ON THE CALCULATION OF THE SPECTRA OF BURNSIDE TAMBARA FUNCTORS

HIROYUKI NAKAOKA

Abstract. For a finite group $G$, a Tambara functor on $G$ is regarded as a $G$-bivariant analog of a commutative ring. In our previous article, we consider a $G$-bivariant analog of the ideal theory for Tambara functors. In this article, we will demonstrate calculations of spectra of Burnside Tambara functors, when $G = \mathbb{Z}/q\mathbb{Z}$.

1. Introduction and Preliminaries

A Tambara functor is firstly defined by Tambara [8] in the name ‘TNR-functor’, to treat the multiplicative transfers of Green functors. (For the definitions of Green and Mackey functors, see [1].) Later it is used by Brun [2] to describe the structure of Witt-Burnside rings.

For a finite group $G$, a Tambara functor is also regarded as a $G$-bivariant analog of a commutative ring, as seen in [9]. As such, for example a $G$-bivariant analog of the fraction ring was considered in [3], and a $G$-bivariant analog of the semigroup-ring construction was discussed in [5] and [6], with relation to the Dress construction [7].

In this analogy, we considered a $G$-bivariant analog of the ideal theory for Tambara functors in our previous article [4]. In this article, we will demonstrate calculations of spectra of Burnside Tambara functors, when $G = \mathbb{Z}/q\mathbb{Z}$ for some prime number $q$.

Throughout this article, the unit of a finite group $G$ will be denoted by $e$. Abbreviately we denote the trivial subgroup of $G$ by $e$, instead of $\{e\}$. $H \leq G$ means $H$ is a subgroup of $G$. $G\set$ denotes the category of finite $G$-sets and $G$-equivariant maps. If $H \leq G$ and $g \in G$, then $gH = gHg^{-1}$ denotes the conjugate $gH = gHg^{-1}$.

A ring is assumed to be commutative, with an additive unit 0 and a multiplicative unit 1. A ring homomorphism preserves 0 and 1.

For any category $\mathcal{C}$ and any pair of objects $X$ and $Y$ in $\mathcal{C}$, the set of morphisms from $X$ to $Y$ in $\mathcal{C}$ is denoted by $\mathcal{C}(X,Y)$.

First we briefly recall the definition of a Tambara functor and its ideal.

Definition 1.1. ([8]) A Tambara functor $T$ on $G$ is a triplet $T = (T^*, T_+, T_\ast)$ of two covariant functors

\[ T_+: G\set \to \text{Set}, \quad T_\ast: G\set \to \text{Set} \]

The author wishes to thank Professor Fumihito Oda for giving him a opportunity to talk at the conference.

The author also wishes to thank Professor Akihiko Hida for his question and comments.

Supported by JSPS Grant-in-Aid for Young Scientists (B) 22740005.
and one contravariant functor

\[ T^*: \text{Gset} \to \text{Set} \]

which satisfies the following. Here Set is the category of sets.

1. \( T^\alpha = (T^*, T_+) \) is a Mackey functor on \( G \).
2. \( T^\mu = (T^*, T_\ast) \) is a semi-Mackey functor on \( G \).
Since \( T^\alpha, T^\mu \) are semi-Mackey functors, we have \( T^*(X) = T_+(X) = T_\ast(X) \)
for each \( X \in \text{Ob}(\text{Gset}) \). We denote this by \( T(X) \).
3. (Distributive law) If we are given an exponential diagram

\[
\begin{array}{ccc}
X & \xrightarrow{p} & A \\
\downarrow f & & \downarrow \lambda \\
Y & \xleftarrow{q} & B
\end{array}
\]

\[
\begin{array}{cccc}
\downarrow \exp & & \downarrow \rho \\
T(X) \xrightarrow{T_+(p)} T(A) & \xrightarrow{T^*(\lambda)} & T(Z) \\
\downarrow T_\ast(f) & & \uparrow T_\ast(\rho) \\
T(Y) & \xleftarrow{T_+(q)} & T(B)
\end{array}
\]

is commutative.

If \( T = (T^*, T_+, T_\ast) \) is a Tambara functor, then \( T(X) \) becomes a ring for each \( X \in \text{Ob}(\text{Gset}) \). For each \( f \in \text{Gset}(X, Y) \),

- \( T^*(f): T(Y) \to T(X) \) is a ring homomorphism.
- \( T_+(f): T(X) \to T(Y) \) is an additive homomorphism.
- \( T_\ast(f): T(X) \to T(Y) \) is a multiplicative homomorphism.
\( T^*(f), T_+(f), T_\ast(f) \) are often abbreviated to \( f^*, f_+, f_\ast \).

In this article, a Tambara functor always means a Tambara functor on some finite group \( G \).

**Example 1.2.** If we define \( \Omega \) by

\[
\Omega(X) = K_0(\text{Gset}/X)
\]

for each \( X \in \text{Ob}(\text{Gset}) \), where the right hand side is the Grothendieck ring of the category of finite \( G \)-sets over \( X \), then \( \Omega \) becomes a Tambara functor on \( G \). This is called the Burnside Tambara functor. For each \( f \in \text{Gset}(X, Y) \),

\[
f_\ast: \Omega(X) \to \Omega(Y)
\]

is the one determined by

\[
f_\ast(A \xrightarrow{p} X) = (\Pi_f(A) \xrightarrow{\varpi} Y) \quad (\forall (A \xrightarrow{p} X) \in \text{Ob}(\text{Gset}/X)),
\]

where \( \Pi_f(A) \) and \( \varpi \) is

\[
\Pi_f(A) = \left\{ (y, \sigma) \mid \begin{array}{c}
y \in Y, \\
f^{-1}(y) \to A \text{ a map of sets,}
\end{array} \sigma \circ p = \text{id}_{f^{-1}(y)} \right\},
\]

\( \varpi(y, \sigma) = y \).
ON THE CALCULATION OF THE SPECTRA OF BURNSIDE TAMBARA FUNCTORS

$G$ acts on $\Pi_f(A)$ by $g \cdot (y, \sigma) = (gy, ^g\sigma)$, where $^g\sigma$ is the map defined by

\[ ^g\sigma(x) = g\sigma(g^{-1}x) \quad (\forall x \in f^{-1}(gy)). \]

**Definition 1.3.** Let $T$ be a Tambara functor. For each $f \in Gset(X, Y)$, define $f_! : T(X) \to T(Y)$ by

\[ f_!(x) = f(x) - f_!(0) \]

for any $x \in T(X)$.

**Remark 1.4.** ([4]) Let $T$ be a Tambara functor. We have the following for any $f \in Gset(X, Y)$.

1. $f_!$ satisfies $f_!(x)f_!(y) = f_!(xy)$ for any $x, y \in T(X)$.
2. If $f$ is surjective, then we have $f_! = f_*$.
3. If

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow \xi & & \downarrow \eta \\
X & \xrightarrow{f} & Y
\end{array}
\]

is a pull-back diagram, then $f_!^*\xi^* = \eta^*f_!$ holds.
4. If

\[
\begin{array}{ccc}
X & \xleftarrow{p} & A & \xleftarrow{\lambda} & Z \\
\downarrow f & & \downarrow \text{exp} & & \downarrow \rho \\
Y & \xleftarrow{\omega} & \Pi
\end{array}
\]

is an exponential diagram, then $\omega_+\rho^*\lambda^* = f_!p_+$ holds.

**Definition 1.5.** ([4]) Let $T$ be a Tambara functor. An ideal $\mathcal{I}$ of $T$ is a family of ideals $\mathcal{I}(X) \subseteq T(X)$ ($\forall X \in \text{Ob}(Gset)$) satisfying

1. $f^*(\mathcal{I}(Y)) \subseteq \mathcal{I}(X)$,
2. $f_+(\mathcal{I}(X)) \subseteq \mathcal{I}(Y)$,
3. $f_!(\mathcal{I}(X)) \subseteq \mathcal{I}(Y)$

for any $f \in Gset(X, Y)$. These conditions also imply

\[ \mathcal{I}(X_1 \coprod X_2) \cong \mathcal{I}(X_1) \times \mathcal{I}(X_2) \]

for any $X_1, X_2 \in \text{Ob}(Gset)$.

Obviously when $G$ is trivial, this definition of an ideal agrees with the ordinary definition of an ideal of a commutative ring.

**Remark 1.6.** For any ideal $\mathcal{I} \subseteq T$, we have $\mathcal{I}(\emptyset) = T(\emptyset) = 0$.

**Definition 1.7.** ([4]) An ideal $p \subseteq T$ is prime if for any transitive $X, Y \in \text{Ob}(Gset)$ and any $a \in T(X), b \in T(Y)$,

\[ \langle a \rangle \langle b \rangle \subseteq p \quad \Rightarrow \quad a \in p(X) \quad \text{or} \quad b \in p(Y) \]

is satisfied. Remark that the converse always holds.

An ideal $m \subseteq T$ is maximal if it is maximal with respect to the inclusion of ideals not equal to $T$. A maximal ideal is always prime.
Definition 1.8. ([4]) For any Tambara functor $T$ on $G$, define $\text{Spec}(T)$ to be the set of all prime ideals of $T$. For each ideal $\mathcal{I} \subseteq T$, define a subset $V(\mathcal{I}) \subseteq \text{Spec}(T)$ by

$$ V(\mathcal{I}) = \{ p \in \text{Spec}(T) \mid \mathcal{I} \subseteq p \}. $$

Remark 1.9. ([4]) For any Tambara functor $T$, we have the following.

(1) $V(\mathcal{I}) = \emptyset$ if and only if $\mathcal{I} = T$.

(2) $V(\mathcal{I}) = \text{Spec}(T)$ if and only if $\mathcal{I} \subseteq \bigcap_{p \in \text{Spec}(T)} p$.

Remark 1.10. ([4]) For any Tambara functor $T$, the family $\{ V(\mathcal{I}) \mid \mathcal{I} \subseteq T \}$ is an ideal of $T$. Thus $\text{Spec} \Omega$ becomes a topological space.

2. SOME PROPOSITIONS

Proposition 2.1. Let $T$ be a Tambara functor. Suppose we are given a family of ideals indexed by the set of finite non-empty transitive $G$-sets

$$(2.1) \quad \{ \mathcal{I}(X_0) \subseteq T(X_0) \}_{\emptyset \neq X_0 \in \text{Ob}(G\text{set})}.$$

For any $X \in \text{Ob}(G\text{set})$, take its orbit decomposition $X = \coprod_{1 \leq i \leq s} X_i$ and put

$$ \mathcal{I}(X) = \mathcal{I}(X_1) \times \cdots \times \mathcal{I}(X_s) \subseteq T(X). $$

(We used the identification $T(X) \cong \prod_{1 \leq i \leq s} T(X_i)$.) Then the following are equivalent.

(1) $\mathcal{I} = \{ \mathcal{I}(X) \}_{X \in \text{Ob}(G\text{set})}$ is an ideal of $T$.

(2) The family (2.1) satisfies

(i) $f^*(\mathcal{I}(Y_0)) \subseteq \mathcal{I}(X_0)$
(ii) $f_+(\mathcal{I}(X_0)) \subseteq \mathcal{I}(Y_0)$
(iii) $f_*(\mathcal{I}(X_0)) \subseteq \mathcal{I}(Y_0)$

for any transitive $X_0, Y_0 \in \text{Ob}(G\text{set})$ and any $f \in G\text{set}(X_0, Y_0)$.

Proof. Remark that for any non-empty transitive $X_0, Y_0 \in \text{Ob}(G\text{set})$ and any $f \in G\text{set}(X_0, Y_0)$, we have $f_* = f_!$. Obviously, (1) implies (2). We will show the converse.

Assume (2) holds. It suffices to show $\mathcal{I}$ satisfies (i), (ii), (iii) in Definition 1.5 for any $f \in G\text{set}(X, Y)$.

First, we reduce to the case where $Y$ is transitive. Take the orbit decomposition $Y = \coprod_{1 \leq j \leq t} Y_j$, put

$$ X_j = f^{-1}(Y_j), \quad f_j = f|_{X_j} : X_j \to Y_j, $$

and suppose (i), (ii), (iii) in Definition 1.5 holds for each $f_j$. Since we have commutative diagrams

$$ T(X) \xrightarrow{\cong} \prod_j T(X_j) \quad T(Y) \xrightarrow{\cong} \prod_j T(Y_j) \quad T(X) \xrightarrow{\cong} \prod_j T(X_j) $$

$$ f_+ \downarrow \circ \quad \downarrow \prod_j f_+ \quad f^* \downarrow \circ \quad \downarrow \prod_j f^* \quad f_* \downarrow \circ \quad \downarrow \prod_j f_* $$

$$ T(Y) \xrightarrow{\cong} \prod_j T(Y_j) \quad T(X) \xrightarrow{\cong} \prod_j T(X_j) \quad T(Y) \xrightarrow{\cong} \prod_j T(Y_j) $$


under the canonical identification, we obtain
\[ f_+(\mathcal{I}(X)) = \prod f_j(\mathcal{I}(X_j)) \subseteq \prod_{j} \mathcal{I}(X_j) = \mathcal{I}(X), \]
\[ f^*(\mathcal{I}(Y)) = \prod f_j^*(\mathcal{I}(Y_j)) \subseteq \prod_{j} \mathcal{I}(X_j) = \mathcal{I}(X), \]
\[ f_1(\mathcal{I}(X)) = \prod f_j(\mathcal{I}(X_j)) \subseteq \prod_{j} \mathcal{I}(X_j) = \mathcal{I}(X). \]

Now it remains to show in the case \( Y \) is transitive. If \( X = \emptyset \), then there is nothing to show. Otherwise, take the orbit decomposition \( X = \bigcup_{1 \leq i \leq s} L \times \mathcal{O}(G) \) and put \( f_i = f|_{X_i} : X_i \to Y \). Remark that in this case, we have \( f_* = f_1 \). By assumption, each \( f_i \) satisfies
\[ f_i(\mathcal{I}(X_i)) = \prod f_j(\mathcal{I}(X_j)) \subseteq \prod_{j} \mathcal{I}(X_j) = \mathcal{I}(X). \]

Under the identification \( T(X) = \prod_{1 \leq i \leq s} T(X_i) \), we obtain \( f^*(\mathcal{I}(Y)) \subseteq \mathcal{I}(X_1) \times \cdots \times \mathcal{I}(X_s) = \mathcal{I}(X) \). Moreover, for any \( x \in \mathcal{I}(X) \), under the identification
\[ \mathcal{I}(X) = \mathcal{I}(X_1) \times \cdots \mathcal{I}(X_s) \]
\[ x = (x_1, \ldots, x_s), \]
we have
\[ f_+(x) = f_1(x_1) + \cdots + f_s(x_s) \in \mathcal{I}(Y), \]
\[ f_*(x) = f_1(x_1) \cdots f_s(x_s) \in \mathcal{I}(Y). \]

Thus it follows \( f_+(\mathcal{I}(X)) \subseteq \mathcal{I}(Y), \) \( f_*(\mathcal{I}(X)) \subseteq \mathcal{I}(Y). \)

\[ \square \]

Corollary 2.2. To give an ideal \( \mathcal{I} \) of a Tambara functor \( T \) on \( G \) is equivalent to give a family of ideals indexed by \( \mathcal{O}(G) \)
\[ \{ \mathcal{I}(G/H) \subseteq T(G/H) \}_{H \in \mathcal{O}(G)} \]
satisfying
\begin{align*}
(\text{i}) & \quad \text{res}_K^H(\mathcal{I}(G/H)) \subseteq \mathcal{I}(G/K), \\
(\text{ii}) & \quad \text{ind}_K^H(\mathcal{I}(G/K)) \subseteq \mathcal{I}(G/H), \\
(\text{iii}) & \quad \text{jnd}_K^H(\mathcal{I}(G/K)) \subseteq \mathcal{I}(G/H), \\
(\text{iv}) & \quad c_{g,H}(\mathcal{I}(G/H)) \subseteq \mathcal{I}(G/\langle g \rangle),
\end{align*}
for any \( K \leq H \leq G \) and \( g \in G \). In particular, \( \mathcal{I}(G/H) \subseteq T(G/H) \) is \( N_G(H)/H \)-invariant.

By construction, for ideals \( \mathcal{I}, \mathcal{J} \subseteq T \), we have
\[ \mathcal{I} \subseteq \mathcal{J} \iff \mathcal{I}(G/H) \subseteq \mathcal{J}(G/H) \; (\forall H \in \mathcal{O}(G)). \]

Corollary 2.3. When \( G = \mathbb{Z}/q\mathbb{Z} \) where \( q \) is a prime number, then to give an ideal \( \mathcal{I} \) of \( T \) is equivalent to give
- a \( G \)-invariant ideal \( \mathcal{I}(G/e) \subseteq T(G/e), \)
- an ideal \( \mathcal{I}(G/G) \subseteq T(G/G), \)
HIROYUKI NAKAOKA

satisfying

(i) \( \pi^*(\mathscr{I}(G/G)) \subseteq \mathscr{I}(G/e) \),
(ii) \( \pi_+(\mathscr{I}(G/e)) \subseteq \mathscr{I}(G/G) \),
(iii) \( \pi_*(\mathscr{I}(G/e)) \subseteq \mathscr{I}(G/G) \),

where \( \pi: G/e \to G/G \) is the unique constant map.

Remark 2.4. (Corollary 4.5 in [4]) An ideal \( \mathscr{I} \subseteq T \) is prime if and only if for any transitive \( X, Y \in \text{Ob}(Gset) \) and any \( a \in T(X), b \in T(Y) \), the following two conditions become equivalent.

(1) \( a \in T(X) \) or \( b \in T(Y) \).
(2) For any \( C \in \text{Ob}(Gset) \) and for any pair of diagrams in \( Gset \)
   \[
   C \xleftarrow{v} D \xrightarrow{w} X, \quad C \xleftarrow{v'} D' \xrightarrow{w'} Y,
   \]
   \( (v\cdot w^*(a)) \cdot (v'\cdot w'^*(b)) \in \mathscr{I}(C) \) is satisfied.

Note that (1) always implies (2).

By the following lemma, it is enough to check (2) only when \( C, D, D' \) are transitive.

Lemma 2.5. Let \( \mathscr{I} \subseteq T \) be an ideal. Condition (2) in Remark 2.4 is equivalent to the following.

(2)' For any transitive \( C \in \text{Ob}(Gset) \) and for any pair of diagrams in \( Gset \)
   \[
   C \xleftarrow{v} D \xrightarrow{w} X, \quad C \xleftarrow{v'} D' \xrightarrow{w'} Y,
   \]
   where \( D \) and \( D' \) are transitive, \( (v\cdot w^*(a)) \cdot (v'\cdot w'^*(b)) \in \mathscr{I}(C) \) is satisfied.

Proof. It suffices to show (2)' implies (2). Assume (2)' holds, take any \( C \in \text{Ob}(Gset) \) and
   \[
   C \xleftarrow{v} D \xrightarrow{w} X, \quad C \xleftarrow{v'} D' \xrightarrow{w'} Y,
   \]
   with not necessarily transitive \( C, D, D' \).

   Let \( C = \coprod_{a \leq i \leq m} C_i \) be the orbit decomposition, and put
   \[
   D_i = v^{-1}(C_i), \quad D'_i = v'^{-1}(C_i),
   \]
   \[
   v_i = v|_{D_i}: D_i \to C_i, \quad v'_i = v'|_{D'_i}: D'_i \to C_i,
   \]
   \[
   w_i = w|_{D_i}: D_i \to X, \quad w'_i = w'|_{D'_i}: D'_i \to Y.
   \]

   Then we have \( v_i w^*(a) = (v_1 w^*_1(a), \ldots, v_m w^*_m(a)) \), where
   \[
   v_i w^*_i(a) = \begin{cases} v_i w^*_i(a) & \text{if } D_i \neq \emptyset \\ 0 & \text{if } D_i = \emptyset.
   \end{cases}
   \]

   Similarly for \( b \). In any case, \( (v_i w^*_i(a)) \cdot (v'_i w'^*_i(b)) \in \mathscr{I}(C_i) \) \( (1 \leq i \leq m) \) follows from (2)' , which means
   \[
   (v_i w^*(a)) \cdot (v'_i w'^*(b)) \in \mathscr{I}(C).
   \]

\[\square\]
Proposition 2.6. Let $T$ be a Tambara functor, and $\mathfrak{p} \subseteq T$ be a prime ideal. Let $T(G/e)^G$ denote the subring of $G$-invariant elements in $T(G/e)$:

$$T(G/e)^G = \{ x \in T(G/e) \mid gx = x \ (\forall g \in G) \}$$

Similarly for $\mathfrak{p}(G/e)^G$:

$$\mathfrak{p}(G/e)^G = \mathfrak{p}(G/e) \cap T(G/e)^G$$

Then, $\mathfrak{p}(G/e)^G \subseteq T(G/e)^G$ is a prime ideal (in the ordinary ring-theoretic meaning).

Proof. Suppose $a, b \in T(G/e)^G$ satisfies $ab \in \mathfrak{p}(G/e)$. By Lemma 2.5, it suffices to show for any transitive $C, D, D'$ and any pair of diagrams in $Gset$

$$C \xleftarrow{v} D \xrightarrow{w} G/e, \quad C \xleftarrow{v'} D' \xrightarrow{w'} G/e,$$

$$(v_*w^*(a)) \cdot (v'_*w'^*(b)) \in \mathfrak{p}(C)$$ is satisfied. Since $D$ and $D'$ are transitive with trivial stabilizers, we may assume $D = D' = G/e$. Furthermore, modifying $v$ and $v'$ by conjugations, we may assume

$$C = G/H, \quad v = v' = p_e^H : G/H \rightarrow G/e$$

for some $H \leq G$. Thus (2.2) is reduced to the case

$$G/H \xrightarrow{p_e^H} G/e \xrightarrow{w} G/e, \quad G/H \xrightarrow{p_e^H} G/e \xrightarrow{w'} G/e,$$

where $w, w'$ are the multiplication by some $g, g' \in G$. Then we have

$$( (p_e^H)_*w^*(a) ) \cdot ( (p_e^H)_*w'^*(b) ) = (p_e^H)_*((ga) \cdot (g'b))$$

$$= (p_e^H)_*(ab) \in \mathfrak{p}(G/H).$$

\qed

Corollary 2.7. If $\mathfrak{p} \subseteq \Omega$ is prime, then $\mathfrak{p}(G/e) \subseteq \Omega(G/e)$ is prime.

Proof. This immediately follows from the fact that $\Omega(G/e) \cong \mathbb{Z}$ has a trivial $G$-action. \qed

3. Spec $\Omega$ for $G = \mathbb{Z}/q\mathbb{Z}$

In the following, we assume $G = \mathbb{Z}/q\mathbb{Z}$ for some prime number $q$, and denote the canonical projection by $\pi = p_e^G : G/e \rightarrow G/G$.

3.1. Structure of $\Omega$.

Proposition 3.1. For $G = \mathbb{Z}/q\mathbb{Z}$, Burnside Tambara functor has the following structure.

1. There are isomorphisms of rings

$$\Omega(G/e) \xrightarrow{\cong} \mathbb{Z} ; \ell G/e \mapsto \ell,$$

$$\Omega(G/G) \xrightarrow{\cong} \mathbb{Z}[X]/(X^2 - qX) ; mG/e + nG/G \mapsto m + nX.$$

2. Under the isomorphisms in (1), the structure morphisms $\pi_+, \pi^*, \pi_*$ are

$$\pi_+ : \mathbb{Z} \rightarrow \mathbb{Z}[X]/(X^2 - qX) ; \ell \mapsto \ell X,$$

$$\pi^* : \mathbb{Z}[X]/(X^2 - qX) \rightarrow \mathbb{Z} ; m + nX \mapsto m + qn,$$

$$\pi_* : \mathbb{Z} \rightarrow \mathbb{Z}[X]/(X^2 - qX) ; \ell \mapsto \ell + \frac{\ell^q - \ell}{q} X.$$
Proof. The only non-trivial part will be
\[ \pi_\bullet(\ell) = \ell + \frac{\ell^q - \ell}{q} X. \]
This is shown by using the following.

**Fact 3.2.** (Proposition 4.17 in [4])
The following diagram is commutative.

\[
\begin{array}{ccc}
\Omega(G/e) & \xrightarrow{\sigma} & \mathbb{Z} \\
\downarrow \pi & & \downarrow m \\
\Omega(G/G) & \ni & m + n X
\end{array}
\]

From this fact, for any \( \ell \in \mathbb{Z} \) we have
\begin{equation}
\pi_\bullet(\ell) = \ell + n X
\end{equation}
for some \( n \in \mathbb{Z} \). Remark that \( n \geq 0 \) holds if \( \ell \geq 0 \).

Besides, by the definition of \( \pi_\bullet \), for any \( \ell \in \mathbb{N}_{\geq 0} \) we have
\[ \pi_\bullet((\Pi G/e \to G/e) = \{ \sigma \mid \sigma: G/e \to \Pi G/e, \text{ a section map for } \nabla \}, \]
and thus
\begin{equation}
\#(\pi_\bullet(\ell)) = \ell^q.
\end{equation}

From (3.1) and (3.2),
\[ \pi_\bullet(\ell) = \ell + \frac{\ell^q - \ell}{q} X \]
for any \( \ell \geq 0 \). As for a negative \( \ell \), since we have
\[ \pi_\bullet(\ell) = \pi_\bullet(-1)\pi_\bullet(|\ell|), \]
it will be enough to determine \( \pi_\bullet(-1) \).

By (3.1), we have \( \pi_\bullet(-1) = -1 + nX \) for some \( n \in \mathbb{Z} \), which should satisfy
\[ 1 = \pi_\bullet(-1)^2 = (-1 + nX)^2 = 1 + n(qn - 2)X. \]
When \( q \) is odd, it follows \( n = 0 \), and \( \pi_\bullet(-1) = -1. \) For \( q = 2 \), both \(-1\) and 
\(-1 + X\) satisfy \((-1)^2 = (-1 + X)^2 = 1.\) However, from the Mackey condition for the pullback
\[
\begin{array}{ccc}
\Pi G/e & \xrightarrow{\nabla} & G/e \\
\downarrow & & \downarrow \pi \\
G/e & \xrightarrow{\pi} & G/G
\end{array}
\]
\( \pi_\bullet(-1) \) should satisfy
\[ \pi^*\pi_\bullet(-1) = 1, \]
which leads to \( \pi_\bullet(-1) = -1 + X. \)

In any case, we obtain
\[ \pi_\bullet(\ell) = \ell + \frac{\ell^q - \ell}{q} X \quad (\forall \ell \in \mathbb{Z}) \]
for any prime \( q \).
3.2. Decomposition into fibers. Using the structural isomorphism in Proposition 3.1, we go on to determine $\text{Spec} \Omega$ for $G = \mathbb{Z}/q\mathbb{Z}$. By Corollary 2.7, any prime ideal $\mathfrak{p} \subseteq \Omega$ satisfies $\mathfrak{p}(G/e) = (p)$ for some prime $p$ or $p = 0$. Thus we have a map

$$F: \text{Spec} \Omega \to \text{Spec} \mathbb{Z}; \quad p \mapsto \mathfrak{p}(G/e).$$

($F$ will be shown to be continuous after $\text{Spec} \Omega$ is determined.)

**Definition 3.3.** Let $p \in \mathbb{Z}$ be prime or $p = 0$. We call an ideal $\mathcal{I} \subseteq \Omega$ is over $p$ if it satisfies $\mathcal{I}(G/e) = (p)$. A prime ideal over $p$ is simply a prime ideal $\mathfrak{p} \subseteq \Omega$ which is over $p$.

**Remark 3.4.** By the above arguments, we have

- $F^{-1}((p)) = \{ \mathfrak{p} \in \text{Spec} \Omega \mid \text{prime ideal over } p \}$,
- $\text{Spec} \Omega = \coprod_{(p) \in \text{Spec} \mathbb{Z}} F^{-1}((p))$.

In the following, we investigate the fibers $F^{-1}((p))$, in the cases $p = 0$, $p = q$, and $p \neq 0, q$.

For each $(p) \in \text{Spec} \mathbb{Z}$, its fiber $F^{-1}((p))$ at least contains one maximal point. In fact, the following was shown in [4].

**Fact 3.5.** (Corollary 4.42 in [4])

$$\text{Spec} \Omega \supseteq \{ \mathcal{I}(p) \mid p \in \mathbb{Z} \text{ is prime} \} \cup \{ \mathcal{I}(0) \} \cup \{ (0) \}.$$ 

Here, for each ideal $I \subseteq \Omega(G/e)$, ideal $\mathcal{I}(I) \subseteq \Omega$ is defined by

$$\mathcal{I}(G/e) = I, \quad \mathcal{I}(G/G) = (\pi^*)^{-1}(I).$$

$\mathcal{I}$ is the largest one, among all ideals $\mathcal{I} \subseteq \Omega$ satisfying $\mathcal{I}(G/e) = I$.

Under the isomorphism in Proposition 3.1, for any $\ell \in \mathbb{Z}$ we have

- $\mathcal{I}(\ell)(G/e) = (\ell) \subseteq \mathbb{Z}$,
- $\mathcal{I}(\ell)(G/G) = \{ m + nX \in \mathbb{Z}[X]/(X^2 - qX) \mid m + qn \in (\ell) \}$
  $$= \{ k\ell + n(X - q) \in \mathbb{Z}[X]/(X^2 - qX) \mid k, n \in \mathbb{Z} \}
  = (\ell, X - q) \subseteq \mathbb{Z}[X]/(X^2 - qX).$$

In this article, we denote $\mathcal{I}(p)$ by $m_p$. For any prime $p \neq 0$, $m_p$ is a maximal ideal of $\Omega$. Namely it is a closed point in $\text{Spec} \Omega$, while $m_0 = \mathcal{I}(0)$ is not. (For this reason, we prefer to use $\mathcal{I}(0)$ rather than $m_0$ only for $p = 0$.) On the other hand, $(0)$ is the smallest ideal of $\Omega$, namely the generic point in $\text{Spec} \Omega$. We have inclusions

$$(0) \subsetneq \mathcal{I}(0) \subsetneq m_p$$

for any prime $p \in \mathbb{Z}$.

3.3. The smallest ideal over $p$.

**Proposition 3.6.** For a prime $p \in \mathbb{Z}$ or $p = 0$, the smallest ideal $I_p \subseteq \Omega$ over $p$ is given by the following.

1. When $p \neq q$ (including the case $p = 0$),
   $$I_p(G/G) = (p) \subseteq \mathbb{Z}[X]/(X^2 - qX).$$

2. When $p = q$,
   $$I_q(G/G) = (qX, X - q) = (q^2, X - q) \subseteq \mathbb{Z}[X]/(X^2 - qX).$$
Proof. (1) \((p) \subseteq I_p(G/e)\) follows from
\[
p = (p + \frac{p^q - p}{q}X) - \frac{p^q - p}{pq} \cdot pX = \pi_+(p) - \frac{p^q - p}{pq} \pi_+(p).
\]
To show the converse, it suffices to show that \(\mathcal{I}(G/e) = (p) \subseteq \mathbb{Z}\) and \(\mathcal{I}(G/G) = (p) \subseteq \mathbb{Z}[X]/(X^2 - qX)\)
in fact form an ideal \(\mathcal{I}\) of \(\Omega\). By Corollary 2.3, this is equivalent to show
\[
\pi^*((p)) \subseteq (p), \\
\pi_+((p)) \subseteq (p), \\
\pi_*(p) \subseteq (p).
\]
However, these immediately follow from
\[
\pi^*(p) = p \in (p)
\]
and
\[
\pi_+(\ell p) = \ell pX \in (p) \\
\pi_*(\ell p) = \ell p + \frac{\ell p^q - \ell p}{q}X \in (p)
\]
for any \(\ell \in \mathbb{Z}\). (Remark that \(\pi^*\) is a ring homomorphism.)

(2) \((qX, X - q) \subseteq I_q(G/e)\) follows from
\[
qX = \pi_+(q)
\]
and
\[
X - q = q^{q-1}X - (q + \frac{q^q - q}{q}X) = \pi_+(q^{q-1}) - \pi_*(q).
\]
To show the converse, it suffices to show
\[
\pi^*((q^2, X - q)) \subseteq (q), \\
\pi_+((q)) \subseteq (qX, X - q), \\
\pi_*(q) \subseteq (qX, X - q).
\]
These follow from
\[
\pi^*(q^2) = q^2, \quad \pi^*(X - q) = 0 \in (q),
\]
and
\[
\pi_+{(\ell q)} = \ell qX \in (qX) \\
\pi_*(\ell q) = \ell(q - X) + \ell q^{q-1}X \in (q - X, qX)
\]
for any \(\ell \in \mathbb{Z}\). \(\square\)
3.4. All ideals over \( p \).

For \( p \neq 0 \), ideals \( \mathcal{I} \subseteq \Omega \) over \( p \) are only \( I_p \) and \( m_p \).

**Claim 3.7.** When \( p \in \mathbb{Z} \) is prime (\( \neq 0 \)), then there is no ideal between \( I_p \subsetneq m_p \).

**Proof.** It suffices to show that there is no element \( f \in \Omega(G/G) \) satisfying

\[
I_p(G/G) \subsetneq I_p(G/G) + (f) \subsetneq (p, X - q).
\]

By \( f \in (p, X - q) \), it should be of the form \( f = kp + n(X - q) \) for some \( k, n \in \mathbb{Z} \).

1. When \( p \neq q \), (3.3) is equal to

\[
(p) \subsetneq (p, f) \subsetneq (p, X - q).
\]

This will mean the existence of \( n \in \mathbb{Z} \) satisfying \( (p) \subsetneq (p, n(X - q)) \subsetneq (p, X - q) \).

However, since

\[
(p, n(X - q)) = \begin{cases} (p) & \text{if } p|n \\ (p, X - q) & \text{if } p \nmid n \end{cases},
\]

there should not exist such \( n \).

2. When \( p = q \), (3.3) is equal to

\[
(q^2, X - q) \subsetneq (q^2, X - q, f) \subsetneq (q, X - q).
\]

This will mean the existence of \( k \in \mathbb{Z} \) satisfying

\[
(q^2, X - q) \subsetneq (q^2, X - q, kq) \subsetneq (q, X - q).
\]

However, since

\[
(q^2, X - q, kq) = \begin{cases} (q^2, X - q) & \text{if } q|k \\ (q, X - q) & \text{if } q \nmid k \end{cases},
\]

there should not exist such \( k \). \( \square \)

On the other hand for \( p = 0 \), there are many ideals between \( (0) \subsetneq \mathcal{I} \).

**Claim 3.8.** If we define \( \mathcal{I}_{(0;n)} \subseteq \Omega \) by

\[
\mathcal{I}_{(0;n)}(G/e) = (0), \quad \mathcal{I}_{(0;n)}(G/G) = n(X - q),
\]

then \( \mathcal{I}_{(0;n)} \subseteq \Omega \) forms an ideal for each \( n \in \mathbb{Z} \). Indeed, these are exactly the all ideals \( \mathcal{I} \subseteq \Omega \) over \( 0 \):

\[
\{ \mathcal{I} \subseteq \Omega \text{ ideal} | \mathcal{I}(G/e) = 0 \} = \{ \mathcal{I}_{(0;n)} | n \in \mathbb{Z} \}
\]

**Proof.** Any ideal between \( (0) \subsetneq (X - q) \) in \( \mathbb{Z}[X]/(X^2 - qX) \) is of the form \( (n(X - q)) \) for some \( n \in \mathbb{Z} \). Since \( \mathcal{I}_{(0;n)}(G/e) = (0) \) and \( \mathcal{I}_{(0;n)}(G/G) = (n(X - q)) \) satisfy

\[
\pi^*(n(X - q)) = 0, \quad \pi_+(0) = 0, \quad \pi_*(0) = 0,
\]

\( \mathcal{I}_{(0;n)} \subseteq \Omega \) gives an ideal for each \( n \in \mathbb{Z} \). \( \square \)
3.5. **Criterion to be prime.** Let $p \in \mathbb{Z}$ be a prime or $p = 0$. Now we give a criterion for an ideal $\mathcal{I} \subseteq \Omega$ over $p$ to be prime.

**Proposition 3.9.** Let $p \in \mathbb{Z}$ be a prime or $p = 0$. Let $\mathcal{I} \subseteq \Omega$ be an ideal over $p$, not equal to $\mathfrak{m}_p$. Then $\mathcal{I}$ is not prime if and only if one of the following conditions is satisfied.

1. There exist $a, b \in \mathfrak{m}_p(G/G)$ satisfying
   
   $a \notin \mathcal{I}(G/G), \ b \notin \mathcal{I}(G/G), \ ab \in \mathcal{I}(G/G)$.

2. There exist $a \in \mathfrak{m}_p(G/G)$ and $b \in \Omega(G/e)$ satisfying
   
   $a \notin \mathcal{I}(G/G), \ \pi^*(b) \notin \mathcal{I}(G/G), \ a \cdot (\pi^*(b)) \in \mathcal{I}(G/G)$.

(Only here, we use the notation $\mathfrak{m}_0 = \mathcal{I}(0)$ for the consistency.) In particular, if $\mathcal{I}(G/G) \subseteq \Omega(G/G)$ is prime, then $\mathcal{I} \subseteq \Omega$ is prime.

More explicitly, these can be written as follows.

1. There exist $k, n, k', n' \in \mathbb{Z}$ satisfying
   
   $kp + n(X - q) \notin \mathcal{I}(G/G), \ k'p + n'(X - q) \notin \mathcal{I}(G/G), \ kk'p^2 + ((n'k + nk')p + nn'q)(X - q) \in \mathcal{I}(G/G)$.

2. There exist $k, n, \ell \in \mathbb{Z}$ satisfying
   
   $kp + n(X - q) \notin \mathcal{I}(G/G), \ \ell + \frac{\ell^{q} - \ell}{q}X \notin \mathcal{I}(G/G), \ kp(\ell + \frac{\ell^{q} - \ell}{q}X) + n\ell(X - q) \in \mathcal{I}(G/G)$.

**Proof.** By Lemma 2.5, $\mathcal{I} \subseteq \Omega$ is not prime if and only if there exist transitive $X, Y \in \text{Ob}(G\text{-set})$ and $a \in \Omega(X), b \in \Omega(Y)$ satisfying $a \notin \mathcal{I}(X), b \notin \mathcal{I}(Y)$ and

1. $(v \cdot w^*(a)) \cdot (v' \cdot w'^*(b)) \in \mathcal{I}(C)$ for any
   
   $C \leftarrow^v D \rightarrow^w X, \ C \leftarrow^{v'} D' \rightarrow^{w'} Y,$

   with $C, D, D'$ transitive.

We may consider this condition in the following three cases.

1. $X = Y = G/e$.
2. $X = Y = G/G$.
3. $X = G/G, \ Y = G/e$.

(1) If $X = Y = G/e$, then $(1)$ is reduced to

   $ab \in \mathcal{I}(G/e) = (p),$

   which implies automatically $a$ or $b$ is in $\mathcal{I}(G/e)$. Thus we can exclude this case.

(2) If $X = Y = G/G$, then condition $(1)$ is equivalent to

   $ab \in \mathcal{I}(G/G), \ \pi^*(a)\pi^*(b) \in \mathcal{I}(G/G), \ (\pi_\cdot\pi^*(a)) \cdot b \in \mathcal{I}(G/G), \ a \cdot (\pi_\cdot\pi^*(b)) \in \mathcal{I}(G/G), \ (\pi_\cdot\pi^*(a)) \cdot (\pi_\cdot\pi^*(b)) \in \mathcal{I}(G/G).$

Since $\mathcal{I}(G/e) = (p)$ is prime, it follows that $\pi^*(a)$ or $\pi^*(b)$ is in $\mathcal{I}(G/e)$. Thus we may assume $\pi^*(a) \in (p)$, namely $a \in \mathfrak{m}_p(G/G)$. Then the above conditions are reduced to

   $ab \in \mathcal{I}(G/G), \ a \cdot (\pi_\cdot\pi^*(b)) \in \mathcal{I}(G/G).$
The existence of such $a$ and $b$ can be divided into the following two cases. Remark that $\pi^*(b) \notin I(G/e)$ will imply $b \notin I(G/G)$.

(2-1) (the case $\pi^*(b) \notin (p)$)
There exist $a \in m_p(G/G)$ and $b \in \Omega(G/e)$ satisfying
\[
a \notin I(G/G), \quad \pi^*(b) \notin I(G/e),
\]
\[
ab \in I(G/G), \quad a \cdot (\pi \pi^*(b)) \in I(G/G).
\]

(2-2) (the case $\pi^*(b) \in (p)$)
There exist $a, b \in m_p(G/G)$ satisfying
\[
a \notin I(G/G), \quad b \notin I(G/G), \quad ab \in I(G/G), \quad a \cdot (\pi.(b)) \in I(G/G).
\]

(3) If $X = G/G$ and $Y = G/e$, then for $a \in \Omega(G/G)$ and $b \in \Omega(G/e)$ which are not in $I$, condition (o) is reduced to
\[
(\pi^*(a)) \cdot b \in I(G/e), \quad a \cdot (\pi \pi^*(b)) \in I(G/G).
\]
Since $b \notin I(G/e) = (p)$, the condition $\pi^*(a) \cdot b \in I(G/e)$ is equivalent to $\pi^*(a) \in I(G/e)$, namely to $a \in m_p(G/G)$. The existence of such $a$ and $b$ can be divided into the following two cases. Remark that $\pi^*(b) \notin I(G/G)$ will imply $b \notin I(G/e)$.

(3-1) (the case $\pi^*(b) \notin I(G/G)$)
There exist $a \in m_p(G/G)$ and $b \in \Omega(G/e)$ satisfying
\[
a \notin I(G/G), \quad \pi^*(b) \notin I(G/e), \quad a \cdot (\pi \pi^*(b)) \in I(G/G).
\]

(3-2) (the case $\pi^*(b) \in I(G/G)$)
There exist $a \in m_p(G/G)$ and $b \in \Omega(G/e)$ satisfying
\[
a \notin I(G/G), \quad b \notin I(G/e), \quad \pi \pi^*(b) \in I(G/G).
\]
Note that, in (3-2), the conditions for $a$ and $b$ are completely separated. Moreover since $I(G/G) \subsetneq m_p(G/G)$, such $a$ always exists. Thus (3-2) is reduced to the following.

(3-2') There exists $b \in \Omega(G/e)$ satisfying
\[
b \notin I(G/e) \quad \text{and} \quad \pi^*(b) \in I(G/G).
\]
However, this never happens. Indeed, since we have
\[
\pi^*(\pi^*(b)) = \ell^q
\]
for any $\ell \in \Omega(G/e)$, we obtain
\[
\pi^*(\ell) \Rightarrow \pi^*(\pi^*(b)) \in I(G/e) \Rightarrow \ell \in I(G/e).
\]

By the arguments so far, $I \subseteq \Omega$ is not prime if and only if one of (2-1), (2-2), (3-1) is satisfied. Furthermore, we see (2-1) implies (3). Indeed if $a$ and $b$ satisfy (2-1), then $a \in \Omega(G/G)$ and $b' = \pi^*(b) \in \Omega(G/e)$ satisfy
\[
a \notin I(G/G), \quad b' \notin I(G/e),
\]
\[
a \cdot (\pi \pi^*(b')) \in I(G/G), \quad \pi^*(a) \cdot b' = \pi^*(ab) \in I(G/e).
\]
Thus, we can conclude that $I \subseteq \Omega$ is not prime if and only if one of (2-2), (3-1) is satisfied. These are respectively the conditions (c1), (c2) in the statement of the proposition.
The latter part can be shown easily by using $m_p(G/G) = (p, X - q)$. An easy observation $X(X - q) = 0$ will help the calculation.

3.6. **Determine each fiber.** Proposition 3.9 enables us to determine the structure of $\text{Spec} \Omega$.

**Corollary 3.10.** Let $p \in \mathbb{Z}$ be a prime or $p = 0$. In each fiber $F^{-1}((p))$ over $p$, we have the following.

1. (the case $p \neq q, 0$)
   - If $p \neq 0$ is a prime other than $q$, then $I_p \subseteq \Omega$ in Proposition 3.9 is prime.
     - For this reason, in the rest we denote $I_p$ by $p_p$.
2. (the case $p = q$)
   - $I_q \subseteq \Omega$ is not prime.
3. (the case $p = 0$)
   - $\mathcal{I}_{(0;n)} \subseteq \Omega$ in Claim 3.8 is prime if and only if $n = 0$ or $n = \pm 1$.

**Proof.** (1) It suffices to show that either of (c1)', (c2)' does not occur. Remark that we have $p_p(G/G) = (p)$.

(c1)' For any $k, n, k', n'$, since
\[
kp + n(X - q) \notin p_p(G/G) \iff p|n,
\]
\[
k'p + n'(X - q) \notin p_p(G/G) \iff p|n',
\]
\[
kk'p^2 + ((n'k + nk')p + nn'q)(X - q) \in p_p(G/G) \iff p|nn',
\]
these never happens simultaneously.

(c2)' For any $k, n, l \in \mathbb{Z}$, since
\[
kp + n(X - q) \notin p_p(G/G) \iff p|n,
\]
\[
\ell + \frac{\ell^q - \ell}{q}X \notin p_p(G/G) \iff p|\ell,
\]
\[
k\ell(X - q) \in p_p(G/G) \iff p|n\ell,
\]
these never happens simultaneously.

(2) We show (c1) holds for $I_q$. Remark that we have $I_q(G/G) = (qX, X - q)$.

For $a = b = X \in m_q(G/G)$, we have
\[
a = b \notin I_q(G/G) \quad \text{and} \quad ab = qX \in I_q(G/G).
\]
Thus $I_q$ is not prime.

(3) We already know $(0) \subseteq \Omega$ and $\mathcal{I}_{(0)} \subseteq \Omega$ are prime. It suffices to show $\mathcal{I}_{(0;n)} \subseteq \Omega$ is not prime for $n \notin \{ -1, 0, 1 \}$. We show (c2) holds for these $n$. Remark that we have $\mathcal{I}_{(0;n)}(G/G) = (n(X - q))$.

For $a = X - q \in \Omega(G/G)$ and $b = n \in \Omega(G/e)$, we have
\[
a \notin \mathcal{I}_{(0;n)}(G/G),
\]
\[
\pi_*(b) = n + \frac{n^q - n}{q}X \notin \mathcal{I}_{(0;n)}(G/G),
\]
\[
(X - q) \cdot (\pi_*(b)) = n(X - q) \in \mathcal{I}_{(0;n)}(G/G).
\]
Thus $\mathcal{I}_{(0;n)}$ is not prime for $n \notin \{ -1, 0, 1 \}$. 

\[\square\]
3.7. Total picture. As a consequence, \( \text{Spec} \Omega \) can be determined as

\[
\text{Spec} \Omega = \{(0)\} \cup \{\mathcal{I}(0)\} \cup \{m_q\} \\
\cup \{(p_p \mid p \in \mathbb{Z} \text{ is prime, } p \neq q) \cup \{m_p \mid p \in \mathbb{Z} \text{ is prime, } p \neq q\}.
\]

Inclusions are

\[
(0) \subsetneq \mathcal{I}(0) \subsetneq m_q \\
\mathfrak{p}_p \subsetneq m_p (p \neq q).
\]

Especially the dimension of \( \text{Spec} \Omega \) is 2.

\( m_q \) and \( m_p \)'s are the closed points, and (0) is the generic point in \( \text{Spec} \Omega \). If we represent the points in \( \text{Spec} \Omega \) by their closures, \( \text{Spec} \Omega \) with fibration \( F \) can be depicted as follows. It can be also easily seen that \( F \) is continuous.

**Figure 1.** \( \text{Spec} \Omega \) for \( G = \mathbb{Z}/q\mathbb{Z} \)

REFERENCES


DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, KAGOSHIMA UNIVERSITY, 1-21-35 KORIMOTO, KAGOSHIMA, 890-0065 JAPAN

_E-mail address: nakaoka@sci.kagoshima-u.ac.jp_