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DRIFT-DIFFUSION MODEL AND 2D BROWNIAN POINT VORTICES

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Abstract. We study the kinetic mean field equation of two-dimensional Brownian vortices, its derivation, similarity to competitive systems of chemotaxis and a drift-diffusion model, and the method of weak scaling limit.

1 INTRODUCTION

It is Onsager [16] that initiated the statistical mechanics of point vortices. It begins with the vortex system

$$\frac{dx_i}{dt} = \nabla_{x_i}^{\perp} H_N, \quad i = 1, 2, \cdots, N \tag{1}$$

associated with the Hamiltonian

$$H_N(x_1, \ldots, x_N) = \frac{1}{2}\sum_i \gamma_i^2 R(x_j) + \sum_{i<j} \gamma_i\gamma_j G(x_i, x_j)$$
$$x_i \in \Omega,\ i = 1, 2, \cdots, N$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$,

$$\nabla^{\perp} = \begin{pmatrix} \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} \end{pmatrix}, \quad x = (x_1, x_2),$$

$G = G(x, x')$ is the Green's function of $-\Delta$ provided with the Dirichlet boundary condition, and

$$R(x) = \left[G(x, x') + \frac{1}{2\pi} \log |x - x'| \right]_{x'=x}$$

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stands for the Robin function. The argument runs on the static theory of Gibbs, converting the micro-canonical measure to the canonical measure using a thermodynamical relation. It drawed an ordered structure in negative temperature. Here, γ_i stands for the intensity of i-th vortex. In the case $\gamma_i = \gamma > 0$, $i = 1, 2, \cdots, N$, the mean field equation

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}, \quad \psi = \int_\Omega G(\cdot, x')\rho(x')dx' \tag{2}$$

arises in the high-energy limit where $\beta = 1/kT$ denotes the renormalized inverse temperature with $k = k_B$ standing for the Boltzmann constant [13, 17, 4]. It is justified under the uniform boundeness of the weight factor and also the uniqueness of the solution to the mean field equation [2, 11]. The derivation of the mean field equation (2) is, consequently, justified in the region $\lambda = -\beta < 8\pi$ [20].

Equation (2), or

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega \tag{3}$$

is the Euler-Lagrange equation of the Vlasov functional

$$\mathcal{F}(\rho) = U(\rho) - TS(\rho)$$

combined with the inner energy and entropy terms

$$U(\rho) = \frac{1}{2}\iint_{\Omega\times\Omega} G(x, x')\rho \otimes \rho \; dxdx', \quad \rho \otimes \rho = \rho(x)\rho(x')$$
$$S(\rho) = -k\int_\Omega \rho(\log\rho - 1).$$

This equation has been called the Boltzmann-Poisson(-Emden) equation in condensed matter physics, astrophysics, quantum chemistry, and information theory [1], There is a *quantized blowup mechanism* and the control of the Hamiltonian concerning the blowup points in the solution sequence to (3) [14]. The latter property is regarded as the *recursive hierarchy* in the original context of [16]. First, the Hamiltonian

$$H_N(x_1, \ldots, x_N) = \frac{1}{2}\sum_i \gamma^2 R(x_j) + \sum_{i<j} \gamma^2 G(x_i, x_j) \tag{4}$$

controls the motion of particles by (1). Next, the continuous distribution of the particle density is derived in the high-energy limit. Finally, the continuous particle density emerged in this mean field limit concentrate on finite points as the critical stage is approaching, with their location identical to the critical point of the normalized Hamiltonian

$$\hat{H}_N(x_1, \ldots, x_N) = \frac{1}{2}\sum_i R(x_j) + \sum_{i<j} G(x_i, x_j).$$

Hence the mean filed limits again exhibit the profile of particles regarded as an equiliburium of (1).

Under long-range interactions, however, the equivalence of thermodynamical relations between statistical ensembles is violated. This problem was approached by Chavanis [3], taking a formal kinetic mean field equation for canonical ensembles. It uses the system of stochastic equations

$$\frac{dx_i}{dt} = \gamma \nabla_{x_i}^{\perp} \hat{H}_N - \mu\gamma^2 \nabla_i \hat{H}_N + \sqrt{2\nu} R_i(t), \quad i = 1, 2, \cdots, N$$

where $\mu > 0$ stands for the mobility in the theory of Brownian motion,

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix},$$

$\nu \geq 0$ denotes the diffusion coefficient describing the viscosity of the system of particles, and $R_i(t)$ is the white noise:

$$\langle R_i(t) \rangle = 0, \quad \langle R_i^{\alpha}(t) R_j^{\beta}(t') \rangle = \delta_{ij}\delta_{\alpha\beta}\delta(t - t').$$

Under this setting, the N-body distribution function $P_N(x_1, \cdots, x_N, t)$ is subject to the Fokker-Planck equation

$$\frac{\partial P_N}{\partial t} + \gamma \nabla^{\perp} \hat{H} \cdot \nabla P_N = \nabla \cdot (\nu \nabla P_N + \mu\gamma^2 P_N \nabla \hat{H}_N)$$

which induces a BBGKY-like hierarchy concerningP_i, $i = 1, 2, \cdots, N$. Then the factorization, or the propagation of chaos,

$$P_N(x_1, x_2, \cdots, x_N, t) = \prod_{i=1}^{N} P_1(x_i, t)$$

is assumed in the high-energy limit, $\beta N \gamma^2 = 1$, $N \uparrow \infty$. The limit equation for $\omega = N\gamma P_1$ arises with the normalized temperature,

$$\begin{aligned} &\frac{\partial \omega}{\partial t} + \nabla^{\perp}\psi \cdot \nabla \omega = \nu \nabla \cdot (\nabla \omega + \beta\gamma\omega\nabla\psi) \\ &-\Delta\psi = \omega, \quad \psi|_{\partial\Omega} = 0. \end{aligned} \tag{5}$$

A similarity of the Smoluchowski-Poisson equation is noticed for (5), that is,

$$\begin{aligned}
&\frac{\partial u}{\partial t} = d\Delta u - \chi\nabla\cdot u\nabla v \\
&-\Delta v = u - \frac{1}{|\Omega|}u && \text{in } \Omega\times(0,T) \\
&d\frac{\partial u}{\partial\nu} - \chi u\frac{\partial v}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T), \quad \int_\Omega v = 0 \\
&u|_{t=0} = u_0(x) \geq 0 && \text{in } \Omega. \qquad (6)
\end{aligned}$$

It is a simplified system of chemotaxis [9] where $d > 0$ and $\chi > 0$ stand for the diffusion and chemotactic sensitivity coefficients, respectively. If the solution to this system blows-up in finite time, then it forms collapses with quantized mass [21, 23]. The stationary state of (6), on the other hand, coincides with the Boltzmann-Poisson equation, especially the one used in gauge theory [26],

$$-\Delta v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right), \quad \left.\frac{\partial v}{\partial\nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v = 0 \tag{7}$$

which is a relative of (3). Actually, if the Poisson part is replaced by

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0 \tag{8}$$

in (6) then the stationary state of this system is exactly the one described by (3). The underlying structure which realizes this profile is a *duality* between u and v (or ρ and ψ). It is associated with the Legendre transformation and may be called the field-particle duality. Hence the kinetic mean field equation of point vorticies (5) is close to the dual form of (2) in the stationary states. This duality, however, is not restricted to the stationary states. Actually, the non-stationary Smoluchowski-Poisson equation is a model (B) equation derived from the Vlasov functional (or the Helmholtz free energy) [22]. For example, system (6) takes the form

$$u_t = \nabla\cdot u\nabla\delta\mathcal{F}(u), \quad \left.u\frac{\partial}{\partial\nu}\delta\mathcal{F}(u)\right|_{\partial\Omega} = 0$$

where

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u\otimes u\; dxdx'$$

with $G = G(x,x')$ standing for the Green's function of the Poisson part

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \quad \left.\frac{\partial v}{\partial\nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v = 0. \tag{9}$$

The above described property of (6) may be thought to be a recursive hierarchy. Thus at least the quantized blowup mechanism of (7) is valid even in the kinetic level (6). However, a more fruitful understanding will be the *nonlinear spectral mechanics*, where the quantized blowup mechanism of stationary states induces that of the kinetic model.

The present paper is concerned with the two-species model of [3] described by

$$\begin{aligned}
&\frac{\partial \omega_\pm}{\partial t} + \nabla \cdot \omega_\pm \nabla^\perp \psi = \nu \nabla \cdot (\nabla \omega_\pm \pm \beta\gamma \nabla \psi) \\
&-\Delta \psi = \omega_+ + \omega_- && \text{in } \Omega \times (0,T) \\
&\frac{\partial \omega_\pm}{\partial \nu} \pm \beta\gamma \omega_\pm \frac{\partial \psi}{\partial \nu} = \psi = 0 && \text{on } \partial\Omega \times (0,T) \\
&\omega_\pm|_{t=0} = \omega_{\pm 0} && \text{in } \Omega
\end{aligned}$$

where $\nu > 0$, $\gamma > 0$, $\beta = -\lambda < 0$, and $\omega_{+0} \geq 0 \geq \omega_{-0}$. Using $u_1 = \omega_+$, $u_2 = -\omega_-$, $v = \psi$, $\chi = -\nu\beta\gamma > 0$, and $d = \nu > 0$, we divide it into the Smoluchowski part

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} + \nabla \cdot u_1 \nabla^\perp v = d\Delta u_1 - \chi \nabla \cdot u_1 \nabla v \\
&\frac{\partial u_2}{\partial t} + \nabla \cdot u_2 \nabla^\perp v = d\Delta u_2 + \chi \nabla \cdot u_2 \nabla v && \text{in } \Omega \times (0,T) \\
&d\frac{\partial u_1}{\partial \nu} - \chi u_1 \frac{\partial v}{\partial \nu} = d\frac{\partial u_2}{\partial \nu} + \chi u_2 \frac{\partial v}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T) \\
&u_i|_{t=0} = u_{i0} \geq 0, \ i = 1,2, && \text{in } \Omega
\end{aligned} \tag{10}$$

and the Poisson part (8) with

$$u = u_1 - u_2. \tag{11}$$

Without the vorticity terms $u_i \nabla^\perp v$, $i = 1,2$, system (10)-(11) is a drift-diffusion model. The forms

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} = d\Delta u_1 - \chi \nabla \cdot u_1 \nabla v \\
&\frac{\partial u_2}{\partial t} = d\Delta u_2 + \chi \nabla \cdot u_2 \nabla v \\
&-\Delta v = u_1 - u_2 && \text{in } \mathbf{R}^2 \times (0,T) \\
&u_i|_{t=0} = u_{i0}(x) \geq 0, \ i = 1,2, && \text{in } \mathbf{R}^2
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} = d\Delta u_1 - \chi\nabla\cdot u_1\nabla v \\
&\frac{\partial u_2}{\partial t} = d\Delta u_2 + \chi\nabla\cdot u_2\nabla v && \text{in } \Omega\times(0,T) \\
&d\frac{\partial u_1}{\partial\nu} - \chi u_1\frac{\partial v}{\partial\nu} = d\frac{\partial u_2}{\partial\nu} + \chi u_2\frac{\partial v}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T) \\
&u_i\big|_{t=0} = u_{i0} \geq 0,\ i=1,2, && \text{in } \Omega
\end{aligned} \tag{13}$$

with (9) and (11) are studied by [12, 7] and [8], respectively.

We can observe two differences between the system of (10) with (8) and (11), and that of (13) with (9) and (11). The first factor is the appearance of the vorticity terms $u_i\nabla^{\perp}v$, $i=1,2$. The Poisson parts are also different, that is, (8) and (9) which cause a technical difficulty to treat the boundary behavior of the solution. Thus, it may be a good strategy to study (13) with (9) and (11) before turning to (10) with (8) and (11). Actually, it has been unsolved for a long while to exclude boundary blowup points in the former case even for the single component case [21, 22].

In this paper, first, we review [5] concerning the competitive system of chemotaxis,

$$\begin{aligned}
&\frac{\partial u_i}{\partial t} = d_i\Delta u_i - \chi_i\nabla\cdot u_i\nabla v && \text{in } \Omega\times(0,T) \\
&d_i\frac{\partial u_i}{\partial\nu} - \chi_i u_i\frac{\partial v}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T) \\
&u_i\big|_{t=0} = u_{i0}(x) \geq 0 && \text{in } \Omega,
\end{aligned} \tag{14}$$

$i=1,2,\cdots,N$, coupled with the Poisson equation (9) with

$$u = \sum_{i=1}^{N} u_i, \tag{15}$$

where $d_i > 0$ and χ_i, $i=1,2,\cdots,N$, are positive constants. Then turning to another competitive system of chemotaxis, we describe basic profiles of (13).

2 COMPETITIVE SYSTEM OF CHEMOTAXIS

System (14) with (9) and (15), henceforth called (ISP), was proposed in [6] to approach the question of cell sorting of Dictyostelium discoideum (Dd) [24]. The other motivation is a competitive feature of chemotaxis observed in cancer cell biology, especially, in tumor microenvironment at the stage of intravasation [25, 10]. Below we assume $N = 2$, but the general case is treated similarly. Given a sufficiently regular initial value (u_0, v_0), the proof of the existence and uniqueness of the local-in-time solution is standard. Then the parabolic regularity guarantees the existence time etimate from below by $\sum_{i=1}^{2} \|u_{i0}\|_\infty$, which implies

$$T = T_{\max} < +\infty \quad \Rightarrow \quad \lim_{t \uparrow T} \sum_{i=1}^{2} \|u_i(\cdot, t)\|_\infty = +\infty. \tag{16}$$

The positivity of each component of the solution is also kept, and hence the component-wise total mass conservation holds by

$$\frac{d}{dt} \int_\Omega u_i dx = 0, \quad i = 1, 2. \tag{17}$$

In [6, 7] the simultaneous blowup

$$T < +\infty \quad \Rightarrow \quad \limsup_{t \uparrow T} \|u_1(\cdot, t)\|_\infty = \limsup_{t \uparrow T} \|u_2(\cdot, t)\|_\infty = +\infty, \tag{18}$$

is proven for the case of $u_i = u_i(|x|, t)$, which does not hold for (12). There is, however, a parameter region which ensures

$$T < +\infty \quad \Rightarrow \quad \lim_{t \uparrow T} \|u_1(\cdot, t)\|_\infty = \lim_{t \uparrow T} \|u_2(\cdot, t)\|_\infty = +\infty, \tag{19}$$

even for non-radially symmetric solutions. Let

$$\xi_i = d_i / \chi_i, \quad \|u_{i0}\|_1 = \lambda_i, \quad i = 1, 2. \tag{20}$$

Theorem 1 ([5]) *If*

$$\lambda_i < 4\pi \xi_i, \quad i = 1, 2 \tag{21}$$

then (19) holds.

Condition (21) is consistent to $T < +\infty$. In fact, if

$$\left(\sum_{i=1}^{2} \lambda_i \right)^2 < 4\pi \sum_{i=1}^{2} \xi_i \lambda_i, \quad \lambda_i < 4\pi \xi_i, \ i = 1, 2$$

$T = +\infty$ always holds, while $T < +\infty$ always can occur in case

$$\left(\sum_{i=1}^{2} \lambda_i \right)^2 > 4\pi \sum_{i=1}^{2} \xi_i \lambda_i. \tag{22}$$

Then the parameter region defined by (21) and (22) in $\lambda_1\lambda_2$ plane in $\lambda_i > 0$, $i = 1,2$, is not empty. Since property (16) means

$$T < +\infty \quad \Rightarrow \quad \lim_{t\uparrow T} \|u(\cdot,t)\|_\infty = +\infty, \tag{23}$$

the blowup set of (u_1, u_2) defined by

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists (x_k, t_k) \to (x_0, T),\ u(x_k, t_k) \to +\infty\} \tag{24}$$

is not empty. In the case of the Smoluchowski-Poisson system for a single unknown species, the formation of collapses occurs with a quantized mass [21, 23]. This property arises also in (ISP) for each component, with possibly degenerate collapses. Here, we say that the collapse $m_i(x_0)\delta_{x_0}(dx)$, $i = 1,2$, in (25) below is degenerate if $m_i(x_0) = 0$.

Theorem 2 ([5]) *If $T < +\infty$, the blowup set $\mathcal{S}$ defined by (24) is finite. It holds that*

$$u_i(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m_i(x_0)\delta_{x_0}(dx) + f_i(x)dx, \quad i = 1,2 \tag{25}$$

in $\mathcal{M}(\overline{\Omega}) = C(\overline{\Omega})'$ as $t \uparrow T = T_{\max} < +\infty$, where $m_i(x_0) \geq 0$, $i = 1,2$, are constants satisfying $(m_1(x_0), m_2(x_0)) \neq (0,0)$, and $0 \leq f_i = f_i(x) \in L^1(\Omega)$, $i = 1,2$, are smooth functions in $\overline{\Omega} \setminus \mathcal{S}$. We have $f_i > 0$ in $\overline{\Omega} \setminus \mathcal{S}$ except for $u_{i0} \equiv 0$.

Equality (26) in the following theorem may be called a *total mass quantization* because it is an identity involving all the collapse masses $m_i(x_0)$, $i = 1,2$.

Theorem 3 ([5]) *It holds that*

$$\left(\sum_{i=1}^{2} m_i(x_0)\right)^2 = m_*(x_0)\sum_{i=1}^{2} \xi_i m_i(x_0) \tag{26}$$

for any $x_0 \in \mathcal{S}$, where

$$m_*(x_0) = \begin{cases} 8\pi, & x_0 \in \Omega \\ 4\pi, & x_0 \in \partial\Omega. \end{cases} \tag{27}$$

The next theorem concerning the formation of subcollapses implies that any blowup rate of (ISP) is type II. To state the result, let

$$(u_1, u_2, v) = (u_1(x,t), u_2(x,t), v(x,t))$$

be a solution to (ISP) satisfying $T = T_{\max} < +\infty$, take $x_0 \in \mathcal{S}$, and let

$$\begin{aligned} &z_i(y,s) = (T-t)u_i(x,t), \quad w(y,s) = v(x,t) \\ &y = (x-x_0)/(T-t)^{1/2}, \quad s = -\log(T-t), \quad i = 1,2. \end{aligned} \tag{28}$$

We assume the 0-extensions of $z_i(y,s)$, $i = 1,2$, where they are not defined. Furthermore, $\mathbf{R}^2 \cup \{\infty\}$ denotes the one-point compactification of $\mathbf{R}^2$, and $C_0(\mathbf{R}^2)$ stands for the set of continuous functions on $\mathbf{R}^2 \cup \{\infty\}$ taking the value 0 at ∞, and $\mathcal{M}_0(\mathbf{R}^2) = C_0(\mathbf{R}^2)'$.

Theorem 4 ([5]) *We have*

$$z_i(y, s+s')dy \rightharpoonup m_i(x_0)\delta_0(dy), \quad i = 1,2 \tag{29}$$

in $C_*(-\infty,+\infty;\mathcal{M}_0(\mathbf{R}^2))$ as $s' \uparrow +\infty$. In particular, it holds that

$$\lim_{t\uparrow T}\, (T-t)\|u(\cdot,t)\|_{L^\infty(\Omega\cap B(x_0,b(T-t)^{1/2})} = +\infty \tag{30}$$

for any $b > 0$.

The blowup set $\mathcal{S}$ coincides with the origin for radially symmetric solutions satisfying $T < +\infty$. The following fact arises for such a case.

Theorem 5 ([5]) *Let Ω be a disc with center at the origin, $u_i = u_i(|x|,t)$, $i = 1,2$, and $T < +\infty$. Then $m_i = m_i(0)$ must satisfy*

$$m_i \le 8\pi\xi_i, \quad i = 1,2 \tag{31}$$

besides (26) with $x_0 = 0$.

Here, first, inequality (31) arises also in the context of global-in-time continuation of the solution associated with the Trudinger-Moser or the logarithmic HLS inequality. Next, (31) is a consequence of (26) if

$$1/2 \le \xi_i/\xi_j \le 2, \quad i,j = 1,2. \tag{32}$$

More precisely, if (32) is the case, the curve (a parabola if $\xi_1 \ne \xi_2$ and a line in the other case) defined by

$$\left(\sum_{i=1}^{2} m_i\right)^2 = m_* \sum_{i=1}^{2} \xi_i m_i, \quad m_* = m_*(x_0), \tag{33}$$

in the m_1m_2-plane in $\{(m_1,m_2) \mid m_i > 0, i = 1,2\}$ does not cross the lines $m_i = \xi_i m_*$, $i = 1,2$. Finally, in the other case of $\xi_i/\xi_j > 2$ or $\xi_i/\xi_j < 1/2$

for $i \neq j$, one of $(m_1, m_2) = (8\pi\xi_1, 0)$ and $(m_1, m_2) = (0, 8\pi\xi_2)$ is an isolated point of (33) in the m_1m_2-plane with $\{(m_1, m_2) \mid m_i \geq 0, i = 1, 2\}$ and (31). The following theorem shows that the mass separation of radially symmetric solutions actually occurs if the total mass of one component is relatively small compared with that of the other. In this context we recall that *simultaneous blowup* (18) is always the case for radially symmetric solutions, regardless of the parameter region indicated by (21). If both simultaneous blowup and mass separation arise, say, $m_i(x_0) = 0$ in (25), then it will hold that $f_i \notin L^\infty(\Omega \cap B(x_0, R))$ for $0 < R \ll 1$, where $B(x_0, R) = \{x \mid |x - x_0| < R\}$.

Theorem 6 **([5])** *Under the assumption of Theorem 5, let $\xi_i/\xi_j > 2$ for some $i \neq j$. Then $m_i = 0$ and hence $m_j = 8\pi\xi_j$ holds, provided that*

$$\|u_{i0}\|_1 < 8\pi(\xi_i - 2\xi_j).$$

A sufficient condition for $T < +\infty$ in the above theorem is

$$\|u_{j0}\|_1 > 8\pi\xi_j, \quad \||x|^2 u_{j0}\|_1 \ll 1$$

([7], Theorem 11 of [5]). Theorem 8 of [5] is also available for this purpose.

We shall review the proof for later use. First, Theorem 1 is proven by a variational structure of (ISP) and the logarithmic HLS inequality [19]. Theorem 2 is obtained by an argument of [18], using an ε-regularity and a monotonicity formula. Then we have the formation of collapses of $\hat{u}(x,t)dx$ as $t \uparrow T$, where $\hat{u} = \sum_{i=1}^{2} \chi_i^{-1} u_i$. A careful analysis then assures this property component-wisely and also

$$m(x_0) \equiv \sum_{i=1}^{2} m_i(x_0) > 0. \tag{34}$$

To prove Theorem 3, first, we apply an argument developed for the single component case [21, 23]. We use the backward self-similar transformation, weak scaling limit, scaling back, and translation limit, to obtain a full-orbit defined on the whole (or the half) space domain. At these precesses, the total masses of the generated weak solutions continue to be the collapse mass because of the *parabolic envelope* and the positivity of the measure. Then, an existence criterion of such orbits follows from the method of local second moments and scaling, which guarantees an estimate of the total collapse mass from above, that is,

$$\left(\sum_{i=1}^{2} m_i(x_0)\right)^2 \leq m_*(x_0) \sum_{i=1}^{2} \xi_i m_i(x_0). \tag{35}$$

We use a different argument than [21, 23] to derive the reverse inequality

$$\left(\sum_{i=1}^{2} m_i(x_0)\right)^2 \geq m_*(x_0)\sum_{i=1}^{2}\xi_i m_i(x_0). \tag{36}$$

Namely, we show the boundedness in time of the total second moment of the rescaled solution and use the scaling limit equation. We have, at the same time, the formation of subcollapses indicated by Theorem 4. We note that a weaker estimate of the total collapse mass from below is obtained similarly to the single component case, that is, either (36) or

$$m_i(x_0) \geq \xi_i m_*(x_0), \quad i = 1, 2 \tag{37}$$

by the logarithmic HLS inequality. Inequality (36), however, is eventually selected for (26) to be established. The proof of Theorem 5 is based on the fact that the interaction between two-components is neglected in the collapse mass estimate from above for radially symmetric solutions. Then Theorem 6 arises with the total mass conservation of each component of the solution.

As we have reviewed, the above theorems are proven by three remarkable structures of the system other than the total mass conservation (17), that is, the decrease of the total free energy

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F}_{\xi_1,\xi_2}(u_1,u_2) \leq 0 \\
&\mathcal{F}_{\xi_1,\xi_2}(u_1,u_2) = \sum_{i=1}^{2}\int_\Omega \xi_i u_i(\log u_i - 1)dx - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle \\
&u = u_1 + u_2,
\end{aligned}$$

the weak form

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left[\sum_{i=1}^{2}\chi_i^{-1}u_i\right]\varphi\ dx - \int_\Omega\left[\sum_{i=1}^{2}\xi_i u_i\right]\Delta\varphi\ dx \\
&\quad = \frac{1}{2}\iint_{\Omega\times\Omega}\rho_\varphi u\otimes u\ dxdx' \\
&\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x')
\end{aligned}$$

valid to

$$\varphi \in C^2(\overline{\Omega}), \quad \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0, \tag{38}$$

and the scaling invariance of the system on the whole space,

$$\begin{aligned} & \frac{\partial u_i}{\partial t} = d_i \Delta u_i - \chi_i \nabla \cdot u_i \nabla v, \quad i = 1,2 \\ & -\Delta v = u, \quad u = \sum_{i=1}^{2} u_i \qquad\qquad \text{in } \mathbf{R}^2 \times (0,\infty) \end{aligned}$$

arising in the scaling limit after several processes, that is,

$$u_i^\mu(x,t) = \mu^2 u_i(\mu x, \mu^2 t), \quad v^\mu(x,t) = v(\mu x, \mu^2 t), \quad \mu > 0. \tag{39}$$

In the final remark the free energy decreasing is used only for the global-in-time existence of the solution. Namely, the weak form, scaling invariance, and ε-regularity are sufficient for the total mass quantization and related other properties to guarantee. It is worth mentioning ithat there is actually a parameter region where the blowup threshold mass has not been known for multi-species model.

3 Cross-chemotaxis model

The above described structures are quite common. A slight modification is the other competitive system of chemotaxis,

$$\begin{aligned} & \frac{\partial u_i}{\partial t} = d_i \Delta u_i - \chi_i \nabla \cdot u_i \nabla v_i && \text{in } \Omega \times (0,T) \\ & d_i \frac{\partial u_i}{\partial \nu} - \chi_i u_i \frac{\partial v_i}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T) \\ & u_i|_{t=0} = u_{i0}(x) \geq 0 && \text{in } \Omega, \end{aligned} \tag{40}$$

$i = 1, 2$, coupled with the Poisson system

$$\begin{aligned} & -\Delta v_1 = u_2 - \frac{1}{|\Omega|} \int_\Omega u_2, \quad \left.\frac{\partial v_1}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v_1 = 0 \\ & -\Delta v_2 = u_1 - \frac{1}{|\Omega|} \int_\Omega u_1, \quad \left.\frac{\partial v_2}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v_2 = 0. \end{aligned} \tag{41}$$

In this system the first species u_1 secretes a chemical v_1 which attracts the second species u_2, and similarly, u_2 secretes v_2 which attracts u_1. In fact, the total mass conservation (17) and the scaling invariance (39) of the limit system

$$\begin{aligned} & \frac{\partial u_i}{\partial t} = d_i \Delta u_i - \chi_i \nabla \cdot u_i \nabla v_i, \quad i = 1,2 \\ & -\Delta v_1 = u_2, \quad -\Delta v_2 = u_1, \qquad \text{in } \mathbf{R}^2 \times (0,T) \end{aligned}$$

are obvious, while the total energy decreasing and the weak form are formulated by

$$\frac{d}{dt}\mathcal{F}_{\xi_1,\xi_2}(u_1,u_2) \leq 0$$
$$\mathcal{F}_{\xi_1,\xi_2}(u_1,u_2) = \sum_{i=1}^{2}\int_\Omega \xi_i u_i(\log u_i - 1)dx - \frac{1}{2}\langle(-\Delta)^{-1}u_1, u_2\rangle$$

and

$$\frac{d}{dt}\int_\Omega \left[\sum_{i=1}^{2}\chi_i^{-1}u_i\right]\varphi\, dx - \int_\Omega \left[\sum_{i=1}^{2}\xi_i u_i\right]\Delta\varphi\, dx$$
$$= \frac{1}{2}\iint_{\Omega\times\Omega}\rho_\varphi u_1\otimes u_2\, dxdx',$$

respectively. Accordingly, we obtain the following results with (20).

Theorem 7 *If*

$$\lambda_1\lambda_2 < 4\pi\sum_{i=1}^{2}\lambda_i\xi_i$$

then it holds that $T = +\infty$ in (40) with (41).

Theorem 8 *Given $x_0 \in \overline{\Omega}$ and $0 < R \ll 1$, if*

$$\lambda_1(x_0)\lambda_2(x_0) > m_*(x_0)\sum_{i=1}^{2}\xi_i\lambda_i(x_0)$$
$$\| |x-x_0|^2 u_{i0}\|_{L^1(B(x_0,2R)\cap\Omega)} \ll 1, \quad i = 1,2$$

then it holds that $T < +\infty$ in (40) with (41), where

$$\lambda_i(x_0) = \|u_{i0}\|_{L^1(B(x_0,R)\cap\Omega)}, \quad i = 1,2.$$

Theorem 9 *If $T < +\infty$, the blowup set $\mathcal{S}$ defined by (24) is finite. It holds that (25) in $\mathcal{M}(\overline{\Omega}) = C(\overline{\Omega})'$ as $t \uparrow T = T_{\max} < +\infty$, where $m_i(x_0) \geq 0$, $i = 1,2$, are constants satisfying $(m_1(x_0), m_2(x_0)) \neq (0,0)$, and $0 \leq f_i = f_i(x) \in L^1(\Omega)$, $i = 1,2$, are smooth functions in $\overline{\Omega}\setminus\mathcal{S}$. We have $f_i > 0$ in $\overline{\Omega}\setminus\mathcal{S}$ except for $u_{i0} \equiv 0$.*

Theorem 10 *The collapse masses $m_i(x_0)$, $i = 1, 2$, satisfy*

$$m_1(x_0)m_2(x_0) = m_*(x_0)\sum_{i=1}^{2}\xi_i m_i(x_0) \tag{42}$$

for any $x_0 \in \mathcal{S}$.

Since (42) with $m_i(x_0) \geq 0$, $i = 1, 2$ and $(m_1(x_0), m_2(x_0) \neq (0, 0)$, it holds that $m_i(x_0) > m_*(x_0)\xi_i$, $i = 1, 2$. Hence any components of the collapse are non-degenerate. In particular, we always obtain simultaneous blowup and non-mass separation in this system.

4 A DRIFT-DIFFUSION MODEL

A slightly generalized system of (13) studied in [8],

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} = d_1\Delta u_1 - \chi_1\nabla\cdot u_1\nabla v \\
&\frac{\partial u_2}{\partial t} = d_2\Delta u_2 + \chi_2\nabla\cdot u_2\nabla v && \text{in } \Omega\times(0,T) \\
&d_1\frac{\partial u_1}{\partial\nu} - \chi_1 u_1\frac{\partial v}{\partial\nu} = d_2\frac{\partial u_2}{\partial\nu} + \chi_2 u_2\frac{\partial v}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T) \\
&u_i|_{t=0} = u_{i0} \geq 0,\ i = 1, 2, && \text{in } \Omega
\end{aligned} \tag{43}$$

with

$$\begin{aligned}
&-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \quad \left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v = 0 \\
&u = u_1 - u_2
\end{aligned} \tag{44}$$

and

$$(u, v)|_{t=0} = (u_0, v_0), \quad u_0 = u_0(x) \geq 0, \quad v_0 = v_0(x) \geq 0 \tag{45}$$

has rather common structures with the competitive system of chemotaxis treated in §2. We shall confirm them to conclude. First, the proof of unique existence of the solution local-in-time for sufficiently regular initial values is standard. We have the positivity and the total mass conservation (17) for the solution. To confirm the free energy decreasing we use

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} = \nabla\cdot u_1\nabla(d_1\log u_1 - \chi_1 v), && \left.\frac{\partial}{\partial\nu}(d_1\log u_1 - \chi_1 v)\right|_{\partial\Omega} = 0 \\
&\frac{\partial u_2}{\partial t} = \nabla\cdot u_2\nabla(d_2\log u_2 + \chi_2 v), && \left.\frac{\partial}{\partial\nu}(d_2\log u_2 + \chi_2 v)\right|_{\partial\Omega} = 0.
\end{aligned}$$

Then it holds that

$$\int_\Omega \frac{\partial u_1}{\partial t}(d_1 \log u_1 - \chi_1 v) = -\int_\Omega u_1|\nabla(d_1 \log u_1 - \chi_1 v)|^2$$
$$\int_\Omega \frac{\partial u_1}{\partial t}(d_2 \log u_2 + \chi_2 v) = -\int_\Omega u_2|\nabla(d_2 \log u_2 + \chi_2 v)|^2$$

and hence

$$\xi_1 \frac{d}{dt}\int_\Omega u_1(\log u_1 - 1) - \int_\Omega \frac{\partial u_1}{\partial t} v = -\chi_1^{-1}\int_\Omega u_1|\nabla(d_1 \log u_1 - \chi_1 v)|^2$$
$$\xi_2 \frac{d}{dt}\int_\Omega u_2(\log u_2 - 1) + \int_\Omega \frac{\partial u_1}{\partial t} v = -\chi_2^{-1}\int_\Omega u_2|\nabla(d_2 \log u_2 + \chi_2 v)|^2$$

where $\xi_i = d_i/\chi_i$, $i = 1, 2$. Writing the Poisson part of (44) as $v = (-\Delta)^{-1}u$, we thus obtain

$$\frac{d}{dt}\left\{\int_\Omega \xi_1 u_1(\log u_1 - 1) + \xi_2 u_2(\log u_2 - 1)\ dx - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle\right\}$$
$$= -\int_\Omega \chi^{-1}u_1|\nabla(d_1 \log u_1 - \chi_1 v)|^2 + \chi_2^{-1}u_2|\nabla(d_2 \log u_2 + \chi_2 v)|^2\ dx.$$

To derive the weak form, on the other hand, we use

$$\chi^{-1}\frac{\partial u_1}{\partial t} = \xi_1 \Delta u_1 - \nabla \cdot u_1 \nabla v, \quad \left.\xi_1 \frac{\partial u_1}{\partial \nu} - u_1 \frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$
$$\chi_2^{-1}\frac{\partial u_2}{\partial t} = \xi_2 \Delta u_2 + \nabla \cdot u_2 \nabla v, \quad \left.\xi_2 \frac{\partial u_2}{\partial \nu} + u_2 \frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

which implies

$$\frac{d}{dt}\int_\Omega (\chi_1^{-1}u_1 + \chi_2^{-1}u_2)\varphi = \int_\Omega (\xi_1 u_1 + \xi_2 u_2)\Delta\varphi + u\nabla v \cdot \nabla\varphi\ dx$$
$$= \int_\Omega (\xi_1 u_1 + \xi_2 u_2)\Delta\varphi + \frac{1}{2}\iint_{\Omega\times\Omega} \rho_\varphi u \otimes u$$
$$\rho_\varphi(x, x') = \nabla_x G(x, x') \cdot \nabla\varphi(x) + \nabla_{x'} G(x, x') \cdot \nabla\varphi(x')$$

where $\varphi \in C^2(\overline{\Omega})$, $\left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0$.

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