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On Construction of Lie Superalgebras from Flexible Lie-admissible Algebras $-sl_2$ loop algebra and tetrahedron algebra– [1]
In memory of Professor Hyo Chul Myung

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Abstract

Construction of some class of Lie superalgebras from non-super flexible Lie-admissible algebras have been studied. Especially, the case of any associative algebra gives Lie superalgebras of some interest.

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1 Introduction and Preamble

It seems that an origin of our construction was appeared in Freudenthal's metasymplectic geometry. That is, it was a construction of Lie algebras from a class of algebraic structure with a ternary product called a triple system.

As a generalization of this notion, in a series of papers [7,8,9], we have studied a construction of simple Lie superalgebras [4,6,16] such as $osp(n,m)$, $A(n,m)$, as well as $D(2,1;\alpha), G(3)$, and $F(4)$ out of some simple non-super algebras by utilizing the well-known construction [19] of Lie superalgebras from $(-1,-1)$-Freudenthal-Kantor triple systems. Moreover, a intimate relationship between balanced $(-1,-1)$-Freudenthal-Kantor triple systems and a class of some Lie superalgebras has been explained in [3].

The main purpose of this paper is however to show that we can also construct a certain type of Lie superalgebras directly from flexible Lie-admissible algebras [e.g. see 12] without resorting to $(-1,-1)$-Freudenthal-Kantor triple systems. Especially, any associative as well as Poisson-Lie algebras can be used for constructions of some Lie superalgebras. These will be the subject of the next two sections. In order to facilitate discussions, let us briefly review here some relevant properties of a flexible Lie-admissible algebra below.

Let A be an algebra over a field F of charachteristic not 2 with bilinear product denoted by juxtaposition, xy for $x,y \in A$. As usual, we introduce a symmetric and skew-symmetric bilinear product, respectively written as $x \cdot y$ and $[x,y]$ in the same vector space A by

$$x \cdot y := xy + yx, \tag{1.1a}$$

[1]This paper is a survey note, which is a talk given in RIMS(Kyoto University) 2012,Feb., also contains some new results, and the details are described in other article.

$$[x, y] := xy - yx, \tag{1.1b}$$

so that

$$xy = \frac{1}{2}\{x \cdot y + [x, y]\}. \tag{1.2}$$

Suppose that these satisfy

(1) $[\circ, \circ]$ defines a Lie algebra A^-, i.e., we have

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \tag{1.3}$$

(2) A is flexible, i.e., $(xy)x = x(yx)$, or equivalently [e.g.12],

$$[z, x \cdot y] = [z, x] \cdot y + x \cdot [z, y] \tag{1.4}$$

for any $x, y, z \in A$. We call A then be a flexible Lie-admissible algebra which we often abbreviate hereafter as to FLAA. We also note the following.

Proposition 1.1(see e.g.12)

A necessary and sufficient condition for A to be a FLAA is to have *ad* z to be a derivation of A for any $z \in A$, i.e.,

$$[z, xy] = [z, x]y + x[z, y]. \tag{1.5}$$

This relation has some relevalance to quantum mechanics [14].

Next, we will list some example of FLAA of some interest.

Example 1.2 (well known algebras)

Any associative, or commutative, or Lie, algebra is a FLAA.

Example 1.3 (Poisson-Lie algebra)

Let a vector space A possess a symmetric and skew-symmetric bi-linear products denoted respectively by $x \cdot y$ and $[x, y]$ for $x, y \in A$. Suppose that they satisfy

(i) $[x, y]$ defines a Lie algebra A^-, i.e. satisfies Eq.(1.3).

(ii) $x \cdot y$ is a commutative and associative product, i.e.$x \cdot y = y \cdot x$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(iii) The validity of Eq.(1.4).

Then, A is called a Poisson-Lie algebra. If we introduce another bi-linear product xy in A by

$$xy := \frac{1}{2}(x \cdot y + [x, y])$$

as in Eq.(1.2), then A becomes a FLAA so that we regard any Poisson-Lie algebra as a FLAA hereafter with this identification.

Example 1.4(Mutation Algebra)

Let A be a FLAA. A new bi-linear product given by

$$x * y := \alpha xy + \beta yx$$

for some $\alpha, \beta \in F$ defines a FLAA, which we may call a mutaton algebra of A.

A interesting example is the associative mutation algebra of Santilli [15].

Let A be an associative algebra, and let $q \in A$ be a fixed element of A. Then, a new product $x \circ y \equiv xqy$ defines also an associative algebra. Hence,

$$x * y = \alpha xqy + \beta yqx (= \alpha x \circ y + \beta y \circ x)$$

gives a FLAA.

Example 1.5

Let A be a FLAA, and let $< \circ|\circ >$ be the Killing form of the associatied Lie algebra A^-,i.e.

$$< x|y >= \text{const. } Tr(ad\ x\ ad\ y).$$

We consider now a larger vector space

$$B = A \oplus F1$$

with a product given by

$$(x \oplus \alpha 1) * (y \oplus \beta 1) = (xy + \alpha y + \beta x) \oplus (\alpha\beta + < x|y >)1,$$

then, B is a FLAA with the unit element $E = 0 \oplus 1$. For many other examples of FLAA, see [12].

2 Construction of Lie superalgebras

Let A be a flexible Lie-admissible algebra (FLAA) over a field F. Let q be a indeterminate satisfying $q^2 = 1$, and consider the extended field $F(q)$ by adjoining q to F. Setting $\bar{x} = qx$ for any $x \in A$, it obviously satisfies

$$x\bar{y} = \bar{x}y = \overline{xy} \tag{2.1a}$$

$$\bar{x}\bar{y} = xy. \tag{2.1b}$$

Viewed from the original field F, we are essentially dealing with a extended new algebra

$$B = A \oplus \bar{A} \tag{2.2}$$

satisfying relations Eqs(2.1) for, $x, y \in A$ and $\bar{x}, \bar{y} \in \bar{A}$. Note that B is then a Z_2-graded algebra with its even and odd parts given respectively by $B_{\bar{0}} = A$ and $B_{\bar{1}} = \bar{A}$. We introduce then two bi-linear products by

$$a \cdot b := ab + (-1)^{ab} ba \tag{2.3a}$$

$$[a, b] := ab - (-1)^{ab} ba \tag{2.3b}$$

for $a, b \in B$,in F, where we have set for simplicity

$$(-1)^{ab} := (-1)^{(grade(a))(grade(b))} \tag{2.4a}$$

with

$$\text{grade}(a) = \begin{cases} 0, & \text{if } a \in B_{\bar{0}} = A \\ 1, & \text{if } a \in B_{\bar{1}} = \bar{A}. \end{cases} \tag{2.4b}$$

Since A is a FLAA, it is easy to verify that $[a, b]$ defines a Lie superalgebra B^- i.e.

$$(-1)^{ac}[[a, b], c] + (-1)^{ba}[[b, c], a] + (-1)^{cb}[[c, a], b] = 0. \tag{2.5}$$

Actually, we have a slightly better result as follows:

Theorem 2.1.
Let A be a FLAA with $B = A \oplus \bar{A}$, where $\bar{A}$ is a copy of A. Then, a larger vector space L in F given by

$$L = B \oplus Fh = A \oplus \bar{A} \oplus Fh \tag{2.6}$$

for a indeterminate h is a Lie superalgebra with

(1)

$$[x, \bar{y}] = -[\bar{y}, x] = \overline{[x, y]} \tag{2.7a}$$

(2)

$$[\bar{x}, \bar{y}] = x \cdot y + < x|y > h \tag{2.7b}$$

(3)

$$h \text{ is a center element of} L \text{ i.e, } [h, L] = [L, h] = 0 \tag{2.7c}$$

with its even part and odd part given by

$$L_{\bar{0}} = A \oplus Fh, \ L_{\bar{1}} = \bar{A} \tag{2.8}$$

where $< x|y >$ is the Killing form of A^{-},i.e.

$$< x|y >= \text{const.Tr}(ad\ x\ ad\ y). \tag{2.9}$$

Proof
Let $J(a, b, c)$ for $(a, b, c) \in L$ be the Jacobian of L, i.e.

$$J(a, b, c) = (-1)^{ac}[[a, b], c] + (-1)^{ba}[[b, c], a] + (-1)^{cb}[[c, a], b].$$

We then have to show $J(a, b, c) = 0$ identically. If $a, b, c \in A$, then this is a simple consequence of A being a FLAA. Similarly, we have $J(x, y, \bar{z}) = 0$. We next calculate

$$\begin{aligned} J(\bar{x}, \bar{y}, z) &= [[\bar{x}, \bar{y}], z] - [[\bar{y}, z], \bar{x}] + [[z, \bar{x}], \bar{y}] \\ &= [x \cdot y + < x|y > h, z] - [\overline{[y, z]}, \bar{x}] + [\overline{[z, x]}, \bar{y}] \\ &= [x \cdot y, z] - [y, z] \cdot x - < [y, z]|x > h + [z, x] \cdot y + < [z, x]|y > h \\ &= 0 \end{aligned}$$

in view of Eq.(1.4) and the fact that $< [x, y]|z >$ is totally skew-symmetric in $x, y, z \in A$.
Finally, we compute

$$\begin{aligned} J(\bar{x}, \bar{y}, \bar{z}) &= -[[\bar{x}, \bar{y}], \bar{z}] - [[\bar{y}, \bar{z}], \bar{x}] - [[\bar{z}, \bar{x}], \bar{y}] \\ &= -[x \cdot y + < x|y > h, \bar{z}] - [y \cdot z + < y|z > h, \bar{x}] - [z \cdot x + < z|x > h, \bar{y}] \\ &= -\overline{[x \cdot y, z]} - \overline{[y \cdot z, x]} - \overline{[z \cdot x, y]} = 0 \end{aligned}$$

when we note Eq.(1.4) again. This completes the proof.

Remark 2.2
Let A be a FLAA, and let $A_2 := A \cdot A$. We can then construct a sub-Lie superalgebra of L in Theorem 2.1 by restricting $L_{\bar{0}} = A$ (or $L_{\bar{1}} = \bar{A}$) to $L'_{\bar{0}} = A_2$ (or $L'_{\bar{1}} = \bar{A}_2$),i.e. $L' = A_2 \oplus \bar{A}$ (or $A \oplus \bar{A}_2$).
The present construction of the Lie superalgebra L is also intimately related to the following. For any FLAA, we introduce a triple product by

$$xyz := [x \cdot y, z] = [x, z] \cdot y + x \cdot [y, z] \tag{2.10}$$

Proposition 2.3
The triple product xyz given by Eq.(2.10) defines an anti-Lie triple system, i.e. it satisfies

(i)
$$xyz = yxz \tag{2.11a}$$

(ii)
$$xyz + yzx + zxy = 0 \tag{2.11b}$$

(iii)
$$uv(xyz) = (uvx)yz + x(uvy)z + xy(uvz). \tag{2.11c}$$

Proof

Eq.(2.11a) and (2.11b) are immediate consequences of Eq.(2.10). In order to prove the validity of Eq.(2.11c), we note that the multiplication operator $L(x,y) \in End\ A$ given by

$$L(x,y)z = xyz \tag{2.12a}$$

has a form of

$$L(x,y) = ad(x \cdot y) \tag{2.12b}$$

by Eq.(2.10). Since $ad\ w(w \in A)$ is a derivation of FLAA, this yields

$$[L(u,v), L(x,y)] = L(L(u,v)x, y) + L(x, L(u,v)y) \tag{2.13}$$

which is equivalent to the validity of Eq.(2.11c). This completes the proof.

We can now construct a Lie superalgebra from this anti-Lie triple system canonically by considering

$$L = L(A,A) \oplus \bar{A} \tag{2.14}$$

with the commutation relations of

$$[L(x,y), \bar{z}] = -[\bar{z}, L(x,y)] = \overline{xyz}$$

$$[\bar{x}, \bar{y}] = L(x,y)$$

and with Eq.(2.13). However, because of Eq.(2.12b), we may identify $L(A,A)$ with $A_2 = A \cdot A$. Then, the Lie superalgebra is essentially isomorphic to that stated in Remark 2.2 with $h = 0$.

As examples of Lie superalgebras constructed in Theorem 2.1, we note the following.

Example 2.4
Let A be a vector space consisting of all $n \times n$ traceless matrices, i.e.,

$$A = \{x | x = n \times n \text{ matrix in } F, \text{and } Tr\ x = 0\},$$

and let $x * y$ denote the ordinary associative matrix product in A. We now set $h = 0$ and

$$[x,y] := x * y - y * x$$

$$x \cdot y = \frac{1}{2}(x * y + y * x) - \frac{1}{n}\, Tr(x * y)E$$

where E is a $n \times n$ unit matrix and we assumed $n \neq 0$ in F. Then, A becomes a FLAA, and the Lie superalgebra constructed as in Theorem 2.1 is of the type $Q(n-1)$ [4,16].

Example 2.5

Let A be any Lie algebra so that we have $x \cdot y = 0$. Then, it yields a Lie superalgebra given by

$$[x, \bar{y}] = -[\bar{y}, x] = \overline{[x, y]},$$
$$[\bar{x}, \bar{y}] = \alpha Tr(ad\ x ad\ y)h, \ \ (\alpha \in F)$$
$$[x, h] = [h, x] = [\bar{x}, h] = [h, \bar{x}] = [h, h] = 0$$

for $L = A \oplus \bar{A} \oplus Fh$.

Remark 2.6

Since $B = A \oplus \overline{A}$ is a field extantion of a FLAA, it is also a FLAA in F. Therefore, we can repeat the same process by replacing A by B by extending the field by adding a indeterminate p satisfying p^2 =1. Setting $B^\star = pB$ and $C = B \oplus B^\star$, we can then construct a larger Lie superalgebra in C. This process can be indefinitely repeated, if we wish.

3 Special Flexible Lie-admissible Algebra

For some special class of FLAA, we can cunstruct more elaborate Lie superalgebra than that given in Theprem 2.1. First, we will prove the following Proposition (for the definition of an anti-Jordan triple system, we refere [9] or [10]).

Proposition 3.1

Let A be a FLAA satisfying a extra condition of

$$([u, v], z, [x, y])^\bullet + [(x, [u, v], y)^\bullet, z] = 0 \tag{3.1}$$

where

$$(x, y, z)^\bullet = (x \cdot y) \cdot z - x \cdot (y \cdot z) \tag{3.2}$$

is the associator of the commutative algebra A^+. Then, a triple product defined by

$$\begin{aligned} xyz &:= [x \cdot y, z] + [x, y] \cdot z \\ &= x \cdot [y, z] + y \cdot [x, z] + z \cdot [x, y] \end{aligned} \tag{3.3}$$

is an anti-Jordan triple system (or equivalently $(1, -1)$ Jordan triple system), i.e.

$$(i) \quad uv(xyz) = (uvx)y + x(vuy)z + xy(uvz) \tag{3.4a}$$

$$(ii) \quad K(x, y)z := xzy + yzx = 0. \tag{3.4b}$$

Proof

Eq.(3.4b) follows immediatly from Eq.(3.3). In order to prove Eq.(3.4a), we caluclate

$$\begin{aligned} &uv(xyz) = [u \cdot v, xyz] + [u, v] \cdot (xyz) \\ &= [u \cdot v, [x \cdot y, z] + [x, y] \cdot z] + [u, v] \cdot \{[x \cdot y, z] + [x, y] \cdot z\} \\ &= [u \cdot v, [x \cdot y, z]] + [u \cdot v, [x, y]] \cdot z + [x, y] \cdot [u \cdot v, z] \\ &+[u, v] \cdot [x \cdot y, z] + [u, v] \cdot \{[x, y] \cdot z\} \end{aligned}$$

so that we find

$$\begin{aligned}
&uv(xyz) - xy(uvz)\\
&= [u \cdot v, [x \cdot y, z]] - [x \cdot y, [u \cdot v, z]]\\
&+[u \cdot v, [x, y]] \cdot z - [x \cdot y, [u, v]] \cdot z\\
&+[u, v] \cdot ([x, y] \cdot z) - [x, y]\{[u, v]] \cdot z\}\\
&= -[z, [u \cdot v, x \cdot y]] + [u \cdot v, [x, y]] \cdot z\\
&-[x \cdot y, [u, v]] \cdot z - ([u, v], z, [x, y])^{\bullet}
\end{aligned}$$

Similary, we find

$$\begin{aligned}
&(uvx)yz + x(vuy)z\\
&= [[u \cdot z, x \cdot y], z] + [u \cdot v, [x, y]] \cdot z\\
&+[[u, v], x \cdot y] \cdot z + [(x, [u, v], y)^{\bullet}, z].
\end{aligned}$$

From these and Eq.(1.4), we obtain Eq.(3.4a). This completes the proof.

Corollary 3.2

Let A be a FLAA satisfying a extra condition of

$$(x, z, y)^{\bullet} = \lambda[z, [x, y]] \tag{3.5}$$

for some $\lambda \in F$. Then, the triple product given by Eq.(3.3) defines an anti-Jordan triple system.

Proof

Suppose that Eq.(3.5) holds. Then, we calculate

$$([u, v], z, [x, y])^{\bullet} = \lambda[z, [[u, v], [x, y]]]$$

and

$$[(x, [u, v], y)^{\bullet}, z] = [\lambda[[u, v], [x, y]], z]$$

so that the condition Eq.(3.1) is identically satisfied. This completes the proof.

Remark 3.3

We call a FLAA satisfying Eq.(3.5) for some $\lambda \in F$ to be a special flexible Lie-admissible algebra hereafter.

Examples are

- (i) Any associative algebra with $\lambda = 1$
- (ii) Any Poisson – Lie algebra with $\lambda = 0$
- (iii) Any associative mutation algebra as in Example 1.4 with $\lambda = (\frac{\alpha+\beta}{\alpha-\beta})^2$, assuming $\alpha - \beta \neq 0$.

Actually, these above three cases are essetially all of special FLAA by the following reason. Assuming the underlying field F to be quadratically closed, and setting $\lambda = \sigma^2$, we introduce a new bilinear product by

$$x * y = \frac{1}{2}\{x \cdot y + \sigma[x, y]\}.$$

Note that $x * y$ is actually a mutation algebra of A and hence a FLAA. Moreover, we can then prove that it is associative, i.e., $(x * y) * z = x * (y * z)$, after some caluclation which we do not give here. The cases of $\sigma = 0$ and 1 then correspond to associative and Poisson-Lie algebras, respectively. For the case of $\sigma \neq 0, 1$, it gives effectively the associative mutation algebra by suitably changing its normalization.

Since the anti-Jordan triple system is equivalent to a $(1, -1)$ Freudenthal-Kantor triple system with $K(x, y) = 0$ (e.g. [9],[10],[19]), we can canonically construct a Lie superalgebra as follows.: We first consider a larger vector space V of form

$$V = \begin{pmatrix} A \\ A \end{pmatrix} \tag{3.6}$$

with

$$X_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix} \tag{3.7}$$

for $x_j, y_j \in A$ $(j = 1, 2, 3)$ and set

$$L(X_1, X_2) = \begin{pmatrix} L(x_1, y_2) + L(x_2, y_1), & 0 \\ 0, & L(y_1, x_2) + L(y_2, x_1) \end{pmatrix} \tag{3.8}$$

where $L(x, y) \in \mathrm{End}\, A$ is the multiplication operator given by

$$L(x, y)z = xyz = [x \cdot y, z] + [x, y] \cdot z \tag{3.9}$$

from Eq.(3.3). Let $\overline{V}$ be a copy of V. Then

$$L = L(V, V) \oplus \overline{V} \tag{3.10a}$$

is a Lie superalgebra with $L_{\bar{0}} = L(V, V)$ and $L_{\bar{1}} = \overline{V}$, provided that we assume cummutation relations of

$$[\overline{X}_1, \overline{X}_2] = L(X_1, X_2) \tag{3.10b}$$

$$[L(X_1, X_2), \overline{X}_3] = \overline{\begin{pmatrix} x_1 y_2 x_3 + x_2 y_1 x_3 \\ y_2 x_1 y_3 + y_1 x_2 y_3 \end{pmatrix}} \tag{3.10c}$$

while $[L(X_1, X_2), L(X_3, X_4)]$ is the ordinary matrix commutator.

Following [9] or [10], we note that if (A, xyz) is an anti Jordan triple system, then $(A, [xyz])$ is an anti Lie triple system with respect to the new product defined by

$$[xyz] = xyz + yxz.$$

There exists another way of constructing a Lie superalgebra from special FLAA by first constructing a Jordan superalgebra as follows. Kantor [11] (see also [17]) already found a construction of a Jordan superalgebra from any Poisson-Lie algebra A by considering the following supercommutative product in $B = A \oplus \bar{A}$

$$x \circ y := x \cdot y \tag{3.11a}$$

$$x \circ \bar{y} := \bar{y} \circ x := \overline{x \cdot y} \tag{3.11b}$$

$$\bar{x} \circ \bar{y} := [x, y]. \tag{3.11c}$$

However, the same construction applies also for both A being associative or its mutation algebra. Let A be an assoicative algebra. Then the bi-linear product given by Eq.(2.1) for B is a Z_2-graded associative algebra. Hence, the super commutative product $a \cdot b$ in Eq.(2.3a) gives a Jordan superalgebra. In terms of x and $\bar{x}$, it reproduces the same relations as in Eq.(3.11). The other case of A being an associative mutation algebra can be similarly verified. Therefore, for any special FLAA, $B = A \oplus \bar{A}$ wiht the bi-linear product given by Eq.(3.11) is a Jordan superalgebra B^+ with $B^+_{\bar{0}} = A$ and $B^+_{\bar{1}} = \bar{A}$.

We can then construct a Lie superalgebra either by the Tits method [18] or by regarding the Jordan superalgebra to be a normal Lie-related triple superalgebra [13]. If A is unital in addition, then B^+ is also structurable algebras ([1],[2]).

Moreover, the case of the associative algebra gives a concise result of some interest. Let B be a Z_2-graded associative algebra which needs not however be of form $B = A \oplus \bar{A}$. Let super commutative and super anti-commutative products, $a \cdot b$ and $[a, b]$ by Eq.(2.3) which define a Jordan superalgebra B^+ and Lie superalgebra B^-. We then find the following.

Theorem 3.4

Let B be a Z_2-graded associative algebra. We introduce 4 copies of B and denote them as $\rho_i(B)$ $(j = 1, 2, 3)$ and $\phi(B)$. Then

$$L = \rho_1(B) \oplus \rho_2(B) \oplus \rho_3(B) \oplus \phi(B) \tag{3.12}$$

is a Lie superalgebra with commutation relations given by

(1)

$$[\phi(a), \phi(b)] = \phi([a, b]) \tag{3.13a}$$

(2)

$$[\rho_i(a), \rho_i(b)] = -\gamma_j \gamma_k^{-1} \phi([a, b]) \tag{3.13b}$$

(3)

$$[\rho_i(a), \rho_j(b)] = -(-1)^{ab}[\rho_j(b), \rho_i(a)] = -\gamma_j \gamma_i^{-1} \rho_k(a \cdot b) \tag{3.13c}$$

(4)

$$[\phi(a), \rho_j(b)] = -(-1)^{ab}[\rho_j(b), \phi(a)] = \rho_j([a, b]) \tag{3.13d}$$

for $a, b \in B$. Here (i, j, k) stands for any cyclic permutation of $(1, 2, 3)$ and $\gamma_j (j = 1, 2, 3)$ denote non-zero constants in F. Note that even and odd parts of L are given by

$$L_{\bar{0}} = \phi(B_{\bar{0}}) \oplus \rho_1(B_{\bar{0}}) \oplus \rho_2(B_{\bar{0}}) \oplus \rho_3(B_{\bar{0}}) \tag{3.14a}$$

$$L_{\bar{1}} = \phi(B_{\bar{1}}) \oplus \rho_1(B_{\bar{1}}) \oplus \rho_2(B_{\bar{1}}) \oplus \rho_3(B_{\bar{1}}). \tag{3.14b}$$

Proof

The validity of the Jacobi identity Eq.(2.5) can be verified as in Theorem 3.1 of [13], if we note

$$[a \cdot b, c] = a \cdot [b, c] + (-1)^{bc}[a, c] \cdot b \tag{3.15a}$$

$$(a \cdot b) \cdot c - a \cdot (b \cdot c) = (-1)^{ab}[b, [a, c]] \tag{3.15b}$$

as well as the grading condition

$$\text{grade}(ab) = \{\text{grade}(a) + \text{grade}(b)\} (\text{mod } 2) \tag{3.15c}$$

for the Z_2-graded associative algebra B. Note that in comparison to the general formula given by [13] for any normal Lie-related triple system, we replaced $T(a, b)$ by $\phi([a, b])$ and

$d(a,b)$ by ad$[a,b]$, respectively therein, just as in section 2 by identifying $L(A,A)$ with $A_2 = A \cdot A$. This completes the proof.

Remark 3.5

Let A be an associative commutative algebra with $B = A \oplus \bar{A}$, then we have $[A,A] = [A,\bar{A}] = 0$ but $[\bar{A},\bar{A}] \neq 0$ in general. Then, $\phi(A)$ commutes with all elements of L so that the quotient algebra

$$L' = L/\phi(A) = L'_{\bar{0}} \oplus L'_{\bar{1}}$$

is a Lie superalgebra with

$$\begin{aligned} L'_{\bar{0}} &\simeq \rho_1(A) \oplus \rho_2(A) \oplus \rho_3(A) \\ L'_{\bar{1}} &\simeq \rho_1(\bar{A}) \oplus \rho_2(\bar{A}) \oplus \rho_3(\bar{A}) \oplus \phi(\bar{A}). \end{aligned}$$

Suppose that A is a infinite dimensional commuttive associative algebra generated by indeterminate t and t^{-1}. Then, $L'_{\bar{0}}$ is isomorphic to the loop algebra of sl_2.

Similarly, if A is generated now by t, t^{-1} and $(1-t)^{-1}$, then, $L'_{\bar{0}}$ is isomorphic to the tetrahedron Lie algebra of Hartwig and Terwillinger [5]. Then, L' may be regarded as super-generalization of these Lie algebras (c.f., section 4 in this note).

Remark 3.6

If we have a bi-linear associative super-symmetric form $< \circ | \circ >$ in B, i.e., if we have

$$\begin{array}{ll} (i) & < a|b >= (-1)^{ab} < b|a > \\ (ii) & < a|b >= 0 \text{ if grade}(a) \neq \text{grade}(b) \\ (iii) & < ab|c >=< a|bc >, \end{array}$$

then we can make as Abelian extension of L by changing Eq.(3.13c) into

$$[\rho_i(a), \rho_j(b)] = -(-1)^{ab}[\rho_j(b), \rho_i(a)] = -\gamma_j\gamma_i^{-1}\rho_k(a \cdot b) + < a|b > f_k$$

where $f_k (k = 1,2,3)$ commute with all other elements of L as well as with themselves.

Since B is an associative Z_2-graded algebras, we can chose

$$< a|b >= \text{const. str}(ab)$$

where str stands for the super trace.

4 sl_2 loop algebra and tetrahedron algebra

In this section, from our methods, we will give examples by means of explicit forms for the sl_2 loop algebra and the tetrahedron algebra over a field k of characteristic not 2.

Example 4.1

Let A be an associative algebra $k[T, T^{-1}]$ and consider basis $\{h, e, f\}$ of the three dimensional simple Lie algebra sl_2, i.e., $[h,e] = 2e$, $[h,f] = -2f$, and $[e,f] = h$.

A particular basis $\{v_0, v_1, v_2\}$ of $g = sl_2 \otimes A$ as a module for A is defined by

$$v_0 = \frac{1}{2}(e \otimes 1 - f \otimes T), \; v_1 = \frac{1}{2}(e \otimes T^{-1} + f \otimes 1), \; v_2 = \frac{1}{2}(h \otimes 1). \tag{4.1}$$

Then it satisfies the relations;

$$[v_0, v_1] = v_2, [v_1, v_2] = -v_0 T^{-1}, [v_2, v_0] = v_1 T. \tag{4.2}$$

Now if we set

$$\rho_1(a) = v_0 a T^{-1}, \rho_2(b) = v_1 b T, \rho_3(c) = -cv_2,$$

$$\rho_3(a \cdot b) = -abv_2 (= -\rho_3(b \cdot a)), \rho_2(c \cdot a) = cav_1 (= -\rho_2(a \cdot c)), \rho_1(b \cdot c) = -bcv_0 (= -\rho_1(c \cdot b)),$$

where $\rho_1(a) \in \rho_1(A), \rho_2(b) \in \rho_2(A), \rho_3(c) \in \rho_3(A), \gamma_1 = \gamma =_2 = \gamma_3 = 1$,

then we have $[\rho_1(a), \rho_2(b)] = -\rho_3(a \cdot b)$, $[\rho_2(b), \rho_3(c)] = -\rho_1(b \cdot c)$, $[\rho_3(c), \rho_1(a)] = -\rho_2(c \cdot a)$.

Indeed, by straightfoward calculations, from (4.2), we obtain $[\rho_1(a), \rho_2(b)] = [av_0 T, bv_1 T^{-1}] = abv_2 = -\rho_3(a \cdot b)$, $[\rho_2(b), \rho_3(c)] = [v_1 T^{-1} b, -v_2 c] = v_0 bc = -\rho_1(b \cdot c)$, $[\rho_3(c), \rho_1(a)] = [-v_2 c, v_0 T a] = -\rho_2(c \cdot a)$.

Thus this means that the sl_2 loop algebra may be constructed by a special case of our methods;

$$L_{\bar{0}}' = \rho_1(A) \oplus \rho_2(A) \oplus \rho_3(A).$$

Example 4.2

Let A be an associative algebra $k[t, t^{-1}, (1-t)^{-1}]$ and consider the basis

$$\{x = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, y = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}, z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$$

of the three dimensional simple Lie algebra sl_2, i.e., $[x, y] = 2(x + y)$, $[y, z] = 2(y + z)$, and $[z, x] = 2(z + x)$.

A particular basis $\{u_0, u_1, u_2\}$ of $g = sl_2 \otimes A$ as a module for A is defined by

$$u_0 = \frac{1}{4}\{x \otimes t'' + y \otimes (t'' - 1) + z \otimes 1\}, \ u_1 = \frac{1}{4}\{y \otimes t + z \otimes (t - 1) + x \otimes 1\},$$

$$u_2 = \frac{1}{4}\{z \otimes t' + x \otimes (t' - 1) + y \otimes 1\}, \tag{4.3}$$

where $t' = 1 - t^{-1}$ and $t'' = (1 - t)^{-1}$. Then it satisfies the relations;

$$[u_0, u_1] = -u_2 t, [u_1, u_2] = -u_0 t', [u_2, u_0] = -u_1 t''. \tag{4.4}$$

Now if we set

$$\rho_1(a) = u_0 a, \rho_2(b) = u_1 b, \rho_3(c) = u_2 c,$$

$$\rho_3(a \cdot b) = abu_2 t (= -\rho_3(b \cdot a)), \rho_2(c \cdot a) = cau_1 t'' (= -\rho_2(a \cdot c)), \rho_1(b \cdot c) = bcu_0 t' (= -\rho_1(c \cdot b)),$$

where $\rho_1(a) \in \rho_1(A), \rho_2(b) \in \rho_2(A), \rho_3(c) \in \rho_3(A), \ \gamma_1 = \gamma_2 = \gamma_3 = 1$.

then we have $[\rho_1(a), \rho_2(b)] = -\rho_3(a \cdot b)$ $[\rho_2(b), \rho_3(c)] = -\rho_1(b \cdot c)$, and $[\rho_3(c), \rho_1(a)] = -\rho_2(c \cdot a)$.

Thus this means that the tetrahedron algebra may be constructed by a special case of our methods;

$$L_{\bar{0}}' = \rho_1(A) \oplus \rho_2(A) \oplus \rho_3(A).$$

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