

DEDEKIND SUMS AND VALUES OF L -FUNCTIONS AT POSITIVE INTEGERS

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ABSTRACT. In this paper, we study Dedekind sums and we connect them to the mean values of Dirichlet L -functions. For this, we introduce and investigate higher order dimensional Dedekind-Rademacher sums given by the expression

$$(1) \quad S_d(\vec{\mathbf{a}}_0, \vec{\mathbf{m}}_0) = \frac{1}{a_0^{m_0+1}} \sum_{k=1}^{a_0-1} \prod_{j=1}^d \cot^{(m_j)} \left(\frac{\pi a_j k}{a_0} \right),$$

where $\vec{\mathbf{a}}_0 = (a_0; a_1, \dots, a_d)$, $\vec{\mathbf{m}}_0 = (m_0; m_1, \dots, m_d)$, a_0, a_1, \dots, a_d are positive integers pairwise coprime and m_0, m_1, \dots, m_d are nonnegative integers. In this paper, we prove that the sums (1) are rational numbers, satisfy a Dedekind reciprocity type law, and their denominators have explicit and universal bounds. Our results recover and improve the well-known reciprocity and rationality theorems in [3, 13] and others. In connection with Dedekind sums we study the mean values of L -functions. For a given positive integer $q \geq 2$ and Dirichlet characters $\chi_1, \dots, \chi_d \pmod{q}$, we investigate the mean value of the twisted product

$$\bar{\chi}_1(a_1) \cdots \bar{\chi}_d(a_d) L(m_1 + 1, \chi_1) \cdots L(m_d + 1, \chi_d),$$

such that m_1, \dots, m_d have the same parity and

$$\chi_i(-1) = (-1)^{m_i+1}, i = 1, \dots, d$$

as an application of our Dedekind reciprocity law, for the non twisted case we give explicit formulae for this mean and we recover and improve the previous works of Walum [11], Louboutin Liu and Zhang [5, 6, 7, 14].

1. Higher dimensional Dedekind-Rademacher sums

Through this paper, for any $\vec{\mathbf{m}} = (m_0, \dots, m_d)$ be a $(d+1)$ -tuple of nonnegative integers, we denote by

$$|\vec{\mathbf{m}}| = \sum_{i=0}^d m_i, \vec{\mathbf{m}}! = \prod_{0 \leq i \leq d} m_i!, M = d + |\vec{\mathbf{m}}|.$$

Let us recall some definitions.

1.1. **Dedekind-Rademacher sums.** Let d, a_i be positive integers, $a_0, \dots, \widehat{a}_i, \dots, a_d$ are positive integers prime to a_i and m_0, \dots, m_d be non-negative integers. For $i = 0, \dots, d$, we consider the multiple Dedekind-Rademacher sum defined by

$$(2) \quad S_d(\vec{\mathbf{a}}_i, \vec{\mathbf{m}}_i) := \begin{cases} \frac{1}{a_i^{m_i+1}} \sum_{k=1}^{a_i-1} \prod_{\substack{j=0 \\ j \neq i}}^d \cot^{(m_j)} \left(\frac{\pi a_j k}{a_i} \right) & \text{if } a_i \geq 2, \\ 0 & \text{if } a_i = 1, \end{cases}$$

where $\vec{\mathbf{a}}_i = (a_i; a_0, \dots, \widehat{a}_i, \dots, a_d)$, $\vec{\mathbf{m}}_i = (m_i; m_0, \dots, \widehat{m}_i, \dots, m_d)$ and as usual \widehat{x}_n means we omit the term x_n . Throughout this paper, we set $M_i = d + \sum_{\substack{j \neq i \\ 0 \leq j \leq d}} m_j$, and \mathbb{N} denotes

the set of nonnegative integers.

1.2. **Bernoulli functions.** The Bernoulli polynomials $B_k(x)$ are defined through the generating function

$$(3) \quad \frac{ze^{xz}}{e^z - 1} = \sum_{k \geq 0} \frac{B_k(x)}{k!} z^k$$

and the Bernoulli numbers are $B_k := B_k(0)$. The Bernoulli functions $\bar{B}_k(x)$ are the periodized Bernoulli polynomials:

$$\bar{B}_k(x) := \begin{cases} 0 & , \text{ if } x \in \mathbb{Z}, k = 1; \\ B_k(\{x\}), & \text{ otherwise.} \end{cases}$$

2. Statement of the results on Dedekind sums

2.1. Rationality theorem.

Theorem 2.1.1. *Let d, a_0 be positive integers, a_1, \dots, a_d be positive integers prime to a_0 , and m_0, \dots, m_d be non-negative integers. We set $H = \frac{2^{M-m_0}}{i^{(M-m_0)(m_1+1)\dots(m_d+1)}}$. Then we have*

$$a_0^{m_0+1} H^{-1} S_d(\vec{a}_0, \vec{m}_0) = a_0^{M-m_0-d+1} \sum_{\substack{0 \leq n_1, \dots, n_d \leq a_0-1 \\ a_0 | n_1 a_1 + \dots + n_d a_d}} \bar{B}_{m_1+1} \left(\frac{n_1}{a_0} \right) \cdots \bar{B}_{m_d+1} \left(\frac{n_d}{a_0} \right) - \bar{B}_{m_1+1}(0) \cdots \bar{B}_{m_d+1}(0).$$

Remarks 2.1.2. *Since the coefficients of Bernoulli polynomials $B_n(x)$ are rationals, then the sum $S_d(\vec{a}_0, \vec{m}_0)$ is a rational number. The denominator of this rational number is given by the Theorem 2.5.1 below.*

2.2. Proof of the Theorem 2.1.1. We use the well-known lemma.

Lemma 2.2.1. *Let m be a non-negative integer, a be an integer ≥ 2 and k be an integer not divisible by a . Then we have*

$$(4) \quad \bar{B}_m(x) = -\frac{m!}{(2\pi i)^m} \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi i \ell x}}{\ell^m}$$

and

$$\cot^{(m-1)} \left(\frac{\pi k}{a} \right) = \frac{1}{ma} \left(\frac{2a}{i} \right)^m \sum_{n=0}^{a-1} e^{-2\pi i kn/a} \bar{B}_m \left(\frac{n}{a} \right).$$

To use this lemma, we set

$$A = \frac{1}{(m_1 + 1) \cdots (m_d + 1) a_0^d} \left(\frac{2a_0}{i} \right)^{M-m_0}$$

Then, we have

$$\begin{aligned} a_0^{m_0+1} S_d(\vec{a}_0, \vec{m}_0) &= A \sum_{t=1}^{a_0-1} \prod_{j=1}^d \sum_{n_j=0}^{a_0-1} \exp\left(\frac{-2\pi i n_j t a_j}{a_0}\right) \bar{B}_{m_j+1} \left(\frac{n_j}{a_0} \right) \\ &= A \sum_{t=1}^{a_0-1} \sum_{0 \leq n_1, \dots, n_d \leq a_0-1} \exp\left(\frac{-2\pi i t}{a_0} \left(\sum_{j=1}^d n_j a_j\right)\right) \prod_{j=1}^d \bar{B}_{m_j+1} \left(\frac{n_j}{a_0} \right). \end{aligned}$$

Since

$$\sum_{t=1}^{a_0-1} \exp\left(\frac{-2\pi i t}{a_0} \left(\sum_{j=1}^d n_j a_j\right)\right) = \begin{cases} a_0 - 1 & \text{if } a_0 | n_1 a_1 + \dots + n_d a_d \\ -1 & \text{otherwise,} \end{cases}$$

it follows that

$$\begin{aligned} a_0^{m_0+1} S_d(\vec{a}_0, \vec{m}_0) &= A \left(\sum_{\substack{0 \leq n_1, \dots, n_d \leq a_0-1 \\ a_0 | n_1 a_1 + \dots + n_d a_d}} (a_0 - 1) \prod_{j=1}^d \bar{B}_{m_j} \left(\frac{n_j}{a_0} \right) - \sum_{\substack{0 \leq n_1, \dots, n_d \leq a_0-1 \\ a_0 | n_1 a_1 + \dots + n_d a_d}} \prod_{j=1}^d \bar{B}_{m_j+1} \left(\frac{n_j}{a_0} \right) \right) \\ &= A \left(a_0 \sum_{\substack{0 \leq n_1, \dots, n_d \leq a_0-1 \\ a_0 | n_1 a_1 + \dots + n_d a_d}} \prod_{j=1}^d \bar{B}_{m_j+1} \left(\frac{n_j}{a_0} \right) - \sum_{0 \leq n_1, \dots, n_d \leq a_0-1} \prod_{j=1}^d \bar{B}_{m_j+1} \left(\frac{n_j}{a_0} \right) \right). \end{aligned}$$

Finally note that this last sum is equal to

$$\prod_{j=1}^d \sum_{n_j=0}^{a_0-1} \bar{B}_{m_j+1} \left(\frac{n_j}{a_0} \right) = \prod_{j=1}^d a_0^{-m_j} \bar{B}_{m_j+1}(0) = a_0^{-m_1 - \dots - m_d} \prod_{j=1}^d \bar{B}_{m_j+1}(0)$$

where we have used the classical Raabe formula [8].

This completes the proof of Theorem 2.1.1.

2.3. Dedekind Reciprocity Law.

Next we state the reciprocity law for these sums that allows us to compute them.

Theorem 2.3.1 ([1]). *Let d be a positive integer, a_0, \dots, a_d be pairwise coprime positive integers and $\vec{m} = (m_0, \dots, m_d)$ be a $(d + 1)$ -tuple of non-negative integers. Assume that $M = d + |\vec{m}|$ is even. Then we have*

$$\sum_{i=0}^d (-1)^{m_i} m_i! \sum_{(\ell_0, \dots, \ell_d)}^{*i} \left(\prod_{\substack{j=0 \\ j \neq i}}^d \frac{a_j^{\ell_j}}{\ell_j!} \right) S_d(\vec{a}_i, \vec{m}_i + \vec{L}_i) = \begin{cases} R + (-1)^{d/2} & , \text{ if all } m_i \text{ are zero;} \\ R & , \text{ otherwise} \end{cases}$$

where \sum^{*i} denotes summation over all $\ell_0, \dots, \ell_i, \dots, \ell_d \geq 0$ such that

$$|\vec{L}_i| = m_i, \vec{L}_i = (\ell_i; \ell_0, \dots, \ell_i, \dots, \ell_d)$$

and

$$(5) \quad R = \frac{(-1)^{M/2} 2^M}{\prod_{i=0}^d a_i^{m_i+1}} \sum_{\substack{j_0, \dots, j_d \geq 0 \\ j_0 + \dots + j_d = M/2}} \prod_{i=0}^d a_i^{2j_i} A_{i, j_i}.$$

and

$$A_{i, j_i} = \begin{cases} \frac{B_{2j_i}}{(2j_i - 1 - m_i)!(2j_i)} & \text{if } j_i \text{ is an integer } \geq (m_i + 1)/2, \\ (-1)^{m_i} m_i! & \text{if } j_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Example. When all m_i are zero, we have $M = d$ and $A_{i, j_i} = \frac{(-1)^{j_i} 2^{2j_i} B_{2j_i}}{(2j_i)!}$, hence the right member of the reciprocity formula in Theorem 2.3.1 becomes

$$(6) \quad R + (-1)^{d/2} = (-1)^{d/2} \left(1 - \frac{2^d}{a_0 \dots a_d} \sum_{\substack{j_0, \dots, j_d \geq 0 \\ j_0 + \dots + j_d = d/2}} \prod_{i=0}^d \frac{B_{2j_i}}{(2j_i)!} a_i^{2j_i} \right).$$

2.4. Proof of the reciprocity Theorem 2.3.1. Let us consider the function f of the complex variable z defined by

$$f(z) = \prod_{j=0}^d \cot^{(m_j)}(\pi a_j z).$$

Let ε be a fixed real number with $\varepsilon \in]0, \min_{0 \leq j \leq d} 1/a_j[$. Let $y > 0$ be a real parameter. We set $A = (1 - \varepsilon) + yi$, $B = -\varepsilon + yi$, $C = \overline{B}$ and $D = \overline{A}$, and we consider the rectangular path $\gamma := [A, B, C, D, A]$. We want to integrate f along γ by applying Cauchy's Residue Theorem. The poles of f lying inside γ are:

- the point $z_0 = 0$ which is a pole of order $M + 1$;
- the points k_j/a_j , where $k_j = 1, \dots, a_j - 1$, $a_j \neq 1$ and $j = 0, \dots, d$, which are distinct since the integers a_j are pairwise coprime. Every point k_j/a_j is a pole of f of order $(m_j + 1)$. By Cauchy's Residue Theorem, we have

$$\frac{1}{2\pi i} \int_{\gamma} f(z) \overline{z} = \text{Res}(f, 0) + \sum_{j=0}^d \sum_{k=1}^{a_j-1} \text{Res}(f, k/a_j).$$

Since 1 is a period of f , we see that

$$\int_{[D,A]} f(z) \overline{z} = - \int_{[B,C]} f(z) \overline{z}.$$

Furthermore, setting $\delta = \pm 1$, we have for all real t

$$\lim_{y \rightarrow +\infty} \cot^{(m)}(t + \delta yi) = \begin{cases} -\delta i & \text{if } m = 0, \\ 0 & \text{if } m \geq 1. \end{cases}$$

Hence

$$\int_{\gamma} f(z) \overline{z} = \begin{cases} 2i^{d+1} & \text{if all } m_i \text{ are zero and } d \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

- Note that if $a_j = 1$ the sum over k is equal to 0.

Therefore, we obtain

$$(7) \quad \sum_{j=0}^d \sum_{k=1}^{a_j-1} \text{Res}(f, k/a_j) = \begin{cases} -\text{Res}(f, 0) + i^d/\pi & \text{if all } m_i \text{ are zero and } d \text{ is even,} \\ -\text{Res}(f, 0) & \text{otherwise.} \end{cases}$$

Now, we need to evaluate the two sides of (7).

1. Residue of f at $z = 0$. The Laurent expansion of the cotangent at 0:

$$\cot(w) = \frac{1}{w} + \sum_{j=1}^{+\infty} \frac{(-1)^j 2^{2j} B_{2j}}{(2j)!} w^{2j-1} \quad (0 < |w| < \pi)$$

implies

$$w^{m+1} \cot^{(m)}(w) = (-1)^m m! + \sum_{\substack{j \text{ integer} \\ 2j-m \geq 1}} \frac{(-1)^j 2^{2j-1} B_{2j}}{(2j-1-m)! j} w^{2j}$$

for every integer $m \geq 1$.

For $i \in \{0, \dots, d\}$, let us set

$$(8) \quad A_{i,j_i} = \begin{cases} \frac{B_{2j_i}}{(2j_i-1-m_i)!(2j_i)} & \text{if } j_i \text{ integer } \geq (m_i+1)/2, \\ (-1)^{m_i} m_i! & \text{if } j_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So, we have

$$\begin{aligned} \text{Res}(f, 0) &= \pi^{-M-1} \prod_{i=0}^d a_i^{-m_i-1} \sum_{\substack{(j_0, \dots, j_d) \in \mathbb{N}^{d+1} \\ j_0 + \dots + j_d = M/2}} \prod_{i=0}^d (-1)^{j_i} (2\pi a_i)^{2j_i} A_{i,j_i} \\ &= \frac{(-1)^{M/2} 2^M}{\pi \prod_{i=0}^d a_i^{m_i+1}} \sum_{\substack{(j_0, \dots, j_d) \in \mathbb{N}^{d+1} \\ j_0 + \dots + j_d = M/2}} \prod_{i=0}^d a_i^{2j_i} A_{i,j_i}. \end{aligned}$$

2. Residue of f at the other poles. For any integer $a_i > 1$ and $1 \leq k \leq a_i - 1$, we have

$$\text{Res}(f, k/a_i) = (-1)^{m_i} \frac{m_i!}{a_i^{m_i+1} \pi} \sum_{\substack{(\ell_0, \dots, \ell_d) \in \mathbb{N}^d \\ \ell_0 + \dots + \ell_i + \dots + \ell_d = m_d}} \prod_{\substack{j=0 \\ j \neq i}}^d \frac{a_j^{\ell_j}}{\ell_j!} \cot^{(m_j+\ell_j)} \left(\frac{\pi k a_j}{a_i} \right).$$

Consequently, we have obtained the relation

$$\sum_{\substack{i=0 \\ a_i \neq 1}}^d \sum_{k=1}^{a_i-1} \text{Res}(f, k/a_i) = \frac{1}{\pi} \sum_{i=0}^d (-1)^{m_i} \frac{m_i!}{a_i^{m_i+1}} \sum_{\substack{(\ell_0, \dots, \ell_d) \in \mathbb{N}^d \\ \ell_0 + \dots + \ell_i + \dots + \ell_d = m_d}} \left(\prod_{\substack{j=0 \\ j \neq i}}^d \frac{a_j^{\ell_j}}{\ell_j!} \right) \sum_{k=1}^{a_i-1} \prod_{\substack{j=0 \\ j \neq i}}^d \cot^{(m_j+\ell_j)} \left(\frac{\pi k a_j}{a_i} \right).$$

2.5. Universal Bounds.

In the following theorem we study the universal bound for the denominator of the higher order dimensional Dedekind sums.

Theorem 2.5.1. *Let d, a_0 be positive integers, a_0, a_1, \dots, a_d be positive integers relatively prime to a_0 and m_0, \dots, m_d be non-negative integers. We set*

$$\mu := \prod_{\substack{3 \leq p \leq M+1 \\ p \text{ prime}}} p^{\lfloor \frac{M}{p-1} \rfloor}, \quad \Delta := \gcd \left(\mu; a_0^{d-1} (m_1+1) \cdots (m_d+1) \prod_{j=1}^d \prod_{\substack{p \leq m_j \\ p \text{ prime} \geq 3}} p \right).$$

Then we have

$$a_0 S_d(\vec{\mathbf{a}}_0, \vec{\mathbf{m}}_0) \in \frac{2^{m_1+\dots+m_d}}{\Delta} \mathbb{Z}.$$

Remark 7. The reason, that we are interested in μ and Δ is that these are the **universal bounds** for the denominator of our higher order dimensional Dedekind sums. For any d, a_0 be a positive integer, a_1, \dots, a_d be positive integers prime to a_0 we obtain

$$a_0 \Delta S_d(\vec{\mathbf{a}}_0, \vec{\mathbf{m}}_0) \in 2^{m_1+\dots+m_d} \mathbb{Z}.$$

For instance, if $m_0 = \dots = m_d = 0$, we obtain $a_0 \mu S_d(\vec{\mathbf{a}}_0, \vec{\mathbf{m}}_0) \in \mathbb{Z}$, this is the rationality theorem of Zagier [13, p.160]. Our method gives us a simple and new way to get this theorem of Zagier.

2.6. **Proof of the universal bound Theorem 2.5.1.** For the classical von Staudt-Clausen theorem we can see [4, 9, 12]. For any non negative integer m and any prime number p , let $v_p(m)$ denote the p -adic valuation of m . We have the useful lemma.

Lemma 2.6.1. *Let n be an integer ≥ 1 . We denote by D_n the denominator of $\frac{B_n}{n!}$. For any prime p , we have*

$$(9) \quad v_p(D_n) \leq \left\lfloor \frac{n}{p-1} \right\rfloor.$$

Proof. Using the classical von Staudt theorem, it's easy to see that

$$v_p(\text{denominator of } B_n) = \begin{cases} 1 & , \text{ if } p-1|n; \\ 0 & , \text{ otherwise.} \end{cases}$$

On the other hand, we have the well known fact. For every prime number p

$$v_p(n!) \leq \begin{cases} \lfloor n/(p-1) \rfloor & , \text{ if } p-1 \text{ does not divide } n; \\ \lfloor n/(p-1) \rfloor - 1 & , \text{ if } p-1 \text{ divides } n. \end{cases}$$

This yields the desired lemma.

Proof of Theorem 2.5.1. From the Theorem 2.1.1 we study the denominator of $a_0^{m_0+1} S_d(\vec{a}_0, \vec{m}_0)$. Let D' be the denominator of $\sum_{\substack{0 \leq n_1, \dots, n_d \leq a_0-1 \\ a_0 | n_1 a_1 + \dots + n_d a_d}} \bar{B}_{m_1+1} \left(\frac{n_1}{a_0} \right) \cdots \bar{B}_{m_d+1} \left(\frac{n_d}{a_0} \right)$.

Then $D'|D_1 \cdots D_d$ where D_j is the denominator of $\bar{B}_{m_j+1} \left(\frac{n_j}{a_0} \right)$ ($j = 1, \dots, d$). If $(m_j, n_j) \neq (0, 0)$, we have

$$\begin{aligned} \bar{B}_{m_j+1} \left(\frac{n_j}{a_0} \right) &= B_{m_j+1} \left(\frac{n_j}{a_0} \right) = \sum_{k=0}^{m_j+1} \binom{m_j+1}{k} \left(\frac{n_j}{a_0} \right)^{m_j+1-k} B_k \\ &= \frac{1}{a_0^{m_j+1}} \left(n_j^{m_j+1} + \sum_{k=1}^{m_j+1} \binom{m_j+1}{k} a_0^k n_j^{m_j+1-k} B_k \right). \end{aligned}$$

By von Staudt's Theorem, we know that if k is even, the denominator of B_k is $\prod_{\substack{p \text{ prime} \\ p-1|k}} p$,

and therefore

$$(10) \quad D_j | a_0^{m_j+1} \prod_{\substack{p \leq m_j+2 \\ p \text{ prime}}} p \quad (j = 1, \dots, d).$$

Thus we obtain

$$(11) \quad D' | 2^d a_0^{\sum_{j=1}^d (m_j+1)} \prod_{j=1}^d \prod_{\substack{p \leq m_j+2 \\ p \text{ prime} \geq 3}} p.$$

Furthermore, if D'' is the denominator of $\bar{B}_{m_1+1}(0) \cdots \bar{B}_{m_d+1}(0)$, all $m_j \neq 0$ and all m_j+1 are even, then we have

$$(12) \quad D'' | 2^d \prod_{\substack{p \leq m_j+2 \\ p \text{ prime} \geq 3}} p.$$

So it follows that

$$(13) \quad a_0^{m_0+1} S_d(\vec{\mathbf{a}}_0, \vec{\mathbf{m}}_0) = \frac{2^{m_1+\dots+m_d} N_0}{(m_1+1)\dots(m_d+1) a_0^{d-1} \prod_{j=1}^d \prod_{\substack{p \leq m_j+2 \\ p \text{ prime} \geq 3}} p},$$

where $N_0 \in \mathbb{Z}$.

Obviously, (13) can be written as

$$a_0^{d+m_0} \frac{(m_1+1)\dots(m_d+1)}{2^{m_1+\dots+m_d}} \left(\prod_{j=1}^d \prod_{\substack{p \leq m_j+2 \\ p \geq 3 \text{ prime}}} p \right) S_d(\vec{\mathbf{a}}_0, \vec{\mathbf{m}}_0) \in \mathbb{Z}.$$

End of the proof of Theorem 2.5.1. We shall now apply Theorem 2.3.1. We begin by giving the denominator of the rational number R defined by (5). Write

$$\prod_{i=0}^d a_i^{2j_i} A_{i,j_i} = \frac{A}{(2j_0)! \dots (2j_d)!} B_{2j_0} \dots B_{2j_d}$$

where $A \in \mathbb{Z}$ and denote by D the denominator of this rational number. By Lemma 2.6.1, we have for all prime numbers p ,

$$v_p(D) \leq \sum_{i=0}^d \left[\frac{2j_0}{p-1} \right] \leq \left[\sum_{i=0}^d \frac{2j_0}{p-1} \right] = \left[\frac{M}{p-1} \right].$$

Let

$$\mu := \prod_{\substack{2 < p \leq M+1 \\ p \text{ prime}}} p^{\left[\frac{M}{p-1} \right]}.$$

It follows that

$$(14) \quad D | 2^M \mu.$$

It is then easy to deduce that the number R defined by (5) can be written as

$$R = \frac{N'_1}{\mu \prod_{i=0}^d a_i^{m_i+1}} \quad (N'_1 \in \mathbb{Z}).$$

Therefore, by Theorem 2.3.1 we can write

$$(15) \quad \sum_{i=0}^d (-1)^{m_i} m_i! \sum_{\substack{\ell_0, \dots, \ell_i, \dots, \ell_d \geq 0 \\ \ell_0 + \dots + \ell_i + \dots + \ell_d = m_i}} \prod_{\substack{j=0 \\ j \neq i}}^d \frac{a_j^{\ell_j}}{\ell_j!} S_d(\vec{\mathbf{a}}_i, \vec{\mathbf{m}}_i + \vec{\mathbf{L}}_i) = \frac{N}{\mu \prod_{i=0}^d a_i^{m_i+1}}, \quad (N \in \mathbb{Z}).$$

If we apply a formula similar to (13), we can write for some $N_i \in \mathbb{Z}$

$$(16) \quad \begin{aligned} & a_i^{m_i+1} S_d(\vec{\mathbf{a}}_i, \vec{\mathbf{m}}_i + \vec{\mathbf{L}}_i) \\ &= \frac{2^{\sum_{j \neq i} m_j + \ell_j} N_i}{a_i^{d-1} \prod_{j \neq i} (m_j + \ell_j + 1) \prod_{j \neq i} \prod_{\substack{p \leq m_j + \ell_j \\ p \text{ prime} \geq 3}} p} \\ &= \frac{2^{m_0 + \dots + m_d} N_i}{a_i^{d-1} \prod_{j \neq i} (m_j + \ell_j + 1) \prod_{j \neq i} \prod_{\substack{p \leq m_j + \ell_j \\ p \text{ prime} \geq 3}} p} \end{aligned}$$

under the condition $\sum_{\substack{j=0 \\ j \neq i}}^d \ell_j = m_i$.

We note that $\mu / \prod_{\substack{j=0 \\ j \neq i}}^d \prod_{\substack{p \leq m_j + \ell_j \\ p \geq 3 \text{ prime}}} p \in \mathbb{N}$, because $m_j + \ell_j \leq M + 1$ and the number of j such

that $p \leq m_j + \ell_j$ is less than $\left\lceil \frac{1}{p} \sum_{j \neq i} m_j + \ell_j \right\rceil = \left\lceil \frac{M-d}{p} \right\rceil$. Therefore, we can write the quantity in (15) as follows

$$2^{m_0 + \dots + m_d} \sum_{i=0}^d (-1)^{m_i} m_i! \sum_{\substack{\ell_0, \dots, \ell_i, \dots, \ell_d \geq 0 \\ \ell_0 + \dots + \ell_i + \dots + \ell_d = m_i}} \prod_{\substack{j=0 \\ j \neq i}}^d \frac{a_j^{m_j + 1 + \ell_j}}{\ell_j!} \frac{1}{\prod_{\substack{j=0 \\ j \neq i}}^d (m_j + \ell_j + 1)} \frac{\mu}{\prod_{\substack{j=0 \\ j \neq i}}^d \prod_{\substack{p \leq m_j + \ell_j \\ p \text{ prime} \geq 3}} p} \frac{N_i}{a_i^{d-1}} \in \mathbb{Z}.$$

This gives

$$2^{m_0 + \dots + m_d} \sum_{i=0}^d (-1)^{m_i} m_i! \sum_{\substack{\ell_0, \dots, \ell_i, \dots, \ell_d \geq 0 \\ \ell_0 + \dots + \ell_i + \dots + \ell_d = m_i}} \prod_{\substack{j=0 \\ j \neq i}}^d \frac{a_j^{m_j + 1 + \ell_j}}{\ell_j!} \frac{T}{\prod_{\substack{j=0 \\ j \neq i}}^d (m_j + \ell_j + 1)} \frac{\mu}{\prod_{\substack{j=0 \\ j \neq i}}^d \prod_{\substack{p \leq m_j + \ell_j \\ p \text{ prime} \geq 3}} p} \frac{N_i}{a_i^{d-1}} \in T\mathbb{Z}$$

where T is the least common multiple of the numbers

$$\prod_{\substack{j=0 \\ j \neq i}}^d (m_j + \ell_j + 1) : i = 0, \dots, d; \ell_0 + \dots + \ell_i + \dots + \ell_d = m_i.$$

Since the integers a_i are pairwise coprime, it follows that for each $i = 0, \dots, d$

$$2^{m_0 + \dots + m_d} \frac{m_i!}{\prod_{\substack{j=0 \\ j \neq i}}^d \ell_j!} \frac{T}{\prod_{\substack{j=0 \\ j \neq i}}^d (m_j + \ell_j + 1)} \frac{\mu}{\prod_{\substack{j=0 \\ j \neq i}}^d \prod_{\substack{p \leq m_j + \ell_j \\ p \text{ prime} \geq 3}} p} \frac{N_i}{a_i^{d-1}} \in T\mathbb{Z}.$$

For instance, if $i = 0$ we have

$$(17) \quad a_0^{d-1} \mid \frac{N_0}{\prod_{j=1}^d (m_j + 1)} \frac{\mu}{\prod_{\substack{j=1 \\ p \text{ prime} \geq 3}}^d \prod_{p \leq m_j} p} \text{ since } m_0 = 0 \text{ and } \ell_1 = \dots = \ell_d = 0.$$

By (13) and (17) we thus arrive at

$$(18) \quad \begin{cases} a_0^{d-1} (m_1 + 1) \dots (m_d + 1) \left(\prod_{j=1}^d \prod_{\substack{p \leq m_j \\ p \geq 3}} p \right) a_0 S_d(\vec{a}_0, \vec{m}_0) \in 2^{m_1 + \dots + m_d} \mathbb{Z} \\ \mu a_0 S_d(\vec{a}_0, \vec{m}_0) \in 2^{m_1 + \dots + m_d} \mathbb{Z}. \end{cases}$$

This clearly implies the desired result that

$$(19) \quad a_0 S_d(\vec{a}_0, \vec{m}_0) \in \frac{2^{m_1 + \dots + m_d}}{\Delta} \mathbb{Z},$$

where $\Delta = \gcd\left(\mu; a_0^{d-1}(m_1 + 1) \dots (m_d + 1) \prod_{j=1}^d \prod_{\substack{p \leq m_j \\ p \text{ prime} \geq 3}} p\right)$.

3. Twisted mean values of L -functions: Introduction

Let q be a positive integer ≥ 2 and χ be a character modulo q , and $L(s, \chi)$ be the Dirichlet L -function corresponding to χ :

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s},$$

where $\Re(s) > 0$ if χ is non principal and $\Re(s) > 1$ if χ is the principal character. Let m_1, \dots, m_d be non negative integers. We shall here be interested by the study of the mean values

$$\sum_{(\chi_1, \dots, \chi_d)}^* \prod_{i=1}^d \bar{\chi}_i(a_i) L(m_i + 1, \chi_i),$$

where \sum^* denotes summation over all characters $\chi_1, \dots, \chi_d \pmod{q}$ such that:

$$\chi_1 \dots \chi_d = 1, \chi_1(-1) = \dots = \chi_d(-1) = (-1)^{m_1+1} = \dots = (-1)^{m_d+1}.$$

Its well-known that in the case $d = 2, m_1 = m_2 = 0$ and $\chi_2 = \bar{\chi}_1$, Walum [11] showed that for prime $q = p \geq 3$, the explicit formula

$$(20) \quad \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1(-1) = -1}} |L(1, \chi_1)|^2 = \frac{\pi^2(p-1)^2(p-2)}{12p^2}.$$

This result has extended by Louboutin [5] and Zhang [14] to any positive integer $q \geq 2$ by the formula as follows

$$(21) \quad \sum_{\substack{\chi_1 \pmod{q} \\ \chi_1(-1) = -1}} |L(1, \chi_1)|^2 = \frac{\pi^2 \varphi^2(q)}{12 q^2} \left(q \prod_{\substack{p|q \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right) - 3 \right)$$

where $\varphi(q)$ is the Euler function. Moreover, Louboutin [6] has considered the case $d = 2, m_1 = m_2 = k$ and proved the formula

$$(22) \quad \frac{2}{\varphi(q)} \sum_{\substack{\chi_1 \pmod{q} \\ \chi_1(-1) = -1}} |L(k, \chi_1)|^2 = \frac{(2\pi)^{2k}}{2((k-1)!)^2} \sum_{l=0}^{2k} r_{k,l} \varphi_l(q) q^{l-2k},$$

where

$$\varphi_l(q) := \prod_{\substack{p|q \\ p \text{ prime}}} \left(1 - \frac{1}{p^l}\right),$$

and the coefficients $r_{k,l}$ are real numbers that were not given explicitly. In 2006, Liu and Zhang [7] treated the mean values of $L(m, \chi_1)L(n, \bar{\chi}_1)$ at positive integers $m, n \geq 1$,

$$(23) \quad \frac{2}{\varphi(q)} \sum_{\substack{\chi_1 \pmod{q} \\ \chi_1(-1) = -1}} L(m, \chi_1)L(n, \bar{\chi}_1) = \frac{(-1)^{\frac{m-n}{2}} (2\pi)^{m+n}}{2(m!n!)} \left(\sum_{l=0}^{m+n} r_{m,n,l} \varphi_l(q) q^{l-m-n} - \frac{\epsilon_{m,n}}{q} B_m B_n \varphi_{m+n-1}(q) \right).$$

where

$$r_{m,n,l} = B_{m+n-l} \sum_{a=0}^m \sum_{b=0}^n B_{m-a} B_{n-b} \frac{\binom{m}{a} \binom{n}{b} \binom{a+b+1}{m+n-l}}{a+b+1}.$$

4. Statement of results on mean values of L -functions

We have the interesting results

Theorem 4.0.2. *Let d be an integer ≥ 1 and $\vec{m} = (m_1, \dots, m_d)$ a d -tuple of positive integers such that $d + |\vec{m}|$ is even. Let q be an integer ≥ 2 . Let a_1, \dots, a_{d-1} be positive such that $(a_i, q) = 1$ ($i = 1, \dots, d - 1$). We set $a_d = 1$. Then we have*

$$\sum_{(\chi_1, \dots, \chi_d)}^* \prod_{i=1}^d \bar{\chi}_i(a_i) L(m_i + 1, \chi_i) = A_q(\vec{m}) \sum_{\substack{b|q \\ b \neq 1}} b \mu\left(\frac{q}{b}\right) S_d(\vec{a}_0, \vec{m}_0)$$

where

$$A_q(\vec{m}) = \frac{(-1)^d}{2^d (\vec{m}!) } \left(\frac{\pi}{q}\right)^M \varphi(q)^{d-1}, \vec{a}_0 = (b; a_1, \dots, a_d), \vec{m}_0 = (0; m_1, \dots, m_d)$$

The above theorem gives immediately

Corollary 4.0.3. *Let m and n be positive having same parity. Let a be a positive integer such that $(a, q) = 1$. Then we have*

$$\sum_{\substack{\chi \pmod{q} \\ \chi(-1) = (-1)^{m+1}}} \bar{\chi}(a) L(m + 1, \chi) L(n + 1, \bar{\chi}) = A \sum_{\substack{b|q \\ b \neq 1}} b \mu(q/b) S_d(\vec{a}_0, \vec{m}_0)$$

where $A = \frac{\varphi(q)}{4m!n!} \left(\frac{\pi}{q}\right)^{m+n+2}$, $\vec{a}_0 = (b; a, 1)$, $\vec{m}_0 = (0; m, n)$.

For every real $\alpha > 0$, let J_α be the Jordan's totient function defined for all positive integer n by :

$$J_\alpha(n) := n^\alpha \sum_{m|n} \frac{\mu(m)}{m^\alpha},$$

where μ is the Mobius function. Since the arithmetical function $J_\alpha(n)/n^\alpha$ is multiplicative, we can write

$$J_\alpha(n) = n^\alpha \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p^\alpha}\right), \text{ see [10, p.11,p.219].}$$

For $\alpha = 1$, this is, of course, Euler's function φ .

For $a_1 = \dots = a_d = 1$ from Theorem 4.0.2 and Theorem 2.3.1, we obtain the following theorem

Theorem 4.0.4. *Let q be an integer ≥ 2 . Let d be an integer ≥ 1 and $\vec{m} = (m_1, \dots, m_d)$ a d -tuple of positive integers such that the number $M := d + |\vec{m}|$ is even. Then*

i) if $(m_1, \dots, m_d) \neq (0, \dots, 0)$ we have

$$\sum_{(\chi_1, \dots, \chi_d)}^* \prod_{i=1}^d L(m_i + 1, \chi_i) = D_q(\vec{m}) \left(\sum_{j_0=1}^{M/2} \left(\sum_{\substack{j_1, \dots, j_d \geq 0 \\ j_1 + \dots + j_d = M/2 - j_0 \\ j_i = 0 \text{ ou } \geq (m_i + 1)/2}} \prod_{i=1}^d A_{i, j_i} \right) \frac{B_{2j_0}}{(2j_0)!} J_{2j_0}(q) \right)$$

where

$$D_q(\vec{m}) = (-1)^{M/2} 2^M A_q(\vec{m}).$$

ii) for $(m_1, \dots, m_d) = (0, \dots, 0)$ we have

$$\sum_{(\chi_1, \dots, \chi_d)}^* \prod_{i=1}^d L(1, \chi_i) = D_q(\vec{0}) \left(2^{-d} \varphi(q) - \sum_{j_0=1}^{d/2} \left(\sum_{\substack{j_1, \dots, j_d \geq 0 \\ j_1 + \dots + j_d = d/2 - j_0}} \prod_{i=1}^d \frac{B_{2j_i}}{(2j_i)!} \right) \frac{B_{2j_0}}{(2j_0)!} J_{2j_0}(q) \right).$$

where $D_q(\vec{0}) = (-1)^{d/2} \left(\frac{\pi}{q}\right)^d \varphi(q)^{d-1}$.

As an immediate consequence, taking $d = 2, m_1 = m_2 = 0$ we obtain a sensitive improvement of Louboutin, Liu and Zhang results [5, 6, 14, 7].

Theorem 4.0.5. *Let m and n be two positive integers having same parity. Then*

• *If $(m, n) \neq (1, 1)$, we have*

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi \\ \chi(-1) = (-1)^m}} L(m, \chi) L(n, \bar{\chi}) = \frac{1}{2} (-1)^{\frac{m+n}{2}} \left(\frac{2\pi}{q}\right)^{m+n} (M_1 + M_2 + M_3)$$

where

$$\begin{aligned} M_1 &= \frac{B_{m+n}}{(m+n)!} J_{m+n}(q), \\ M_2 &= \frac{(-1)^{m-1}}{(n-1)!m!} \sum_{j=1}^{[m/2]} \binom{m}{2j} \frac{B_{m+n-2j}}{m+n-2j} B_{2j} J_{2j}(q), \\ M_3 &= \frac{(-1)^{n-1}}{(m-1)!n!} \sum_{j=1}^{[n/2]} \binom{n}{2j} \frac{B_{m+n-2j}}{m+n-2j} B_{2j} J_{2j}(q). \end{aligned}$$

• *If $m = n = 1$, we have*

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi \\ \chi(-1) = -1}} |L(1, \chi)|^2 = \frac{\pi^2}{6} \frac{\varphi(q)}{q^2} \left(q \prod_{\substack{p|q \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right) - 3 \right).$$

Proof. For the proof we refer to [2]. □

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