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A computer experiment on primitive stable representations

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1. Character variety and mapping class group

In this section, we briefly introduce the notion of character variety and the actions on it.

1.1. Character variety. Let π be a finitely generated group, and G a Lie group. Let us consider the representation space $\mathrm{Hom}(\pi, G)$. Note that G acts on $\mathrm{Hom}(\pi, G)$ by post-composition of conjugation, that is, for $g \in G$ and $\rho \in \mathrm{Hom}(\pi, G)$, $g(\rho) := \iota_g \circ \rho$, where $\iota_g : x \mapsto gxg^{-1}$.

The *character variety* is the (categorical) quotient of $\mathrm{Hom}(\pi, G)$ by this G action and denoted by $\mathcal{X} = \mathcal{X}(\pi) = \mathrm{Hom}(\pi, G)/G$.

Each element α of the automorphism group $\mathrm{Aut}(\pi)$ of π also acts on the representation space $\mathrm{Hom}(\pi, G)$ by pre-composition $\alpha(\rho) := \rho \circ \alpha^{-1}$. Note that the inner-automorphism group $\mathrm{Inn}(\pi)$ acts trivially on the quotient $\mathrm{Hom}(\pi, G)/G$. Thus, the outer-automorphism group

$$\mathrm{Out}(\pi) := \mathrm{Aut}(\pi)/\,\mathrm{Inn}(\pi)$$

acts on the character variety.

We are interested in the dynamics of this action.

1.2. Surface groups and mapping class groups. From now on, we assume that the group π is the fundamental group of a surface Σ, that is $\pi = \pi_1(\Sigma)$. The mapping class group $\mathrm{MCG}(\Sigma) := \pi_0(\mathrm{Diff}^+(\Sigma))$ of Σ is a index two subgroup of $\mathrm{Out}(\pi)$.

When Σ has $n > 0$ boundaries $\partial_1, \partial_2, \ldots, \partial_n$, there is a boundary restriction map

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G \to \prod_i^n \mathrm{Hom}(\pi_1(\partial_i), G)/G$$

The fibers of this map are called *relative character variety.* Now, We have:

Proposition 1.1. *The action of* $\mathrm{MCG}(\Sigma)$ *on* $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ *preserves the fibers.*

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2. One-holed torus and $\mathrm{SL}(2,\mathbb{C})$-character variety

In this section, we consider the case when Σ is a one-holed torus $\Sigma_{1,1}$ and $G = \mathrm{SL}(2,\mathbb{C})$. Then, $\pi = \pi_1(\Sigma_{1,1}) = F_2$, the free group of rank two. We fix a standard generators for $F_2 = \langle X, Y\rangle$, so that the commutator $XYX^{-1}Y^{-1}$ corresponds to the boundary (hole). Then the character variety is identified with $\mathbb{C}^3$ by the following identification:

$$\begin{aligned}\chi : \mathrm{Hom}(\pi, G)/G &\to \mathbb{C}^3\\ \chi([\rho]) &\mapsto (\mathrm{tr}(\rho(X)), \mathrm{tr}(\rho(Y)), \mathrm{tr}(\rho(XY))).\end{aligned}$$

Remark 2.1. *Let us consider the lift of the identification map:* $\widetilde{\chi} : \mathrm{Hom}(\pi, G) \to \mathbb{C}^3$ *so that* $\widetilde{\chi}(\rho) := (\mathrm{tr}(\rho(X)), \mathrm{tr}(\rho(Y)), \mathrm{tr}(\rho(XY)))$.

(1) If $\chi(\rho) = \chi(\rho') = (x, y, z)$ *with* $\kappa(x, y, z) \neq 2$, *then* $g(\rho) = \rho'$ *for some* $g \in G$. *(Recall that g acts on the representation space by post-composition of conjugation.)*

(2) Let us identify $\mathrm{Hom}(\pi, G)$ *with* $G \times G$ *using the standard generating system. If* $f : \mathrm{Hom}(\pi, G) = G \times G \to \mathbb{C}$ *is a regular function which is invariant under the G action on* $\mathrm{Hom}(\pi, G) = G \times G$, *then there exists a polynomial function* $F(x, y, z) \in \mathbb{C}[x, y, z]$ *such that* $f(\xi, \eta) = F(\mathrm{tr}(\xi), \mathrm{tr}(\eta), \mathrm{tr}(\xi\eta))$.*)*

After this identification, the boundary restriction map is given as:

$$\begin{aligned}\mathrm{Hom}(\pi, G)/G \cong \mathbb{C}^3 &\mapsto \mathrm{Hom}(\partial, G)/G \cong \mathbb{C}\\ \kappa(x, y, z) &= x^2 + y^2 + z^2 - xyz - 2.\end{aligned}$$

The (extended) mapping class group is generated by two Dehn twist maps and an involution. The following is the corresponding actions on $\pi = \pi_1(\Sigma_{1,1})$.

$$\begin{aligned}T_X &: X \mapsto X, Y \mapsto YX,\\ T_Y &: X \mapsto XY^{-1}, Y \mapsto Y,\\ \iota &: X \mapsto X^{-1}, Y \mapsto Y.\end{aligned}$$

This induces the polynomial automorphisms on $\mathbb{C}^3$.

$$\begin{aligned}\phi_X &: (x, y, z) \mapsto (x, z, zx - y)\\ \phi_Y &: (x, y, z) \mapsto (xy - z, y, x)\\ \phi_\iota &: (x, y, z) \mapsto (x, y, xy - z)\end{aligned}$$

Remark 2.2. *Let* $\mathrm{Aut}(\kappa)$ *be the group of polynomial automorphisms of* $\mathbb{C}^3$ *which leave invariant the fibers of* $\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$ *constant. By Proposition 1.1, we see that* $\mathrm{MCG}(\Sigma_{1,1}) \subset \mathrm{Aut}(\kappa)$. *In fact, it is known that* $\mathrm{MCG}(\Sigma)$ *is commensurable with* $\mathrm{Aut}(\kappa)$. *Therefore, our study in this note can be considered as a study of dynamical system* $\mathrm{Aut}(\kappa)$. *See* [2] *for the dynamics of* $\mathrm{Aut}(\kappa)$.

3. Three decompositions of the character variety

In this section, we assume that the group π is a free group of rank n. The Lie group G is always equal to $\mathrm{SL}(2,\mathbb{C})$.

3.1. **Geometric decomposition.** Let $\mathcal{D}(F_n)$ be the set of characters corresponding to discrete faithful representations and $\mathcal{E}(F_n)$ the set of characters corresponding to representations with dense image in G. Then, the following fact is known:

Fact 3.1. *(1) $\mathcal{E}(F_n)$ is nonempty and open.*
(2) $\mathcal{D}(F_n)$ is closed.
(3) $\mathcal{X}(F_n) \setminus (\mathcal{D}(F_n) \cup \mathcal{E}(F_n))$ has measure 0.
(4) This decomposition is $\mathrm{Out}(F_n)$*-invariant.*

Thus, we can say that $\mathcal{E}(F_n)$ and $\mathcal{D}(F_n)$ give *geometric decomposition* of the character variety. In Kleinian group theory, we study $\mathcal{D}(F_n)$.

In order to describe $\mathcal{D}(F_n)$, let us introduce one notion from Kleinian group theory.

Definition 3.2. *Suppose that $D_1, D_1', \dots, D_n, D_n'$ are $2n$ disjoint closed topological disks in $\partial\mathbb{H}^3 = \mathbb{C}$ and $g_1, \dots, g_n \in \mathrm{PSL}(2, \mathbb{C})$ are isometries such that $g_i(D_i)$ is the closure of the complement of D_i'. Then $[g_1, \dots, g_n]$ generate a free discrete group of rank n, called a* Schottky group.

The representation sending $x_i \mapsto g_i$ is discrete and faithful, and moreover, an open neighborhood of it in $\mathrm{Hom}(F_n, G)$ consists of similar representations. We let $\mathcal{S}(F_n)$ denote the open set of all characters of Schottky representations.

Proposition 3.3. *$\mathcal{S}(F_n)$ is the interior of $\mathcal{D}(F_n)$.*

Note that $\mathrm{Out}(F_n)$ acts properly discontinuously on $\mathcal{S}(F_n)$, i.e., $\{\phi \in \mathrm{Out}(F_n) \mid \psi(K) \cap K \neq \emptyset\}$ is finite for any compact $K \subset \mathcal{S}(F_n)$.

3.2. **Dynamical decomposition.** Minsky [4] and Lubotzky introduced another decomposition of $\mathcal{X}(F_n)$ by primitive stable ($\mathcal{PS}(F_n)$) and redundant ($\mathcal{R}(F_n)$) characters. (See also [3].)

Remark 3.4. *It is not known whether $\mathcal{X}(F_n) \setminus (\mathcal{PS}(F_n) \cup \mathcal{R}(F_n))$ has measure zero in $\mathcal{X}(F_n)$ or not.*

$\mathrm{Out}(F_n)$ acts ergodically on $\mathcal{R}(F_n)$ and acts properly discontinuously on $\mathcal{PS}(F_n)$. (Gelander, Minsky). This is why this decomposition is called dynamical decomposition.

Now, following Minsky [4], let us define primitive stableness. We fix the standard generating system $F_n = \langle X_1, \dots, X_n \rangle$. We denote by X the set of generators and inverses $\{X_1, \dots, X_n, X_1^{-1}, \dots, X_n^{-1}\}$.

Let Γ be the Cayley graph of $F_n = \langle X_1, \dots, X_n \rangle$, i.e., the graph with vertex set Γ and edge set $\{(w_1, w_2) \in \Gamma \times \Gamma \mid w_1 x = w_2 \text{ for some } x \in X\}$. An element of F_n is called *primitive* if it is the image of an element in X by some element of $\mathrm{Aut}(F_n)$. Any primitive element can be expressed as a cyclically reduced word. Using any cyclically reduced word w, we can define a bi-infinite reduced word $\cdots www \cdots$ by repeating infinitely many times in both direction. Set $P = \{\cdots www \cdots \mid w \text{ is a primitive word in } F_n\}$.

Given a representation $\rho : F_n \to \mathrm{SL}(2, \mathbb{C})$ and a basepoint $x \in \mathbb{H}^3$, there is a unique map $\tau_{\rho,x} : \Gamma \to \mathbb{H}^3$ mapping the origin of Γ to x, ρ-equivariant, and mapping each edge to a geodesic segment. More explicitly, if a vertex v of Γ corresponds to a word $X_{i_i} X_{i_2} \dots X_{i_k}$ in F_n, then,

$$\tau_{\rho,x}(v) = \rho(X_{i_k}) \circ \cdots \circ \rho(X_{i_1})(x).$$

(Note that the order of the elements X_{i_j} is reversed.) Here, an element of $\mathrm{SL}(2,\mathbb{C})$ is understood to act on $\mathbb{H}^3$ by Poincaré extension.

By this map $\tau_{\rho,x}$, every element of P is mapped to a (family of) broken geodesic path(s) in $\mathbb{H}^3$.

Recall that a map $f : \mathbb{R} \to X$ is called (K,δ)-quasigeodesic if

$$\frac{1}{K}d(x,y) - \delta \leq d(f(x), f(y)) \leq Kd(x,y) + \delta$$

for any $x, y \in X$. Here is the definition of the primitive stableness.

Definition 3.5. *A representation $\rho : F_n \to SL(2,\mathbb{C})$ is* primitive-stable *if there are constants K, δ and a basepoint $x \in \mathbb{H}^3$ such that $\tau_{\rho,x}$ takes all elements of P to (K,δ)-quasigeodesics.*

We denote by $\mathcal{PS}(F_n)$ the set of primitive stable characters in $\mathcal{X}(F_n)$.

Here is a brief summary of results of Minsky's paper [4].

Theorem 3.6 (Minsky [4]). *(1) If ρ is Schottky, then it is primitive-stable.*
(2) Primitive-stability is an open condition in $\mathcal{X}(F_n)$.
(3) $\mathcal{PS}(F_n)$ contains a point on the boundary of the Schottky space.
(4) The action of $\mathrm{Out}(F_n)$ *on $\mathcal{PS}(F_n)$ is properly discontinuous.*
(5) $\mathcal{PS}(F_n)$ is strictly larger than the set of Schottky characters, which is $\mathrm{Out}(F_n)$ *invariant, and on which* $\mathrm{Out}(F_n)$ *acts properly discontinuously.*

The last statement is quite surprising and this was the main result of [4].

3.3. **Bowditch's Q-condition.** In this subsection, we consider only the free group of rank two. In other words, $\pi = \pi_1(\Sigma_{1,1})$.

Bowditch defined the following condition on $[\rho] \in \mathcal{X}(F_2)$ [1], which Tan-Wong-Zhang call condition BQ [5]:

(1) $\rho(x)$ is loxodromic for all primitive $x \in F_2$.
(2) The number of conjugacy classes of primitive elements x such that $|\operatorname{tr}(\rho(x))| \leq 2$ is finite.

The following is a quite interesting conjecture.

Conjecture 3.7 (Bowditch[1]). *$BQ \cap \kappa^{-1}(-2)$ is exactly the set of quasi-fuchsian groups (punctured torus groups).*

Roughly speaking, the above conjecture states that the geometric decomposition and the condition BQ coincides in the relative character variety $\kappa^{-1}(-2)$.

3.4. **Questions.** We want to investigate the relations among three decompositions of the character variety. (BQ is defined only in F_2.)

Here is a part of the specific questions that Minsky asked in [4].

(1) Is $\mathcal{PS}(F_2)$ dense in $\mathcal{X}(F_2)$?
(2) Is BQ equal to $\mathcal{PS}(F_2)$?
(3) How do we produce computer pictures of $\mathcal{PS}(F_n)$?

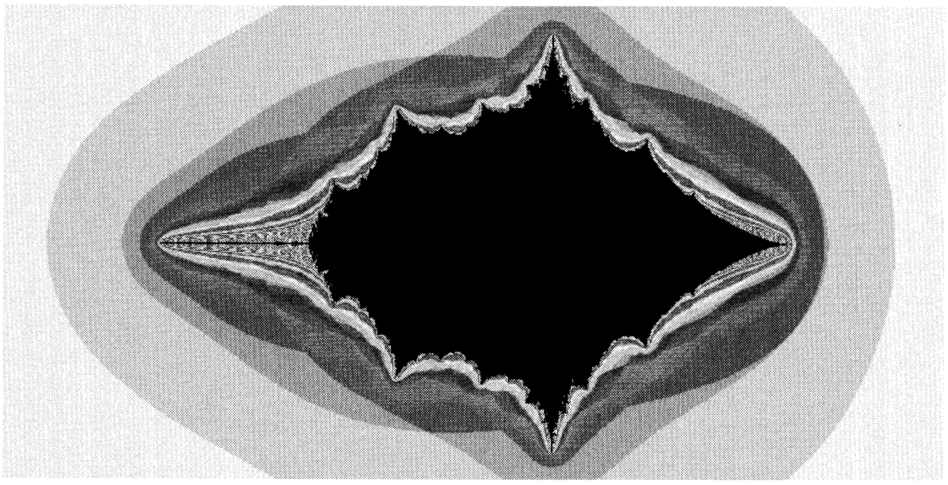

FIGURE 1. Bowditch's Q-condition: Diagonal slice

4. A COMPUTER EXPERIMENT

4.1. **Pictures.** In this section, we show some computer pictures of $\mathcal{PS}(F_2)$. The details of the algorithm will be described elsewhere in the future. This gives (a first step to) a partial answer to the question (3) by Minsky.

We call $\{(x,y,z)\,|\,x=y=z\} \subset \mathrm{Hom}(\pi_1(\Sigma_{1,1}), \mathrm{SL}(2,\mathbb{C})) = \mathbb{C}^3$ the *diagonal slice.* Also, for a constant $C \in \mathbb{C}$, we call $\{(x,y,z) \mid \kappa(x,y,z) = -2, x = C\} \subset \mathbb{C}^3$ a *linear slice* with $x = C$.

Remark 4.1. *For linear slices, we can change the constant value of $\kappa(x,y,z)$ if we want. But, $\kappa(x,y,z) = -2$ is the case most studied.*

Figure 1 is a picture of the diagonal slice for Bowditch's Q-condition and Figure 2 is a picture of the diagonal slice for primitive stableness. In both pictures, the (center) black regions correspond to "NOT BQ" and "Not primitive stable" parts.

Figure 3 is a picture of the linear slice with $X = 100$ for Bowditch's Q-condition and Figure 4 is a picture of the linear slice with $x = 100$ for primitive stableness. In both pictures, the black regions correspond to "NOT BQ" and "Not primitive stable" parts.

4.2. **Some observations.** By comparing these pictures naively by our eyes, the pictures for Bowditch's Q-condition and for primitive stable look almost the same. We'd like to say that this suggests that the answer to the Minsky's question (2) seems to be positive.

Also, we believe that these pictures suggest that $\mathcal{PS}(F_2)$ is not dense in $\mathcal{X}(F_2)$. That is, the answer to Minsky's question (1) should be no in this case.

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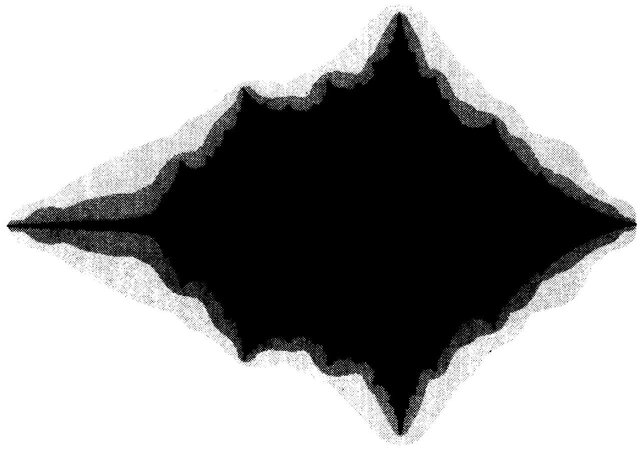

FIGURE 2. Primitive stable: Diagonal slice

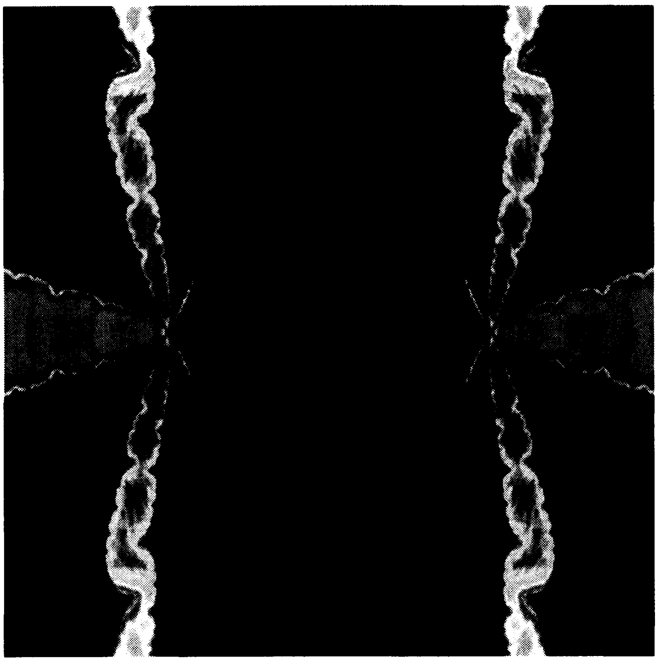

FIGURE 3. Bowditch's Q-condition: linear slice

[3] T Gelander and Y. N Minsky. The dynamics of $\mathrm{Aut}(F_n)$ on redundant representations, To appear in Groups, Geometry and Dynamics, `arXiv:1104.4774 [math.GT]`.
[4] Y. N. Minsky. On dynamics of $\mathrm{Out}(F_n)$ on PSL(2,C) characters, `arXiv:0906.3491v2 [math.GT]`.
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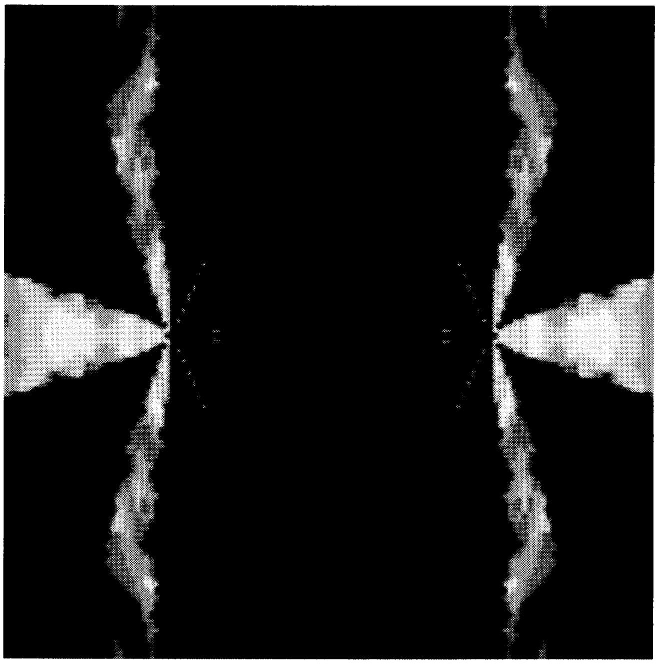

FIGURE 4. Primitive stable: linear slice