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The fractional map by two and the parent map of numerical semigroups [1]

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Abstract

We consider the two kinds of maps between numerical semigroups. One sends a numerical semigroup to the one whose genus is decreased by 1. The other is the quotient map by two. We investigate Weierstrass (reps. non-Weierstrass) numerical semigroups whose images by the two maps are Weierstrass(reps. non-Weierstrass).

1 Introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0\backslash H$ is finite. The cardinality of $\mathbb{N}_0\backslash H$ is called the *genus* of H, denoted by $g(H)$. For a numerical semigroup H we set

$$c(H)=\min\{c\in\mathbb{N}_0 \mid c+\mathbb{N}_0\subseteqq H\},$$

which is called the *conductor* of H. We note that $c(H)-1\not\in H$. We set $p(H)=H\cup\{c(H)-1\}$, which is a numerical semigroup of genus $g(H)-1$. The numerical semigroup $p(H)$ is called the *parent* of H. For a numerical semigroup H we set

$$d_2(H)=\left\{\frac{h}{2}\,\middle|\, h\in H\ \text{ is even}\right\},$$

which is a numerical semigroup. We call d_2 the *fractional map by two.* We set

$$n=n(H)=\min\{h\in H\mid h \text{ is odd}\}.$$

Then we get

$$\frac{g(H)}{2}-\frac{n-1}{4}\leqq g(d_2(H))\leqq g(H)-\frac{n-1}{2}$$

(see [3]).

Lemma 1.1 *Let H be a numerical semigroup with odd $c(H)$. Then we have $d_2\circ p(H)=p\circ d_2(H)$. Namely we have a commutative diagram*

$$\begin{array}{ccc} H & \overset{p}{\longmapsto} & p(H) \\ \downarrow^{d_2} & \circlearrowleft & \downarrow^{d_2} \\ d_2(H) & \overset{p}{\mapsto} & p(d_2(H))=d_2(p(H)) \end{array}$$

[1]This paper is an extended abstract and the details will appear elsewhere.

Example 1. 1 Let $H = \langle 6,7,8,9,11\rangle$. Then $c(H) = 11$, which is odd. In fact, we have

$$\begin{array}{ccc} \langle 6,7,8,9,11\rangle & \overset{p}{\mapsto} & \langle 6,7,8,9,10,11\rangle \\ \downarrow^{d_2} & \circlearrowleft & \downarrow^{d_2} \\ \langle 3,4\rangle & \overset{p}{\mapsto} & \langle 3,4,5\rangle \end{array}$$

Lemma 1.2 . *Let H be a numerical semigroup with even $c(H)$. Then we have $d_2 \circ p(H) = d_2(H)$. Namely we have a commutative diagram*

$$\begin{array}{ccc} H & \overset{p}{\mapsto} & p(H) \\ \downarrow^{d_2} & \overset{d_2}{\swarrow} & \\ d_2(H) & & \end{array}$$

Example 1. 2 Let $H = \langle 6,7,8\rangle$. Then $c(H) = 18$, which is even. In fact, we have

$$\begin{array}{ccc} \langle 6,7,8\rangle & \overset{p}{\mapsto} & \langle 6,7,8,17\rangle \\ \downarrow^{d_2} & \overset{d_2}{\swarrow} & \\ \langle 3,4\rangle & & \end{array}$$

2 Weierstrass Diagrams

A *curve* means a complete non-singular irreducible algebraic curve over an algebraically closed field k of characteritic 0. For a pointed curve (C, P) we set

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP\}$$

where $k(C)$ is the field of rational functions on C. A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) with $H(P) = H$. A diagram

$$\begin{array}{ccc} H & \overset{p}{\mapsto} & p(H) \\ \downarrow^{d_2} & & \\ d_2(H) & & \end{array}$$

is said to be *Weierstrass* if H, $d_2(H)$ and $p(H)$ are Weierstrass.

Example 2. 1 If a numerical semigroup H is of genus $\leqq 8$, then the diagram

$$\begin{array}{ccc} H & \overset{p}{\mapsto} & p(H) \\ \downarrow^{d_2} & & \\ d_2(H) & & \end{array}$$

is Weierstrass, because H, $d_2(H)$ and $p(H)$ are Weierstrass (see [2] and [7]).

Example 2. 2 Let g be odd. The diagram

$$\begin{array}{ccc} \langle g+1 \mapsto 2g+1\rangle & \overset{p}{\mapsto} & \langle g \mapsto 2g-1\rangle \\ \downarrow{\scriptstyle d_2} & \overset{d_2}{\swarrow} & \\ \langle \frac{g+1}{2} \mapsto g\rangle & & \end{array}$$

is Weierstrass, because the above three numerical semigroups are the Weierstrass semigroups of ordinary points.

Example 2. 3 Let g be even. The diagram

$$\begin{array}{ccc} \langle g+1 \mapsto 2g+1\rangle & \overset{p}{\mapsto} & \langle g \mapsto 2g-1\rangle \\ \downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_2} \\ \langle \frac{g+2}{2} \mapsto g+1\rangle & \overset{p}{\mapsto} & \langle \frac{g}{2} \mapsto g-1\rangle \end{array}$$

is Weierstrass, because the above four numerical semigroups are the Weierstrass semigroups of ordinary points.

3 Double Covering Diagrams

A numerical semigroup H is said to be *of double covering type* if there exists a double covering $\pi : C \longrightarrow C'$ with a ramification point P such that $H(P) = H$. In this case we have $d_2(H(P)) = H(\pi(P))$. A diagram

$$\begin{array}{ccc} H & \overset{p}{\mapsto} & p(H) \\ \downarrow{\scriptstyle d_2} & & \\ d_2(H) & & \end{array}$$

is said to be *of double covering type* if H, $d_2(H)$ and $p(H)$ are of double covering type.

Example 3. 1 Let H' be a numerical semigroup of double covering type and n an odd integer $\geqq 4g(H') + 1$. The diagram

$$\begin{array}{ccc} H = 2H' + n\,\mathbb{N}_0 & \overset{p}{\mapsto} & p(H) \\ \downarrow{\scriptstyle d_2} & \overset{d_2}{\swarrow} & \\ d_2(H) = H' & & \end{array}$$

is of double covering type (see [4] and [8]).

4 Buchweitz Diagrams

Let H be a numerical semigroup. For any $m \geqq 2$ we set

$$L_m(H) = \{\ell' + \ell'' + \cdots + \ell^{(m)} \mid \ell^{(i)} \in \mathbb{N}_0 \backslash H, \text{ all } i\ \}.$$

The numerical semigroup H is said to be *Buchweitz* if for some m we have

$$\sharp L_m(H) > (2m-1)(g(H)-1).$$

In this case, H is non-Weierstrass (see [1]). A diagram

$$\begin{array}{ccc} H & \overset{p}{\mapsto} & p(H) \\ \downarrow^{d_2} & & \\ d_2(H) & & \end{array}$$

is said to be *Buchweitz* if the above three numerical semigroups H, $d_2(H)$ and $p(H)$ are Buchwietz. It is not easy to construct Buchweitz diagrams. So, we prepare some terminologies and lemmas. A numerical semigroup H is called an *m-semigroup* if $m = \min\{h \in H \mid h > 0\}$. An m-semigroup H is said to be *primitive* if we have $\gamma < 2m$ for any $\gamma \notin H$.

Example 4.1 A subset H of $\mathbb{N}_0$ with $\mathbb{N}_0 \backslash H = \{1, 2, \ldots, m-1\} \cup \{m+1\}$ is a primitive m-semigroup.

Remark 4.1 *We have a bijective correspondence between the set of subsets S of the set $S_{m+1} = \{m+1, m+2, \ldots, 2m-1\}$ and the set of primitive m-semigroups sending S to H_S with*

$$\mathbb{N}_0 \backslash H_S = \{1, 2, \ldots, m-1\} \cup S.$$

Lemma 4.2 *Let H be a primitive n-semigroup with $g(H) \geqq n+5$. Let $\overline{H}$ be a primitive $2n$-semigroup with*

$$\mathbb{N}_0 \backslash \overline{H} = \{1, \ldots, 2n-1\} \cup \{2\ell_n, 2\ell_{n+1}, \ldots, 2\ell_{g(H)}\} \cup \{4n-3, 4n-1\}$$

where $\mathbb{N}_0 \backslash H = \{1, \ldots, n-1, \ell_n < \ldots < \ell_{g(H)}\}$. Assume that $\sharp L_2(H) \geqq 3g(H) - 2$. Then we have $d_2(\overline{H}) = H$, $\sharp L_2(\overline{H}) \geqq 3g(\overline{H}) - 2$ and $\sharp L_2(p(\overline{H})) \geqq 3g(p(\overline{H})) - 2$ (see [5]).

Example 4.2 Let $t \geqq 5$ and $n \geqq 4t+1$. Let H be a primitive n-semigroup with

$$\mathbb{N}_0 \backslash H = \{1, \ldots, n-1\} \cup$$

$$\{2n-2t-1, 2n-2t-1+2\cdot 1, \ldots, 2n-2t-1+2\cdot(t-2)\} \cup \{2n-2, 2n-1\}.$$

Then H satisfies $\sharp L_2(H) = 3g(H) - 2$. For example, if we set $t = 5$ and $n = 21$, we have $\{1, \ldots, 20\} \cup \{31, 33, 35, 37, 40, 41\}$.

Example 4.3 A diagram

$$\begin{array}{ccc} \overline{H} & \overset{p}{\mapsto} & p(\overline{H}) \\ \downarrow^{d_2} & \overset{d_2}{\swarrow} & \\ d_2(\overline{H}) = H & & \end{array}$$

is Buchweitz where H is one of the above examples and $\overline{H}$ is as in the above Lemma. In this case $\overline{H}$, $H = d_2(\overline{H})$ and $p(\overline{H})$ are Buchweitz.

5 Quasi-Stöhr-Torres Diagrams

For a numerical semigroup H we have $c(H) \leqq 2g(H)$. The numerical semigroup H is said to be *symmetric* if $c(H) = 2g(H)$.

Remark 5.1 *For a numerical semigroup H and $g \geqq 4g(H)$ we set*

$$S(H,g) = 2H \cup \{2g-1-2t \mid t \in \mathbb{Z}\backslash H\}.$$

Then $S(H,g)$ is a symmetric numerical semigroup of genus g (see [10]).

Theorem 5.2 (Oliveira [9]) *Every symmetric numerical semigroup is non-Buchweitz.*

A numerical semigroup H is said to be *quasi-Stöhr-Torres* if it is non-Weierstrass and non-Buchweitz and $d_2(H)$ is non-Weierstrass.

Theorem 5.3 (Stöhr-Torres [10]) *Let H be a non-Weierstrass numerical semigroup, for example a Buchweitz numerical semigroup. Assume that $g \geqq 6g(H)+4$. Then the symmetric numerical semigroup $S(H,g)$ is quasi-Stöhr-Torres.*

Theorem 5.4 *Let H be a non-Buchweitz numerical semigroup of genus g with $\mathbb{N}_0\backslash H = \{\ell_1 < \ldots < \ell_g\}$. Assume that $\ell_g + \ell_{g-2} > 2\ell_{g-1}$. Then $p(H)$ is non-Buchweitz* (see [5]).

Lemma 5.5 *Let H be a symmetric numerical semigroup of genus g. We set $n_1 = \min\{h \in H \mid h > 0\}$ and $n_2 = \min\{h \in H \mid h > n_1\}$. Assume that $n_2 < 2n_1$. Then we get $\ell_g + \ell_{g-2} > \ell_{g-1}$* (see [5]).

A diagram

$$\begin{array}{ccc} H & \overset{p}{\mapsto} & p(H) \\ \downarrow^{d_2} & & \\ d_2(H) & & \end{array}$$

is said to be *quasi-Stöhr-Torres* if H, $d_2(H)$ and $p(H)$ are quasi-Stöhr-Torres.

Example 5. 1 Let H be a Buchweitz n-semigroup with $h \in H$ satisfying $n < h < 2n$. Let $g \geqq 6g(H)+4$ and $\tilde{g} \geqq 6g+5$. Then we have a quasi-Stöhr-Torres diagram

$$\begin{array}{ccc} S(S(H,g),\tilde{g}) & \overset{p}{\mapsto} & p(S(S(H,g),\tilde{g})). \\ \downarrow^{d_2} & \overset{d_2}{\swarrow} & \\ S(H,g) & & \end{array}$$

6 Mixed Diagrams

A diagram consisting of numerical semigroups

$$\begin{array}{ccc} H & \overset{p}{\mapsto} & p(H) \\ \downarrow^{d_2} & & \\ d_2(H) & & \end{array}$$

is said to be *mixed* if H, $d_2(H)$ and $p(H)$ are not the same type.

Example 6.1 A diagram

$$\begin{array}{ccc} B = \langle 13 \mapsto 18, 20, 22, 23 \rangle & \overset{p}{\mapsto} & \langle 13 \mapsto 18, 20, 22, 23, 25 \rangle \\ \downarrow^{d_2} & \swarrow^{d_2} & \\ W = \langle 7 \mapsto 11, 13 \rangle & & \end{array}$$

is mixed, because B is Buchweitz and W is Weierstrass.

Example 6.2 A diagram

$$\begin{array}{ccc} H = \langle 8, 12, 8\ell+2, 8\ell+6, n, n+4 \rangle & \overset{p}{\mapsto} & H + (n + 8\ell - 2)\mathbb{N}_0 \\ \downarrow^{d_2} & \swarrow^{d_2} & \\ W = \langle 4, 6, 4\ell+1, 4\ell+3 \rangle & & \end{array}$$

with $\ell \geqq 2$ and odd $n \geqq 16\ell + 19$ is mixed, because H is non-Weierstrass, non-Buchweitz and non-quasi-Stöhr-Torres (see [4]), and W is Weierstrass. Moreover, $p(H) = H + (n + 8\ell - 2)\mathbb{N}_0$ is of double covering type (see [6]).

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