数理解析研究所講究録
第 1802 巻 2012 年 288-294

On optimal selling strategy for assets driven by exponential Lévy process [1]

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1 Introduction

Throughout this paper we consider optimal stopping problems

$$\sup_{0\le\tau\le T} \mathrm{E}U\left(\frac{S_\tau}{\max\limits_{0\le t\le T} S_t}\right) \tag{1}$$

and

$$\inf_{0\le\tau\le T} \mathrm{E}U\left(\frac{\max\limits_{0\le t\le T} S_t}{S_\tau}\right), \tag{2}$$

where process $S=(S_t)_{t\le T}$, $T<\infty$, is an exponential Lévy process; $S_t=e^{H_t}$ and $U=U(x)$ is an utility function. These two problems were discussed primarily in papers [5], [16] and [26] in connection with stochastic problem of optimal liquidation of stock.

If $U(x)=\log x$ and S is the exponential of the Brownian motion with randomly changing drift, solution of (1) and (2) leads to an optimal detection problem, see [5] and [13]. For utility function $U(x)=x$, problems (1) and (2) were discussed firstly in [16] and [26]. In both papers it it supposed that stock price S is evaluated as a geometric Brownian motion, $dS_t=rS_tdt+\sigma S_tdB_t$, $S_0=1$, $t\le T$, where $B=(B_t)_{t\le T}$ is a standard Brownian motion. In [26], authors consider problem (1) in cases $r\ge\sigma^2$ and $r\le\sigma^2/2$. In the first case, the solution of (1), i. e. a stopping moment $0\le\tau^*\le T$ such that

$$\sup_{0\le\tau\le T} \mathrm{E}U\left(\frac{S_\tau}{\max\limits_{0\le t\le T} S_t}\right)=\mathrm{E}U\left(\frac{S_{\tau^*}}{\max\limits_{0\le t\le T} S_t}\right),$$

is $\tau^*=T$, and the optimal liquidation strategy is "buy and hold" here. If $r\le\sigma^2/2$, then $\tau^*=0$ and the optimal strategy is "stop immediately". The authors of [16] discuss the case $\sigma^2/2\le r\le\sigma^2$ in (1), obtaining that the solution is $\tau^*=T$ as well. Moreover, they consider problem (2) for all ratios of r and σ deriving that its solution $\tau_*=T$ if $r\ge\sigma^2$, $\tau_*=0$ if $r\le\sigma^2/2$ and there exists an increasing random boundary function which determines the

[1] The authors would like to thank professor A. Shiryaev for valuable comments.

optimal stopping moment if $\sigma^2/2 < r < \sigma^2$. In [7], authors extend these results to a problem of optimal buying of an asset, considering the minimum of the process together with its maximum. Concerning other works at the same direction, let us mention a paper [10], where geometric and arithmetic averages are discussed instead of the maximum in the problems (1) and (2), and works [8] and [9], in which authors solve an infinite time horizon problem of stopping as close as possible to the zero hitting time considering a mean-reverting diffusion process. As we mentioned above, instead of the geometric Brownian motion, in this paper we discuss exponential Lévy processes, which are very popular as a model of dynamics of assets (among others, see, for example, recent papers [1], [6], [15], [17], [18], [28] on pricing and hedging theory). The results relate to the exponentials of the Lévy processes, both problems (1) and (2), logarithmic and linear utilities. On empiric tests which support a suggestion that log-returns of assets have α-stable or generalized hyperbolic distributions, see papers [3], [11], [19], [22].

The paper is constructed as follows. Section 2 is dedicated to α-stable Lévy processes with drift and problem (1) is solved for linear utility in a case of a nonnegative drift. All $0 < \alpha \leq 2$ are discussed. Our result extends the result of [26] (a Brownian motion, $\alpha = 2$). In Section 3, we consider a time-changed Brownian motion. The results give full solution of problem (1) and extend results on (2) for geometric Brownian motion. Section 4 discusses a simpler case of logarithmic utility. Proofs are set in Section 5. The paper is completed by a list of literature.

2 α-stable Lévy processes

Let $Z = (Z_t)_{t\leq T}$ be a symmetric α-stable Lévy process with characteristic function, $\varphi_t(\theta) = \mathrm{E}e^{i\theta Z_t} = e^{-t|\theta|^\alpha}$, where $0 < \alpha < 2$. If $X^{(\alpha)}$ is the positive random variable with Laplace transform, $\mathrm{E}e^{-\lambda X^{(\alpha)}} = e^{-\lambda^\alpha}$, $\lambda > 0$, $0 < \alpha < 1$, it is not difficult to prove (see e. g. [25])

$$Z_t = B_{\tilde{T}(t)}, \quad t \leq T, \tag{3}$$

where $\tilde{T}(t)$ an $\alpha/2$ -stable subordinator with

$$Law(\tilde{T}(1)) = Law(X^{(\alpha/2)}). \tag{4}$$

Throughout this section, we model the price process of the asset S by the exponential Lévy process of the symmetric α-stable Lévy process with drift, i.e.,

$$H_t = Z_t + \mu t, \quad \mu \in \mathbb{R}. \tag{5}$$

For reasons of using of H as a model of evolution of log-returns of stock prices, we refer to [22]. Recalling proofs of results for the geometric Brownian motion ([16] and [26]), one can see that it is based on exploiting the closed form expression of the density of the maximum of the Brownian motion. However, some results which concern stable Lévy processes could be obtained without using of such forms.

Theorem 1 *Assume that H is an α-stable symmetric Lévy process with $0 < \alpha \leq 2$ and drift μ. If $\mu \geq 0$, the solution of (1) for $U(x) = x$ is time T.*

Example 1. If $\alpha = 2$, the 2-stable symmetric Lévy process is a Brownian motion $B = (B_t)_{t \leq T}$. As it is mentioned above, if the price of asset is supposed to be a geometric Brownian motion, i.e., $S_t = e^{\mu t + B_t}$, it was established (see [16] and [26]) that for $\mu \leq 0$ the optimal stopping moment is $\tau^* = 0$ and for $\mu \geq 0$ $\tau^* = T$ in (1). Therefore, the result of theorem 1 extends this result of [16] and [26] if $\mu \geq 0$.

3 Time-changed Brownian motion

Let $H = (H_t)_{t \leq T}$ be a time-changed Brownian motion with drift, i.e.,

$$H_t = \beta\gamma(t) + \sigma B_{\gamma(t)}, \tag{6}$$

where $\beta \in \mathbb{R}$, $\sigma > 0$ and random change of time (in sense of definition (a)–(b), p.109, [24]) γ is independent with B and satisfies condition $\mathrm{P}(\gamma(T) < \infty) = 1$.The next theorem follows.

Theorem 2 *Let U(x)=x. The solution of (1) is $\tau^* = T$ if $\beta \geq 0$ and $\tau^* = 0$ if $\beta < 0$. For problem (2), solution $\tau_* = T$ if $\beta \geq \sigma^2/2$ and $\tau_* = 0$ if $\beta \leq 0$.*

Example 2. Normal-inverse Gaussian process. A normal-inverse Gaussian distribution (NIG), introduced in [2] (see also [3] and [25]), is a normal variance-mean mixture where the mixing density is an independent inverse Gaussian distribution, i.e., the NIG random variable $H = H(\alpha, \beta, \delta)$ is defined as $H = \beta X + \sqrt{X}N$, where N is normally distributed and the density of X is

$$p_X(x) = \sqrt{\frac{b}{2\pi}} e^{\sqrt{ab}} \frac{1}{x^{3/2}} \exp\left(-\frac{1}{2}\left(ax + \frac{b}{x}\right)\right),$$

where $a = \alpha^2 - \beta^2$, $b = \delta^2$. Parameters α, β, δ are suggested to satisfy conditions; $\alpha > 0$, $0 \leq |\beta| < \alpha$ and $\delta \geq 0$.The density function f of H is

$$f(x) = \frac{\alpha\delta K_1\left(\alpha\sqrt{\delta^2 + x^2}\right)}{\pi\sqrt{\delta^2 + x^2}} e^{\delta\sqrt{\alpha^2 - \beta^2} + \beta x}, \tag{7}$$

where K_1 is modified Bessel function of the second type. The NIG process H_t is defined as a Lévy process such that H_1 has density (7). It is known, see for details [25], that for a Brownian motion $\tilde{B} = (\tilde{B}_s)_{s \geq 0}$, a change of time, $\tilde{T}(t) = \inf\{s > 0 : \tilde{B}_s + \sqrt{a}s \geq \sqrt{b}t\}$ and an independent Brownian motion $B = (B_t)_{t \geq 0}$ process H_t can be represented in form $H_t = B_{\tilde{T}(t)} + \beta\tilde{T}(t)$. Therefore, solutions of (1) and (2) for a NIG process do not depend on parameters α and δ, due to theorem 2.

Example 3. Variance-gamma process. A variance-gamma (VG) process $Y = (Y_t)_{t\leq T}$ can be written (see e. g. [21]) as a time-changed Brownian motion $B = (B_t)_{t\geq 0}$, where the random time change follows a gamma process $\Gamma(t;1,\nu)$, $\nu > 0$, i.e., $Y_t = \beta\Gamma(t;1,\nu)+\sigma B_{\Gamma(t;1,\nu)}$. Despite the fact that parameters $\beta \in \mathbb{R}$, $\sigma > 0$ and ν reflect only indirectly such parameters of the VG distribution as variance, skewness and kurtosis (it can be shown by straightforward calculation of moments of Y), we immediately use such parametrization of the VG process as above since it is usually used in literature, see [14], [19], [21]). As a model of distribution of market returns, the symmetric VG distribution was primarily studied in [19] and [20]. In [21], the general case of VG process with application to option pricing was discussed. For further investigations on the VG process, see [14] and [27].

In context of theorem 2, we have solutions of (1) and (2) for a VG process with respect to value of parameter β.

4 Logarithmic utility

In case of logarithmic utility, problems (1) and (2) can be rewritten as

$$\sup_{0\leq\tau\leq T} \mathrm{E}(H_\tau - \overline{H}_T)^q \text{ and } \inf_{0\leq\tau\leq T} \mathrm{E}(\overline{H}_T - H_\tau)^q,$$

respectively, with $q = 1$. For $q = 2$ these problems were discussed in [12] for a Brownian motion. Their result was extended to all $q > 0$ by [23]. For $q = 1$ and a Brownian motion with spontaneously changing drift, see [5] and [13].

Assume that H is a Lévy process which has decomposition

$$H_t = \mu t + \beta\varphi(t) + \sigma B_{\varphi(t)}, \tag{8}$$

where $\mu \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\sigma > 0$ and stochastic change of time (in sense of definition (a)–(b), p.109, [24]) φ satisfies condition $\mathrm{E}\sqrt{\varphi(T)} < \infty$. Since H is a Lévy process (8), it is submartingale if $\mathrm{E}H_t \geq 0$ and it is supermartingale if $\mathrm{E}H_t \leq 0$. Keeping in mind Hunt's stopping time theorem ((A.2), p.60, [24]) and Wald identity ((3.2.5), p.61, [24]), we conclude that solution of both problems (1) and (2) for logarithmic utility here is

$$\tau^* = \tau_* = T \quad \text{ if } \mathrm{E}\varphi(1) \geq -\frac{\mu}{\beta} \text{ and } \tau^* = \tau_* = T \quad \text{ if } \mathrm{E}\varphi(1) \leq -\frac{\mu}{\beta}.$$

In particular, for the VG process the solutions are time T if $\beta \geq -\mu$ and time T if $\beta \leq -\mu$.

5 Proofs

Proof of theorem 1. Set

$$\overline{H}_t = \max_{0\leq u\leq t} H_u \qquad \text{and} \qquad \overline{S}_t = \max_{0\leq u\leq t} S_u = e^{\overline{H}_t}. \tag{9}$$

Then problem (1) can be rewritten as $\sup_{0\le\tau\le T}\mathrm{E}(S_\tau/\overline{S}_T)$. Let $(\tilde{\Omega},\tilde{\mathcal{F}},(\tilde{\mathcal{F}}_t)_{t\le T})$ be a measurable space with filtration generated by $\alpha/2$-stable subordinator $\tilde{T}(t)$ defined by (4), i.e., $\tilde{\mathcal{F}}_t=\sigma\left(\tilde{T}(s),s\le t\right)$. Then due to (3) and (5) for any $\tau\le T$

$$\mathrm{E}\left(S_\tau/\overline{S}_T\right)=\mathrm{E}\left(\mathrm{E}\left(\frac{S_\tau}{\overline{S}_T}|\tilde{\mathcal{F}}_T\right)\right) \tag{10}$$

and for $\tilde{\omega}\in\tilde{\Omega}$,

$$\mathrm{E}\left(\frac{S_\tau}{\overline{S}_T}|\tilde{\mathcal{F}}_T\right)=\mathrm{E}\left(\exp\left(H_\tau-\overline{H}_T\right)|\tilde{\mathcal{F}}_T\right)=\mathrm{E}\left(\exp\left(B_{\tilde{T}(\tau)}+\mu\tau-\max_{t\le T}(B_{\tilde{T}(t)}+\mu t)\right)\right)(\tilde{\omega}).$$

One could observe that for a fixed $\tilde{\omega}\in\tilde{\Omega}$ there is bijection, $t\longleftrightarrow\tilde{T}(t)$ and we can define a time-deterministic for a.a. fixed $\tilde{\omega}$ process $\xi(s)=\mu\frac{\tilde{T}^{-1}(s)}{s}$, $s\le\tilde{T}(T)$. Next, for a deterministic drift $\lambda(t)$ and Brownian motion B set $B_t^\lambda=B_t+\lambda(t)t$ and $\overline{B}_t^\lambda=\max_{u\le t}(B_u+\lambda(u)u)$ and define by $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0})$ a measurable space which is determined by Brownian motion $(B_t)_{t\ge0}$. Then

$$\begin{aligned}
&\mathrm{E}\left(\exp\left(B_{\tilde{T}(\tau)}+\mu\tau-\max_{t\le T}(B_{\tilde{T}(t)}+\mu t)\right)\right)(\tilde{\omega})\\
&=\mathrm{E}\left(\exp\left(B_{\tilde{T}(\tau)}+\xi(\tilde{T}(\tau))\tilde{T}(\tau)-\max_{t\le T}(B_{\tilde{T}(t)}+\xi(\tilde{T}(t))\tilde{T}(t))\right)\right)(\tilde{\omega})\\
&=\mathrm{E}\left(\exp\left(B_{\tilde{T}(\tau)}+\xi(\tilde{T}(\tau))\tilde{T}(\tau)-\max_{t\le\tilde{T}(T)}(B_t+\xi(t)t)\right)\right)(\tilde{\omega})\\
&=\mathrm{E}\mathrm{E}\left(\min\left\{e^{B^\xi_{\tilde{T}(\tau)}-\overline{B}^\xi_{\tilde{T}(\tau)}},e^{-\max\limits_{\tilde{T}(\tau)\le t\le\tilde{T}(T)}\left(B^\xi_t-B^\xi_{\tilde{T}(\tau)}\right)}\right\}|\mathcal{F}_{\tilde{T}(\tau)}\right)(\tilde{\omega}).
\end{aligned}$$

Set

$$G^\xi(t,x)=\mathrm{E}\left(\min\left\{e^{-x},e^{-\max\limits_{t\le u\le\tilde{T}(T)}(B^\xi_u-B^\xi_t)}\right\}\right)(\tilde{\omega}).$$

Then

$$\mathrm{E}\left(\exp\left(H_\tau-\overline{H}_T\right)|\tilde{\mathcal{F}}_T\right)=\mathrm{E}\left(G^\xi\left(\tilde{T}(\tau),\overline{B}^\xi_{\tilde{T}(\tau)}-B^\xi_{\tilde{T}(\tau)}\right)\right). \tag{11}$$

At first, one can notice that for any $x\ge0$, some positive time-dependent drift $\eta=(\eta(s))_{s\ge0}$ and $t\le\tilde{T}(T)$

$$\begin{aligned}
G^\xi(t,x)=&\mathrm{E}\left(\min\left\{e^{-x},e^{-\max\limits_{t\le u\le\tilde{T}(T)}(B^\xi_u-B^\xi_t)}\right\}\right)(\tilde{\omega})=\\
&\mathrm{E}\left(\min\left\{e^{-x},e^{-\overline{B}^\eta_{\tilde{T}(T)-t}}\right\}\right)(\tilde{\omega})\le\mathrm{E}\left(\min\left\{e^{-x},e^{-\overline{B}_{\tilde{T}(T)-t}}\right\}\right)(\tilde{\omega})=G(t,x),
\end{aligned} \tag{12}$$

where $G(t,x)$ is defined by the last equality in (12) (and actually $G(t,x)=G^0(t,x)$). Next, as long as

$$\begin{aligned}
&G^\xi\left(\tilde{T}(T),\overline{B}^\xi_{\tilde{T}(T)}-B^\xi_{\tilde{T}(T)}\right)=\min\left\{e^{B^\xi_{\tilde{T}(T)}-\overline{B}^\xi_{\tilde{T}(T)}},1\right\}(\tilde{\omega})\ge\\
&\min\left\{e^{B_{\tilde{T}(T)}-\overline{B}_{\tilde{T}(T)}},1\right\}(\tilde{\omega})=G\left(\tilde{T}(T),\overline{B}_{\tilde{T}(T)}-B_{\tilde{T}(T)}\right),
\end{aligned}$$

we conclude that

$$\mathrm{E}\left(G^{\xi}(\tilde{T}(T),\overline{B}^{\xi}_{\tilde{T}(T)}-B^{\xi}_{\tilde{T}(T)})|\overline{B}^{\xi}_{t}-B^{\xi}_{t}=x\right)\geq\mathrm{E}\left(G(\tilde{T}(T),\overline{B}_{\tilde{T}(T)}-B_{\tilde{T}(T)})|\overline{B}_{t}-B_{t}=x\right). \tag{13}$$

Note that Proposition (i), Theorem 2.1, [26] ensures that

$$\mathrm{E}\left(G(\tilde{T}(T),\overline{B}_{\tilde{T}(T)}-B_{\tilde{T}(T)})|\overline{B}_{t}-B_{t}=x\right)\geq G(t,x).$$

Therefore, exploiting (12) and (13), we get that

$$\mathrm{E}\left(G^{\xi}(\tilde{T}(T),\overline{B}^{\xi}_{\tilde{T}(T)}-B^{\xi}_{\tilde{T}(T)})|\overline{B}^{\xi}_{t}-B^{\xi}_{t}=x\right)\geq G^{\xi}(t,x). \tag{14}$$

It follows from (14) that for all stopping times $\theta\leq\tilde{T}(T)$

$$\mathrm{E}\left(G^{\xi}(\tilde{T}(T),\overline{B}^{\xi}_{\tilde{T}(T)}-B^{\xi}_{\tilde{T}(T)})|\overline{B}^{\xi}_{\theta}-B^{\xi}_{\theta}\right)\geq G^{\xi}(\theta,\overline{B}^{\xi}_{\theta}-B^{\xi}_{\theta}). \tag{15}$$

holds. Therefore, we have from (11) that for all $\tau\leq T$

$$\mathrm{E}\left(\exp\left(H_{T}-\overline{H}_{T}\right)|\tilde{\mathcal{F}}_{T}\right)\geq\mathrm{E}\left(\exp\left(H_{\tau}-\overline{H}_{T}\right)|\tilde{\mathcal{F}}_{T}\right)$$

which concludes our proof because of (10). □

Proof of theorem 2. We omit it, because of the restriction of total pages.

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