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The Eulerian Recurrent Lengths of Complete Graphs

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Abstract

The maximum of the shortest cycle length of Eulerian circuits of an Eulerian graph is called the Eulerian recurrent length of the graph. Let n be a positive odd integer, and let $\mathrm{ERL}(n)$ denote the Eulerian recurrent length of the complete graph K_n with n vertices. Previously, the following conjecture has been proposed by the author et al: For any odd integer $n \geqq 7$, $\mathrm{ERL}(n) < n - 2$ holds. In this paper, improvements of the algorithm to seek evidence for the conjecture are described. With the algorithm improved, $\mathrm{ERL}(21) < 19$ has been proved. Furthermore, a new conjecture on Eulerian recurrent lengths of complete graphs is proposed. The aim of the new conjecture is to help solve the previous one.

KEYWORDS. Eulerian circuit, complete graph, shortest cycle length, computational experiment.

1 Introduction

It is well known that finding an Eulerian circuit of a graph is a fundamental problem since the dawn of graph theory. It is easy to determine whether a graph given has an Eulerian circuit, that is, the graph is Eulerian, or not. Methods for finding an Eulerian circuit of an Eulerian graph given is also known and easy. We call the length of a shortest subcycle in the Eulerian circuit the shortest cycle length of the Eulerian circuit. We also define the Eulerian recurrent length of a graph as the maximum of the shortest cycle length of Eulerian circuits of the graph.

Finding the Eulerian recurrent length of graphs is useful to the following situation. Assume that there is a set of samples and that it is necessary to test a number of pairs of those samples with some inspection device. Furthermore, assume the following: there is some cost to input a sample to the device; a sample is effected by the device, where, the larger the effect is, the worse the accuracy of the test is, and the effects decrease as time passes. We consider the whole test a graph. Each sample corresponds to a vertex, and each pair of samples to test with the device corresponds to an edge. If cutting down the cost takes precedence over everything else, and the graph is an Eulerian graph, then the process of the whole test should make an Eulerian circuit. Furthermore, since it is desirable for inaccuracy of the result to decrease as small as possible, the Eulerian recurrent length of the graph should be found.

In this paper, we investigate the Eulerian recurrent length of complete graphs with odd numbers of vertices. Previously, we proposed the following conjecture.

Conjecture 1 *The Eulerian recurrent length of a complete graph with n vertices, K_n, is at most $n-3$, where n is a positive odd integer with $n \geqq 7$.*

The conjecture has not been proved yet as far as we know. We, therefore, have verified the conjecture by computational experiments. In current experiments, we have succeeded in verifying that Conjecture 1 holds for $n = 21$. Furthermore, we propose a new conjecture to approach the proof of Conjecture 1. In this paper, we describe the mechanisms of the algorithm used in the computational experiments for verifying Conjecture 1 and the new conjecture.

In the next section, we shall define several notions necessary for the arguments that follow. In Section 3, we shall describe known results and conjectures on the Eulerian recurrent length of complete graphs. In Section 4, we shall describe the improved algorithm to verify the conjectures made in our previous work, and present a new conjecture based on the results of computational experiments with the improved algorithm. The aim of the new conjecture is to help solve one of the previous conjectures. In the last section, we provide a suggestion to further improve the computational experiments.

2 Definitions

The order and size of a graph are the number of vertices and edges of the graph, respectively. A walk is an alternating sequence of vertices and edges such that the end vertices of each edge are the vertices next to the edge on the walk. A trail is a walk such that all its edges are distinct. Every graph that appears in this paper is a simple undirected graph. We may, therefore, express a walk W with only its vertices as $v_0 \to v_1 \to \cdots \to v_m$, where v_0 is the initial vertex and v_m the terminal vertex. The walk W is said to be a v_0-v_m walk, or a walk from v_0 to v_m. The length of a walk is the number of edges on the walk, even if the walk is closed. If a walk is closed, then the walk is expressed as $W = v_0 \to v_1 \to \cdots \to v_m \to v_0$. A closed trail is said to be a circuit. A path is a walk such that all its vertices are distinct except that the initial and terminal vertices may be identical. A closed path of positive length is said to be a cycle . A circuit in a graph G containing all the edges is said to be an Eulerian circuit of G. A graph is Eulerian if it has an Eulerian circuit. It is a well known fact that a graph is Eulerian if and only if each vertex of the graph has even degree. Let G be a graph, and W_1 and W_2 walks in G. If W_1 is a subsequence of W_2, then W_1 is said to be a subwalk of W_2. Terms subtrail, subcircuit, subpath, and subcycle are defined as the same manner.

We call the length of a shortest subcycle in a trail the shortest cycle length of the trail. Here, the trail may be non-closed. However, if the trail is a path and is not closed, then the shortest cycle length of the trail cannot be defined. Clearly, the shortest cycle length of an Eulerian circuit is always defined. We call the maximum of the shortest cycle length of the Eulerian circuits in an Eulerian graph G the Eulerian recurrent length (ERL) of G. The Eulerian recurrent length of G is denoted by $\mathrm{ERL}(G)$. Note that, since the length of any cycle in G is not greater than the order of G, the order of G is an upper bound on the Eulerian recurrent length of G.

The number of elements in a finite set S is denoted by $|S|$. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The order and size of a graph G are, therefore, denoted by $|V(G)|$ and $|E(G)|$, respectively. For simplicity, we usually assume that the vertex set of a graph G consists of consecutive integers that stars from 0, that is, the vertex set is $\{0, 1, \ldots, |V(G)| - 1\}$.

Let S be a set of vertices of G, T a set of edges of G. The subgraph of G obtained by deleting all vertices in S is denoted by $G - S$. The subgraph of G obtained by deleting all edges in T is also denoted by $G - T$. Furthermore, if v and e denote a vertex and an edge of G, then $G - v$ and $G - e$ denote $G - \{v\}$ and $G - \{e\}$, respectively.

3 Conjectures on the Eulerian recurrent lengths of complete graphs

It is clear that if a complete graph is Eulerian then the order is odd. As the following theorem states, the Eulerian recurrent length of a complete graph is very close to its order[3].

Theorem 1 *Let n be an odd integer with $n \geqq 11$. Then, there is an Eulerian circuit C of K_n such that the shortest cycle length of C is exactly $n-4$ if there is an integer m with $n = 4m+3$, and exactly $n-6$ otherwise.*

Theorem 1 is proved by decomposition of the edge set of K_n into Hamiltonian cycles $H_k \to n-1$, where H_k is a Hamiltonian path for $k = 0, 1, 2, \ldots, n-2$ described as in Figure 1. To construct the Eulerian circuit in Theorem 1, if $(n-1)/2$ is odd then $(n-1)/2$ Hamiltonian cycles $H_0, H_2, H_4, \ldots, H_{n-3}$ are used, else $H_0, H_2, H_4, \ldots, H_{((n-1)/2)-2}$, $H_{((n-1)/2)+1}, H_{((n-1)/2)+3}, \ldots, H_{n-2}$ used. The decomposition above is described by Bollobás [1].

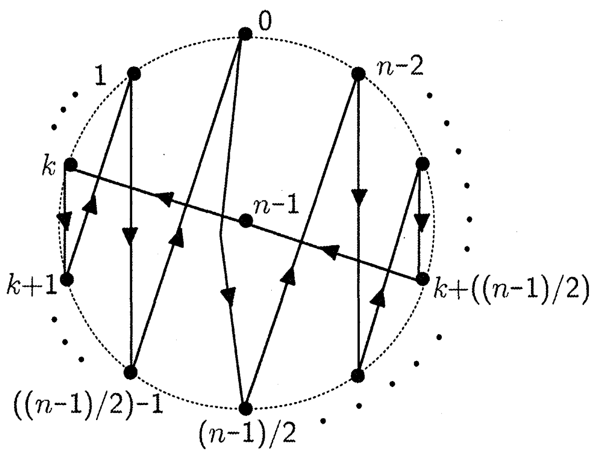

Figure 1: Structure of Hamiltonian path H_k.

The following theorem slightly improves the trivial upper bound on $\mathrm{ERL}(K_n)$[3].

Theorem 2 *Let n be an odd integer with $n \geqq 5$. Then, every Eulerian circuit of K_n has a subcycle of length at most $n-2$.*

You could expect that the upper bound on $\mathrm{ERL}(K_n)$ in Theorem 2 will be improved by using similar techniques in the known proof more elaborately. However, we are pessimistic about the achievement of such an improvement because of the following facts. Let H denote the trail of K_n with length $|E(K_n)| - 1$ defined by

$$H = H_0 \to H_1 \to \cdots \to H_{((n-1)/2)-1},$$

where H_j's are defined above. Then, the shortest cycle length of H is $n-2$. However, the Eulerian circuit $H \to n-1$ contains a triangle $n-2 \to n-1 \to 0 \to n-2$.

We have obtained that $\mathrm{ERL}(K_5) = \mathrm{ERL}(K_3) = 3$, and $\mathrm{ERL}(K_n) = n-3$ for every $n \in \{7, 9, 11, 13\}$, by computational experiments. The following are examples of Eulerian circuits of K_7, K_9, K_{11}, and K_{13} such that, for each example, the shortest cycle length of the Eulerian circuit is equal to the Eulerian recurrent length of the complete graph.

An example of Eulerian circuit of K_7 with the shortest cycle length 4:

$$\begin{array}{l} 0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 0 \to 2 \to 4 \to 1 \to 3 \to 5 \to 2 \to \\ 6 \to 4 \to 0 \to 5 \to 1 \to 6 \to 3 \to 0. \end{array}$$

An example of Eulerian circuit of K_9 with the shortest cycle length 6:

$$\begin{array}{l} 0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 0 \to 6 \to 1 \to \\ 3 \to 7 \to 2 \to 0 \to 4 \to 1 \to 8 \to 5 \to 6 \to \\ 7 \to 4 \to 2 \to 8 \to 3 \to 0 \to 7 \to 5 \to 2 \to \\ 6 \to 4 \to 8 \to 7 \to 1 \to 5 \to 3 \to 6 \to 8 \to 0. \end{array}$$

An example of Eulerian circuit of K_{11} with the shortest cycle length 8:

$$\begin{array}{l} 0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 0 \to 8 \to 1 \to \\ 3 \to 9 \to 2 \to 4 \to 6 \to 0 \to 5 \to 1 \to 7 \to 8 \to 10 \to \\ 9 \to 4 \to 0 \to 2 \to 5 \to 3 \to 6 \to 10 \to 1 \to 4 \to 8 \to \\ 9 \to 0 \to 3 \to 7 \to 2 \to 10 \to 5 \to 8 \to 6 \to 1 \to 9 \to \\ 7 \to 4 \to 10 \to 3 \to 8 \to 2 \to 6 \to 9 \to 5 \to 7 \to 10 \to 0. \end{array}$$

An example of Eulerian circuit of K_{13} with the shortest cycle length 10:

$$\begin{array}{l} 0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 0 \to 10 \to 1 \to \\ 3 \to 11 \to 2 \to 4 \to 6 \to 8 \to 5 \to 0 \to 7 \to 1 \to 12 \to 10 \to 2 \to \\ 9 \to 6 \to 11 \to 8 \to 0 \to 3 \to 5 \to 12 \to 7 \to 10 \to 4 \to 9 \to 11 \to \\ 1 \to 6 \to 0 \to 2 \to 8 \to 12 \to 3 \to 7 \to 4 \to 11 \to 10 \to 9 \to 1 \to \\ 5 \to 2 \to 6 \to 12 \to 0 \to 4 \to 8 \to 10 \to 3 \to 9 \to 5 \to 11 \to 7 \to \\ 2 \to 12 \to 4 \to 1 \to 8 \to 3 \to 6 \to 10 \to 5 \to 7 \to 9 \to 12 \to 11 \to 0. \end{array}$$

On the basis of the experiments above, we currently have the following conjecture stronger than Conjecture 1.

Conjecture 2 *For any odd integer n with $n \geqq 7$, $\mathrm{ERL}(K_n) = n - 3$ holds, that is to say the Eulerian recurrent length of a complete graph with n vertices is $n-3$.*

4 Computational experiments to verify the conjectures

In this section, we describe the mechanism of our algorithm for the computational experiments to verify Conjectures 1 and 2. We use the programming language C in the computational experiments for simplicity of coding and efficiency of execution.

In what follows, for every positive integer n, we regard $\{0, 1, 2, \ldots, n-1\}$ as the vertex set of K_n. The name of only one core function in the program for the experiments is `srceuler`. The behavior of `srceuler` is viewed as searching a tree consisting of growing trails in K_n by backtracking.

Suppose that there is an Eulerian circuit C whose shortest cycle length is $n-2$. Without loss of generality, we describe a shortest subcycle of C as

$$1 \to 2 \to \cdots \to n-2 \to 1.$$

Furthermore, since the shortest cycle length of C is $n-2$, and since any edge of K_n must not appear more than once in C, we may describe C as

$$C = 0 \to 1 \to 2 \to \cdots \to n-3 \to n-2 \to 1 \to n-1 \to \cdots \to 0.$$

We therefore set the root node of the search tree the following trail of length n:

$$T_0 = 0 \to 1 \to 2 \to \cdots \to n-3 \to n-2 \to 1 \to n-1.$$

Let $T = v_0 \to v_1 \to \cdots \to v_k$ be a trail of K_n such that $K_n - E(T)$ is connected. Note that there is an Eulerian circuit C such that T is a subtrail of C, if $K_n - E(T)$ is connected. Furthermore, let $T' = T \to v_{k+1} \to v_{k+2} \to \cdots \to v_{k+l-1} \to v_{k+l} = v_k$ be a trail of K_n such that T is a prefix subtrail of T', the terminal vertex of T' is the same as that of T, and the vertex v_k, the terminal vertex of T and T', does not appear in subtrail $v_{k+1} \to v_{k+2} \to \cdots \to v_{k+l-1}$. By the following theorem, if T' is a subtrail of an Eulerian circuit of K_n, then $n-2 \leqq l \leqq n+3$ must hold.

Theorem 3 *[2] Let m, n be integers with $4 \leqq n \leqq m-1$. Let $C = y \to x_1 \to x_2 \to \cdots \to x_{m+n+1} \to y$ be a circuit of K_{m+2} whose length is $m+n+2$. Then, if y does not appear in the subcircuit $C' = x_1 \to x_2 \to \cdots \to x_{m+n+1}$ of C then C' has a subcycle of length less than m.*

From the arguments above, we make `srceuler` set T_0 as the initial trail, that is the root node, then look at each vertex on the growing trail in turn from the initial vertex 0. Let `srceuler` look at the k-th vertex $v = v_k$ on the growing trail. Then, it determines interval l with $n-2 \leqq l \leqq n+3$ between the current occurrence of v and the next one such that the $(k+l)$-th vertex v_{k+l} on the growing trail is undetermined, and sets v_{k+l} as v. In this way, `srceuler` tries to extend the growing trail so that an Eulerian circuit may be obtained. Notice that the growing trail may have undetermined vertices. For example, if `srceuler` determines $n+2$ as the interval between the initial vertex 0 and the next occurrence of 0 just after it starts, and extends the growing trail, then the $n+2$-th vertex v_{n+2} is the terminal vertex of the growing trail and the $n+1$-th vertex v_{n+1} is undetermined. Since `srceuler` tries to make an Eulerian circuit whose shortest cycle length is $n-2$, if it looks at the k-th vertex $v_k = v$ on the growing trail and the $k+n-2$-th vertex v_{k+n-2} is undetermined, then it must determine $n-2$ as the interval between the current occurrence of v and the next one so that $v_{k+n-2} = v$. When `srceuler` cannot extend the trail, it backtracks.

The length of each Eulerian circuit C of K_n is $|E(K_n)| = n(n-1)/2$, and each vertex v of K_n occurs exactly $(n-1)/2$ times on C. Furthermore, if the shortest cycle length of C is $n-2$, then any interval between an occurrence of a vertex v and the next one of v on C lies in $\{n-2, n-1, n, n+1, n+2, n+3\}$ by Theorem 3. The following Theorem 4 follows from those facts.

Theorem 4 *Let n be an integer with $n \geqq 7$, Let L be an Eulerian circuit of K_n whose shortest cycle length is $n-2$ represented by*

$$L = v_0 \to v_1 \to v_2 \to \cdots \to v_{(n(n-1)/2)-1} \to v_0,$$

and C a subcircuit of L expressed as:

$$C = v_m \to v_{m+1} \to \cdots \to v_{m+l},$$

where $v_m = v_{m+l}$. Let k denote

$$|\{i \in \{m, m+1, m+2, \ldots, m+l-1\} \mid v_i = v_m\}|.$$

Then, l, the length of C, must satisfy the following inequalities:

$$kn - 3\left(\frac{n-1}{2} - k\right) \leqq l \leqq kn + 2\left(\frac{n-1}{2} - k\right).$$

In our current computational experiments, the backtracking condition derived from Theorem 4 causes a reduction of the search tree that `srceuler` builds. By computational experiments with `srceuler` improved by Theorem 4, we have verified that Conjecture 1 holds for $n = 21$. Table 1 shows the number of times that recurrent procedure `srceuler` is invoked, where $N_1(n)$ is the number for the improved `srceuler`, and $N_2(n)$ the one for the previous `srceuler`. We expect that the execution time of one trial of the experiment is approximately in proportion to the number of times that `srceuler` is invoked. As Table 1 shows, the truth of Conjecture 1 for $n = 21$ is also verified with the previous `srceuler`. The reason for that is chiefly the computers used in the current experiments being more powerful than the previous `srceuler`. For $n = 21$, the improved `srceuler` is about three times faster than the previous one. Currently, we intend to verify Conjecture 1 for $n \in \{23, 25, 27\}$ by large-scale computation.

Table 1: Results of the verification experiments for Conjecture 1.

n	$N_1(n)$	$N_2(n)$
17	3776653933	18490379207
19	52896311490	473349261179
21	2318764728335	7815975519946

Variation in the intervals between two successive identical vertices on the growing trail that `srceuler` extends leads us to the following new conjecture that can prove Conjecture 1.

Conjecture 3 *Let n be an integer with $n \geqq 7$, and $L = v_0 \to v_1 \to v_2 \to \cdots \to v_{((n-1)/2)-1}$ a non-closed trail with length $((n-1)/2)-1$ of a complete graph with n vertices, K_n. For each vertex $v \in V(K_n)$ and each integer $i \in \{1, 2, \ldots, (n-1)/2\}$, let $p(v, i)$ denote the length from the initial vertex v_0 of L to the i-th occurrence of v, where $p(v_0, 1) = 0$. Furthermore, let sequence $\{\Delta(i)\}_{i=1}^{(n-1)/2}$ be defined as $\Delta(i) = |p(v, i) - p(v, 1) - (i-1)n|$ for $i \in \{1, 2, \ldots, (n-1)/2\}$.*

Then, $\{\Delta(i)\}_{i=1}^{(n-1)/2}$ is monotone increasing.

5 Remarks

We hope that the algorithm for verifying Conjecture 1 and 2 improved by exploiting an idea derived from the following proposition are orders of magnitude faster than the current one.

Proposition 1 *Let n be an integer with $n \geqq 7$, and $\varphi : \{0,1,2,\ldots,n-1\} \to \{0,1,2,\ldots, n-1\}$ denote the one-to-one and onto mapping defined as*

$$\varphi(i) = \begin{cases} n-1 & \text{if } i = 0, \\ 1 & \text{if } i = 1, \\ 0 & \text{if } i = n-1, \\ n-i & \text{otherwise.} \end{cases}$$

Suppose that $\{0,1,2,\ldots,n-1\}$ is $V(K_n)$, the vertex set of complete graph K_n with n vertices. For a trail $T = t_0 \to t_1 \to t_2 \to \cdots \to t_s$ of K_n, let $r(T)$ be defined as

$$r(T) = \varphi(t_s) \to \varphi(t_{s-1}) \to \varphi(t_{s-2}) \to \cdots \to \varphi(t_1) \to \varphi(t_0).$$

Let k and l be positive integers such that $l \geqq n-2$ and $2k+l+1-n = |E(K_n)| = n(n-1)/2$.

Then, the following is a necessary and sufficient condition for existence of an Eulerian circuit of K_n whose shortest cycle length is $n-2$: There are two trails of length $k+l$,

$$S = s_0 \to s_1 \to s_2 \to \cdots \to s_{k+l} \qquad \text{and} \qquad T = t_0 \to t_1 \to t_2 \to \cdots \to t_{k+l},$$

such that

1. *the shortest cycle lengths of S and T are both $n-2$,*
2. *$s_0 = t_0 = 0, s_1 = t_1 = 1, s_2 = t_2 = 2, \ldots, s_{n-2} = t_{n-2} = n-2, s_{n-1} = t_{n-1} = 1$, and $s_n = t_n = n-1$,*
3. *$s_{k+1} = \varphi(t_{k+l}), s_{k+2} = \varphi(t_{k+l-1}), s_{k+3} = \varphi(t_{k+l} - 2), \ldots, s_{k+l} = \varphi(t_{k+1})$, and*
4. *$E(S) \cup E(T) = E(K_n)$.*

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References

[1] Béla Bollobás. *Modern Graph Theory.* Springer-Verlag New York, Inc., New York, 1998.

[2] S. Jimbo, K. Hashiguchi, and M. Han. On the recurrent intervals of complete graphs and complete bipartite graphs. *Technical Report of IEICE*, 101(707):49–56, 2002. In Japanese, COMP2001-89.

[3] S. Jimbo, Y. Inui, and K. Hashiguchi. On the intervals between identical points on Eulerian trails. *RIMS Kôkyûroku*, 1106:25–36, 1999. In Japanese.