the same sign is attributed to steadiness of a non-viscous fluid.

Construction of steady solutions in a rest frame or in a steadily moving frame are problems of the first variation. Investigation of the linear (= spectral) and weakly nonlinear stability requires the second variation. For this purpose, the kinematically accessible or isovortical disturbances are indispensable, and the Lagrangian approach has turned out to offer a vital tool [35, 36, 37, 38, 39, 40, 41].

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