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Extending the Erdős–Ko–Rado theorem

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1 Introduction

Leonard systems [23] naturally arise in representation theory, combinatorics, and the theory of orthogonal polynomials (see e.g. [25, 28]). Hence they are receiving considerable attention. Indeed, the use of the name "Leonard system" is motivated by a connection to a theorem of Leonard [11], [2, p. 260], which involves the q-Racah polynomials [1] and some related polynomials of the Askey scheme [9]. Leonard systems also play a role in coding theory; see [10, 18].

Let $\Phi = \big(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d\big)$ be a Leonard system over a field $\mathbb{K}$, and V the vector space underlying Φ (see Section 2 for formal definitions). Then $V = \bigoplus_{i=0}^d E_i^* V$ and $\dim E_i^* V = 1$ $(0 \leqslant i \leqslant d)$. We have a "canonical" (ordered) basis for V associated with this direct sum decomposition, called a *standard basis*. There are 8 variations for this basis. Next, let $U_\ell = \big(\sum_{i=0}^\ell E_i^* V\big) \cap \big(\sum_{j=\ell}^d E_j V\big)$ $(0 \leqslant \ell \leqslant d)$. Then, again it follows that $V = \bigoplus_{\ell=0}^d U_\ell$ and $\dim U_\ell = 1$ $(0 \leqslant \ell \leqslant d)$. We have a "canonical" basis for V associated with this *split decomposition*, called a *split basis*. The split decomposition is crucial in the theory of Leonard systems,[1] and there are 16 variations for the split basis. Altogether, Terwilliger [24] defined 24 bases for V and studied in detail the transition matrices between these bases as well as the matrices representing A and A^* with respect to them.

In this article, we introduce another basis for V, which we call an *Erdős–Ko–Rado* (or *EKR*) *basis* for V, under a mild condition on the eigenvalues of A and A^*. As its name suggests, this basis arises in connection with the famous *Erdős–Ko–Rado theorem* [6] in extremal set theory. Indeed, Delsarte's *linear programming method* [4], which is closely related to Lovász's ϑ-function bound [12, 15] on the Shannon capacity of graphs, has been successfully used in the proofs of the "Erdős–Ko–Rado theorems" for certain families of *Q-polynomial distance-regular graphs*[2] [29, 7, 16, 19] (including the original 1961 theorem of Erdős et al.), and constructing appropriate feasible solutions to the dual programs amounts to describing the EKR bases for the Leonard systems associated with these graphs; see Section 4. It seems that the previous constructions of the feasible solutions depend on the geometric/algebraic structures which are more or less specific to the family of graphs in question. Our results give a uniform description of such feasible solutions in terms of the *parameter arrays* of Leonard systems. We refer the reader to [20] for more details.

2 Leonard systems

Let $\mathbb{K}$ be a field, d a *positive* integer, $\mathscr{A}$ a $\mathbb{K}$-algebra isomorphic to the full matrix algebra $\mathrm{Mat}_{d+1}(\mathbb{K})$, and V an irreducible left $\mathscr{A}$-module. We remark that V is unique up to isomorphism, and that V has dimension $d+1$. An element A of $\mathscr{A}$ is said to be *multiplicity-free* if it has $d+1$ mutually distinct eigenvalues in $\mathbb{K}$. Let A be a multiplicity-free element of $\mathscr{A}$ and $\{\theta_i\}_{i=0}^d$ an ordering of the eigenvalues of A. Let $E_i : V \to V(\theta_i)$ $(0 \leqslant i \leqslant d)$ be the projection map onto $V(\theta_i)$ with respect to $V = \bigoplus_{i=0}^d V(\theta_i)$, where $V(\theta_i) = \{\boldsymbol{u} \in V : A\boldsymbol{u} = \theta_i \boldsymbol{u}\}$. We call E_i the *primitive idempotent* of A associated with θ_i. We note that the E_i are polynomials in A.

A *Leonard system* in $\mathscr{A}$ ([23, Definition 1.4]) is a sequence

$$\Phi = \big(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d\big) \tag{1}$$

satisfying the following axioms (LS1)–(LS5):

(LS1) Each of A, A^* is a multiplicity-free element in $\mathscr{A}$.[3]

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[1] In some cases, V has the structure of an evaluation module of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, and the split decomposition corresponds to its weight space decomposition; see e.g. [8].

[2] Q-polynomial distance-regular graphs are thought of as finite/combinatorial analogues of compact symmetric spaces of rank one; see [2, pp. 311–312].

[3] It is customary that A^* denotes the conjugate transpose of A. It should be stressed that we are *not* using this convention.

(LS2) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A.

(LS3) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^*.

(LS4) $E_i^*AE_j^* = \begin{cases} 0 & \text{if } |i-j|>1 \\ \neq 0 & \text{if } |i-j|=1 \end{cases} \quad (0 \leqslant i,j \leqslant d).$

(LS5) $E_iA^*E_j = \begin{cases} 0 & \text{if } |i-j|>1 \\ \neq 0 & \text{if } |i-j|=1 \end{cases} \quad (0 \leqslant i,j \leqslant d).$

We call d the *diameter* of Φ, and say that Φ is *over* $\mathbb{K}$. We refer the reader to [23, 26, 28] for background on Leonard systems.

For the rest of this article, $\Phi = \left(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d\right)$ shall always denote the Leonard system (1). Note that the following are Leonard systems in $\mathscr{A}$:

$$\begin{aligned} \Phi^* &= \left(A^*; A; \{E_i^*\}_{i=0}^d; \{E_i\}_{i=0}^d\right), \\ \Phi^{\downarrow} &= \left(A; A^*; \{E_i\}_{i=0}^d; \{E_{d-i}^*\}_{i=0}^d\right), \\ \Phi^{\Downarrow} &= \left(A; A^*; \{E_{d-i}\}_{i=0}^d; \{E_i^*\}_{i=0}^d\right). \end{aligned}$$

Viewing $*, \downarrow, \Downarrow$ as permutations on all Leonard systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \quad \Downarrow * = * \downarrow, \quad \downarrow * = * \Downarrow, \quad \downarrow\Downarrow = \Downarrow\downarrow.$$

The group generated by the symbols $*, \downarrow, \Downarrow$ subject to the above relations is the dihedral group D_4 with 8 elements. We shall use the following notational convention:

Notation 2.1. For any $g \in D_4$ and for any object f associated with Φ, we let f^g denote the corresponding object for $\Phi^{g^{-1}}$; an example is $E_i^*(\Phi) = E_i(\Phi^*)$.

It is known [26, Theorem 6.1] that there is a unique antiautomorphism $\dagger$ of $\mathscr{A}$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. For the rest of this article, let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ be a nondegenerate bilinear form on V such that ([26, Section 15])

$$\langle X\boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = \langle \boldsymbol{u}_1, X^\dagger \boldsymbol{u}_2 \rangle \quad (\boldsymbol{u}_1, \boldsymbol{u}_2 \in V, \ X \in \mathscr{A}).$$

We shall write

$$||\boldsymbol{u}||^2 = \langle \boldsymbol{u}, \boldsymbol{u} \rangle \quad (\boldsymbol{u} \in V).$$

Notation 2.2. Throughout the article, we fix a nonzero vector $\boldsymbol{v}^g$ in $E_0^g V$ for each $g \in D_4$. We abbreviate $\boldsymbol{v} = \boldsymbol{v}^1$ where 1 is the identity of D_4. For convenience, we also assume $\boldsymbol{v}^{g_1} = \boldsymbol{v}^{g_2}$ whenever $E_0^{g_1}V = E_0^{g_2}V$ $(g_1, g_2 \in D_4)$. We may remark that $||\boldsymbol{v}^g||^2$, $\langle \boldsymbol{v}^g, \boldsymbol{v}^{*g} \rangle$ are nonzero for any $g \in D_4$; see [26, Lemma 15.5].

We now recall a few direct sum decompositions of V, as well as (ordered) bases for V associated with them. First, $\dim E_i V^* = 1$ $(0 \leqslant i \leqslant d)$ and $V = \bigoplus_{i=0}^d E_i^* V$. By [26, Lemma 10.2], $E_i^* \boldsymbol{v} \neq 0$ $(0 \leqslant i \leqslant d)$, so that $\{E_i^* \boldsymbol{v}\}_{i=0}^d$ is a basis for V, called a *Φ-standard basis* for V. Next, let $U_\ell = \left(\sum_{i=0}^\ell E_i^* V\right) \cap \left(\sum_{j=\ell}^d E_j V\right)$ $(0 \leqslant \ell \leqslant d)$. Then, again $\dim U_\ell = 1$ $(0 \leqslant \ell \leqslant d)$ and $V = \bigoplus_{\ell=0}^d U_\ell$, which is referred to as the *Φ-split decomposition* of V [28]. We observe $U_0 = E_0^* V$ and $U_d = E_d V$. For $0 \leqslant i \leqslant d$, let θ_i be the eigenvalue of A associated with E_i. Then it follows that $(A - \theta_\ell I)U_\ell = U_{\ell+1}$ and $(A^* - \theta_\ell^* I)U_\ell = U_{\ell-1}$ for $0 \leqslant \ell \leqslant d$, where $U_{-1} = U_{d+1} = 0$ [23, Lemma 3.9]. For $0 \leqslant i \leqslant d$, let τ_ℓ, η_ℓ be the following polynomials in $\mathbb{K}[z]$:

$$\tau_\ell(z) = \prod_{i=0}^{\ell-1}(z - \theta_i), \quad \eta_\ell(z) = \tau_\ell^{\Downarrow}(z) = \prod_{i=0}^{\ell-1}(z - \theta_{d-i}).$$

By the above comments it follows that $\tau_\ell(A)\boldsymbol{v}^* \in U_\ell$ $(0 \leqslant \ell \leqslant d)$ and $\{\tau_\ell(A)\boldsymbol{v}^*\}_{\ell=0}^d$ is a basis for V, called a *Φ-split basis* for V. Moreover, there are nonzero scalars φ_ℓ $(1 \leqslant \ell \leqslant d)$ in $\mathbb{K}$ such that $A^*\tau_\ell(A)\boldsymbol{v}^* = \theta_\ell^* \tau_\ell(A)\boldsymbol{v}^* + \varphi_\ell \tau_{\ell-1}(A)\boldsymbol{v}^*$.

Let $\phi_\ell = \varphi_\ell^{\Downarrow}$ $(1 \leqslant \ell \leqslant d)$. The *parameter array* of Φ is

$$p(\Phi) = \left(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\right).$$

Terwilliger [23, Theorem 1.9] showed that the isomorphism class[4] of Φ is determined by $p(\Phi)$ and gave a classification of the parameter arrays of Leonard systems; cf. [27, Section 5]. In particular, the sequences $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ are recurrent in the following sense:

$$\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i},\ \frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i}\ \text{are equal and independent of } i \quad (2 \leqslant i \leqslant d-1). \tag{2}$$

It also follows that

$$\phi_i = \varphi_1 \vartheta_i + (\theta_i^* - \theta_0^*)(\theta_{d-i+1}-\theta_0) \quad (1 \leqslant i \leqslant d), \tag{3}$$

where

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (1 \leqslant i \leqslant d).$$

Note that $\vartheta_1 = \vartheta_d = 1$. Moreover,

$$\vartheta_{d-i+1} = \vartheta_i, \quad \vartheta_i^* = \vartheta_i \quad (1 \leqslant i \leqslant d). \tag{4}$$

The parameter array behaves nicely with respect to the D_4 action:

Lemma 2.3 ([23, Theorem 1.11]). *The following hold.*

(i) $$p(\Phi^*) = \left(\{\theta_i^*\}_{i=0}^d; \{\theta_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d\right).$$

(ii) $$p(\Phi^\downarrow) = \left(\{\theta_i\}_{i=0}^d; \{\theta_{d-i}^*\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d\right).$$

(iii) $$p(\Phi^\Downarrow) = \left(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d\right).$$

3 The Erdős–Ko–Rado basis

We shall mainly work with the $\Phi^\downarrow$-split decomposition $V = \bigoplus_{\ell=0}^d U_\ell^\downarrow$, where we recall

$$U_\ell^\downarrow = \left(\sum_{i=d-\ell}^d E_i^* V\right) \cap \left(\sum_{j=\ell}^d E_j V\right) \quad (0 \leqslant \ell \leqslant d).$$

We now "modify" the $U_\ell^\downarrow$ and introduce the subspaces W_t $(0 \leqslant t \leqslant d)$ defined by[5]

$$W_t = \left(E_0^* V + \sum_{i=d-t+1}^d E_i^* V\right) \cap \left(E_0 V + \sum_{j=t+1}^d E_j V\right) \quad (0 \leqslant t \leqslant d).$$

Observe $W_t \neq 0$ $(0 \leqslant t \leqslant d)$, $W_0 = E_0^* V$, and $W_d = E_0 V$. Note also that

$$W_t^* = W_{d-t} \quad (0 \leqslant t \leqslant d). \tag{5}$$

Proposition 3.1. *Let $\boldsymbol{w} \in W_t$. Then the following hold.*

(i) $$\begin{aligned} \boldsymbol{w} =& E_0 \boldsymbol{w} + \frac{\langle \boldsymbol{w}, E_0 \boldsymbol{v}^* \rangle}{\|E_0 \boldsymbol{v}^*\|^2} \cdot \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_t^*(\theta_0^*)} \\ &\times \sum_{j=t+1}^d \frac{\phi_{d-j+1}\cdots\phi_d}{\varphi_2\cdots\varphi_j(\theta_j-\theta_0)} \left(\sum_{\ell=t+1}^j \frac{\tau_\ell(\theta_j)\eta_{\ell-1}^*(\theta_0^*)\vartheta_\ell}{\phi_{d-\ell+1}\cdots\phi_{d-t}}\right) E_j \boldsymbol{v}^*. \end{aligned}$$

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[4] A Leonard system Ψ in a $\mathbb{K}$-algebra $\mathscr{B}$ is *isomorphic* to Φ if there is a $\mathbb{K}$-algebra isomorphism $\gamma : \mathscr{A} \to \mathscr{B}$ such that $\Psi = \Phi^\gamma := \left(A^\gamma; A^{*\gamma}; \{E_i^\gamma\}_{i=0}^d; \{E_i^{*\gamma}\}_{i=0}^d\right)$.

[5] The subscript t is used in accordance with the concept of *t-intersecting families* in the Erdős–Ko–Rado theorem; see Section 4.

(ii)
$$\begin{aligned} \boldsymbol{w} =& E_0^*\boldsymbol{w} + \frac{\langle \boldsymbol{w}, E_0^*\boldsymbol{v}\rangle}{||E_0^*\boldsymbol{v}||^2} \cdot \frac{\eta_t^*(\theta_0^*)}{\eta_d^*(\theta_0^*)\eta_{d-t}(\theta_0)} \\ & \times \sum_{i=d-t+1}^{d} \frac{\phi_1 \dots \phi_i}{\varphi_2 \dots \varphi_i(\theta_i^* - \theta_0^*)} \left(\sum_{\ell=d-t+1}^{i} \frac{\tau_\ell^*(\theta_i^*)\eta_{\ell-1}(\theta_0)\vartheta_\ell}{\phi_{d-t+1}\dots\phi_\ell} \right) E_i^*\boldsymbol{v}. \end{aligned}$$

In particular, $E_0^*W_t \neq 0$, $E_0W_t \neq 0$, *and* $\dim W_t = 1$.

Notation 3.2. Henceforth we let q be a nonzero scalar in the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$ such that $q+q^{-1}+1$ is equal to the common value of (2) for $2 \leqslant i \leqslant d-1$. We call q a *base* for Φ.[6] By convention, if $d < 3$ then q can be taken to be *any* nonzero scalar in $\overline{\mathbb{K}}$.

Lemma 3.3 (cf. [17, (6.4)]). *For* $1 \leqslant i \leqslant d$, *we have* $\vartheta_i = 0$ *precisely when* $q = -1$, d *is odd, and* i *is even.*

By Proposition 3.1 and Lemma 3.3, it follows that

Lemma 3.4. *Let* q *be as above. Then for* $1 \leqslant t \leqslant d-1$, *the following hold.*
(i) *Suppose* $q \neq -1$, *or* $q = -1$ *and* d *is even. Then* $E_{d-t+1}^*W_t \neq 0$, $E_{t+1}W_t \neq 0$.
(ii) *Suppose* $q = -1$ *and* d *is odd. Then* $E_{d-t+1}^*W_t \neq 0$ *(resp.* $E_{t+1}W_t \neq 0$*) if and only if* t *is odd (resp. even).*

Corollary 3.5. *Let* q *be as above. Then the following hold.*
(i) *Suppose* $q \neq -1$, *or* $q = -1$ *and* d *is even. Then* $V = \bigoplus_{t=0}^d W_t$. *Moreover,* $\sum_{t=0}^h W_t = E_0^*V + \sum_{i=d-h+1}^d E_i^*V$, $\sum_{t=h}^d W_t = E_0V + \sum_{j=h+1}^d E_jV$ $(0 \leqslant h \leqslant d)$.
(ii) *Suppose* $q = -1$ *and* d *is odd. Then* $W_{2s-1} = W_{2s}$ *for* $1 \leqslant s \leqslant \lfloor d/2 \rfloor$.

Proof. (i): Immediate from Lemma 3.4 (i).

(ii): By Lemma 3.4 (ii), we find

$$W_{2s-1} = \left(E_0^*V + \sum_{i=d-2s+2}^{d} E_i^*V \right) \cap \left(E_0V + \sum_{j=2s+1}^{d} E_jV \right) = W_{2s}$$

for $1 \leqslant s \leqslant \lfloor d/2 \rfloor$. □

By virtue of Corollary 3.5, we make the following assumption.

Assumption 3.6. With reference to Notation 3.2, for the rest of this article we shall assume $q \neq -1$, or $q = -1$ and d is even.[7]

Now we are ready to introduce an Erdős–Ko–Rado basis for V.

Definition 3.7. With reference to Assumption 3.6, for $0 \leqslant t \leqslant d$ let $\boldsymbol{w}_t$ be the (unique) vector in W_t such that $E_0\boldsymbol{w}_t = E_0\boldsymbol{v}^*$. We call $\{\boldsymbol{w}_t\}_{t=0}^d$ a (Φ-)*Erdős–Ko–Rado* (or *EKR*) basis for V.

We note that the basis $\{\boldsymbol{w}_t\}_{t=0}^d$ linearly depends on the choice of $\boldsymbol{v}^* \in E_0^*V$. In particular, we have $\boldsymbol{w}_0 = \boldsymbol{v}^*$ and $\boldsymbol{w}_d = E_0\boldsymbol{v}^*$. Our preference for the normalization $E_0\boldsymbol{w}_t = E_0\boldsymbol{v}^*$ comes from the applications to the Erdős–Ko–Rado theorem; see Section 4. The following theorem gives the transition matrix from each of the $\Phi^{\downarrow}$-split basis $\{\tau_\ell(A)\boldsymbol{v}^{*\downarrow}\}_{\ell=0}^d$, the Φ^*-standard basis $\{E_j\boldsymbol{v}^*\}_{j=0}^d$, and the Φ-standard basis $\{E_i^*\boldsymbol{v}\}_{i=0}^d$, to the EKR basis $\{\boldsymbol{w}_t\}_{t=0}^d$.

Theorem 3.8. *The following hold for* $0 \leqslant t \leqslant d$.

(i)
$$\boldsymbol{w}_t = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^*\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle} \left\{ \sum_{\ell=0}^{t} \frac{\eta_{d-\ell}(\theta_0)}{\eta_d(\theta_0)} \tau_\ell(A)\boldsymbol{v}^{*\downarrow} + \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_t^*(\theta_0^*)} \sum_{\ell=t+1}^{d} \frac{\eta_\ell^*(\theta_0^*)}{\phi_{d-\ell+1}\dots\phi_{d-t}} \tau_\ell(A)\boldsymbol{v}^{*\downarrow} \right\}.$$

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[6] We may remark that if $d \geqslant 3$ then Φ has at most two bases, i.e., q and q^{-1}.

[7] The Leonard systems with $d \geqslant 3$ that do *not* satisfy this assumption are precisely those of "Bannai/Ito type" [27, Example 5.14] with d odd, and those of "Orphan type" [27, Example 5.15].

(ii)
$$\boldsymbol{w}_t = E_0\boldsymbol{v}^* + \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_t^*(\theta_0^*)} \sum_{j=t+1}^{d} \frac{\phi_{d-j+1}\cdots\phi_d}{\varphi_2\cdots\varphi_j(\theta_j-\theta_0)} \left(\sum_{\ell=t+1}^{j} \frac{\tau_\ell(\theta_j)\eta_{\ell-1}^*(\theta_0^*)\vartheta_\ell}{\phi_{d-\ell+1}\cdots\phi_{d-t}} \right) E_j\boldsymbol{v}^*.$$

(iii)
$$\begin{aligned}\boldsymbol{w}_t =& \frac{\langle \boldsymbol{v}, \boldsymbol{v}^*\rangle}{||\boldsymbol{v}||^2} \left\{ \frac{\eta_d^*(\theta_0^*)\eta_{d-t}(\theta_0)}{\phi_1\cdots\phi_{d-t}\eta_t^*(\theta_0^*)} E_0^*\boldsymbol{v} \right. \\ &\left. + \sum_{i=d-t+1}^{d} \frac{\phi_{d-t+1}\cdots\phi_i}{\varphi_2\cdots\varphi_i(\theta_i^*-\theta_0^*)} \left(\sum_{\ell=d-t+1}^{i} \frac{\tau_\ell^*(\theta_i^*)\eta_{\ell-1}(\theta_0)\vartheta_\ell}{\phi_{d-t+1}\cdots\phi_\ell} \right) E_i^*\boldsymbol{v} \right\}.\end{aligned}$$

Corollary 3.9. *Let $\{\boldsymbol{w}_t^*\}_{t=0}^d$ be the Φ^*-EKR basis for V normalized so that $E_0^*\boldsymbol{w}_t^* = E_0^*\boldsymbol{v}$ $(0 \leqslant t \leqslant d)$. Then*
$$\boldsymbol{w}_t^* = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^*\rangle}{||\boldsymbol{v}^*||^2} \cdot \frac{\phi_1\cdots\phi_t\eta_{d-t}^*(\theta_0^*)}{\eta_d^*(\theta_0^*)\eta_t(\theta_0)} \boldsymbol{w}_{d-t} \quad (0 \leqslant t \leqslant d).$$

Proof. By (5), $\boldsymbol{w}_t^*$ is a scalar multiple of $\boldsymbol{w}_{d-t}$, and the scalar is found by looking at the coefficient of $E_0^*\boldsymbol{v}$ in $\boldsymbol{w}_{d-t}$ as given in Theorem 3.8 (iii), and by noting that $\langle \boldsymbol{v}, \boldsymbol{v}^*\rangle^2 = ||\boldsymbol{v}||^2||\boldsymbol{v}^*||^2$. □

Next we give the transition matrix from $\{\boldsymbol{w}_t\}_{t=0}^d$ to each of the three bases $\{\tau_\ell(A)\boldsymbol{v}^{*\downarrow}\}_{\ell=0}^d$, $\{E_i^*\boldsymbol{v}\}_{i=0}^d$, and $\{E_j\boldsymbol{v}^*\}_{j=0}^d$.

Theorem 3.10. *Setting $\boldsymbol{w}_{-1} = \boldsymbol{w}_{d+1} = 0$, the following hold.*[8]

(i)
$$\begin{aligned}\tau_\ell(A)\boldsymbol{v}^{*\downarrow} =& \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^*\rangle} \cdot \frac{\eta_d(\theta_0)}{\varphi_1} \left\{ \frac{\phi_{d-\ell+1}(\theta_\ell-\theta_0)}{\eta_{d-\ell+1}(\theta_0)\vartheta_\ell} \boldsymbol{w}_{\ell-1} \right. \\ &\left. + \frac{1}{\eta_{d-\ell}(\theta_0)} \left(\frac{\phi_{d-\ell}}{\vartheta_{\ell+1}} + \frac{\phi_{d-\ell+1}}{\vartheta_\ell} - \varphi_1 \right) \boldsymbol{w}_\ell + \frac{\theta_{d-\ell}^* - \theta_0^*}{\eta_{d-\ell-1}(\theta_0)\vartheta_{\ell+1}} \boldsymbol{w}_{\ell+1} \right\}\end{aligned}$$

for $0 \leqslant \ell \leqslant d$, where we interpret $\phi_0/\vartheta_{d+1} = \phi_{d+1}/\vartheta_0 = \varphi_1$.

(ii)
$$\begin{aligned}E_j\boldsymbol{v}^* =& \frac{\varphi_2\cdots\varphi_j\eta_d(\theta_0)}{\phi_{d-j+1}\cdots\phi_d\tau_j(\theta_j)\eta_{d-j}(\theta_j)} \left\{ -\frac{\phi_{d-j+1}\eta_{d-j}(\theta_j)}{\eta_{d-j}(\theta_0)\vartheta_j} \boldsymbol{w}_{j-1} \right. \\ &+ (\theta_j-\theta_0) \sum_{t=j}^{d-1} \frac{\eta_{d-t-1}(\theta_j)}{\eta_{d-t}(\theta_0)} \left(\frac{\phi_{d-t}}{\vartheta_{t+1}} + \frac{(\theta_j-\theta_{t+1})(\theta_{d-t+1}^*-\theta_0^*)}{\vartheta_t} \right) \boldsymbol{w}_t \\ &\left. + \big(\varphi_1 + (\theta_1^*-\theta_0^*)(\theta_j-\theta_0)\big)\boldsymbol{w}_d \right\}\end{aligned}$$

for $1 \leqslant j \leqslant d$, and $E_0\boldsymbol{v}^* = \boldsymbol{w}_d$.

(iii)
$$\begin{aligned}E_i^*\boldsymbol{v} =& \frac{\langle \boldsymbol{v}, \boldsymbol{v}^*\rangle}{||\boldsymbol{v}^*||^2} \cdot \frac{\varphi_2\cdots\varphi_i}{\tau_i^*(\theta_i^*)\eta_{d-i}^*(\theta_i^*)} \left\{ \frac{\phi_{i+1}\cdots\phi_d}{\eta_d(\theta_0)} \big(\varphi_1 + (\theta_1-\theta_0)(\theta_i^*-\theta_0^*)\big)\boldsymbol{w}_0 \right. \\ &+ (\theta_i^*-\theta_0^*) \sum_{t=1}^{d-i} \frac{\phi_{i+1}\cdots\phi_{d-t}\eta_{t-1}^*(\theta_i^*)}{\eta_{d-t}(\theta_0)} \left(\frac{\phi_{d-t+1}}{\vartheta_t} \right. \\ &\left.\left. + \frac{(\theta_i^*-\theta_{d-t+1}^*)(\theta_{t+1}-\theta_0)}{\vartheta_{t+1}} \right) \boldsymbol{w}_t + \frac{\eta_{d-i}^*(\theta_i^*)(\theta_i^*-\theta_0^*)}{\eta_{i-1}(\theta_0)\vartheta_i} \boldsymbol{w}_{d-i+1} \right\}\end{aligned}$$

for $1 \leqslant i \leqslant d$, and $E_0^*\boldsymbol{v} = \langle \boldsymbol{v}, \boldsymbol{v}^*\rangle||\boldsymbol{v}^*||^{-2}\boldsymbol{w}_0$.

Finally, we describe the matrices representing A and A^* with respect to the EKR basis $\{\boldsymbol{w}_t\}_{t=0}^d$. We use the following notation:

$$\Delta_s = \frac{\eta_{s-1}^*(\theta_0^*)\big((\theta_{d-s+1}^*-\theta_0^*)\vartheta_{s+1} - (\theta_{d-s}^*-\theta_0^*)\vartheta_s\big)}{\phi_{d-s+1}\cdots\phi_d\eta_{d-s-1}(\theta_0)\vartheta_{s+1}} \quad (1 \leqslant s \leqslant d-1).$$

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[8] We also interpret the coefficients of $\boldsymbol{w}_{-1}$ and $\boldsymbol{w}_{d+1}$ as zero, whenever these terms appear.

We note that

$$\Delta_s^* = \frac{\eta_{s-1}(\theta_0)\big((\theta_{d-s+1}-\theta_0)\vartheta_{s+1}-(\theta_{d-s}-\theta_0)\vartheta_s\big)}{\phi_1\ldots\phi_s\eta^*_{d-s-1}(\theta^*_0)\vartheta_{s+1}} \quad (1 \leqslant s \leqslant d-1).$$

by virtue of Theorem 2.3 (i) and (4).

Theorem 3.11. *With the above notation, the following hold.*

(i)
$$\begin{aligned} A\boldsymbol{w}_t =& \theta_{t+1}\boldsymbol{w}_t + \left(\frac{\phi_{d-t+1}\ldots\phi_d\eta_{d-t}(\theta_0)}{\eta^*_t(\theta^*_0)}\Delta_{t+1} - (\theta_{t+1}-\theta_0)\right)\boldsymbol{w}_{t+1} \\ &+ \frac{\phi_{d-t+1}\ldots\phi_d\eta_{d-t}(\theta_0)}{\eta^*_t(\theta^*_0)}\left\{\sum_{s=t+2}^{d-1}(\Delta_s-\Delta_{s-1})\boldsymbol{w}_s - \Delta_{d-1}\boldsymbol{w}_d\right\}\end{aligned}$$

for $0 \leqslant t \leqslant d-2$, $A\boldsymbol{w}_{d-1} = \theta_d\boldsymbol{w}_{d-1} - (\theta_d-\theta_0)\boldsymbol{w}_d$, *and* $A\boldsymbol{w}_d = \theta_0\boldsymbol{w}_d$.

(ii)
$$\begin{aligned} A^*\boldsymbol{w}_t =& -\frac{\phi_1\ldots\phi_d}{\eta_d(\theta_0)}\Delta^*_{d-1}\boldsymbol{w}_0 + \sum_{s=1}^{t-2}\frac{\phi_1\ldots\phi_{d-s}\eta^*_s(\theta^*_0)}{\eta_{d-s}(\theta_0)}(\Delta^*_{d-s}-\Delta^*_{d-s-1})\boldsymbol{w}_s \\ &+ \left(\frac{\phi_1\ldots\phi_{d-t+1}\eta^*_{t-1}(\theta^*_0)}{\eta_{d-t+1}(\theta_0)}\Delta^*_{d-t+1} - \frac{\phi_{d-t+1}}{\theta_t-\theta_0}\right)\boldsymbol{w}_{t-1} + \theta^*_{d-t+1}\boldsymbol{w}_t\end{aligned}$$

for $2 \leqslant t \leqslant d$, $A^*\boldsymbol{w}_1 = \theta^*_d\boldsymbol{w}_1 - (\theta^*_d-\theta^*_0)\boldsymbol{w}_0$, *and* $A^*\boldsymbol{w}_0 = \theta^*_0\boldsymbol{w}_0$.

We end this section with an attractive formula for Δ_s.

Lemma 3.12. *For* $1 \leqslant s \leqslant d-1$, *we have*

$$(\theta_{d-s+1}-\theta_0)\vartheta_{s+1} - (\theta_{d-s}-\theta_0)\vartheta_s = \frac{\big(\theta_{d-\lfloor\frac{s}{2}\rfloor}-\theta_{\lfloor\frac{s}{2}\rfloor}\big)\big(\theta_{d-\lfloor\frac{s-1}{2}\rfloor}-\theta_{\lfloor\frac{s+1}{2}\rfloor}\big)}{\theta_d-\theta_0}.$$

Proof. This is verified using [23, Lemma 10.2]. □

Corollary 3.13. *For* $1 \leqslant s \leqslant d-1$, *we have*

$$\Delta_s = \frac{\eta^*_{s-1}(\theta^*_0)\big(\theta^*_{d-\lfloor\frac{s}{2}\rfloor}-\theta^*_{\lfloor\frac{s}{2}\rfloor}\big)\big(\theta^*_{d-\lfloor\frac{s-1}{2}\rfloor}-\theta^*_{\lfloor\frac{s+1}{2}\rfloor}\big)}{\phi_1\ldots\phi_s\eta^*_{d-s-1}(\theta^*_0)(\theta^*_d-\theta^*_0)\vartheta_{s+1}}.$$

Proof. Immediate from Lemma 3.12 and (4). □

4 Applications to the Erdős–Ko–Rado theorems

The Erdős–Ko–Rado type theorems for various families of Q-polynomial distance-regular graphs provide one of the most successful applications of Delsarte's linear programming method [4].[9]

Let Γ be a Q-polynomial distance-regular graph with vertex set $X = V(\Gamma)$. (We refer the reader to [2, 3, 21] for the background material.) Pick a "base vertex" $x \in X$ and let $\Phi = \Phi(\Gamma)$ be the Leonard system (over $\mathbb{K} = \mathbb{R}$) afforded on the primary module of the Terwilliger algebra $\boldsymbol{T}(x)$; cf. [18, Example (3.5)].[10] The second eigenmatrix $Q = (Q_{ij})_{i,j=0}^d$ of Γ is defined by[11]

$$E_j\boldsymbol{v}^* = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^*\rangle}{||\boldsymbol{v}||^2}\sum_{i=0}^d Q_{ij}E^*_i\boldsymbol{v} \quad (0 \leqslant j \leqslant d).$$

As summarized in [19], every "t-intersecting family" $Y \subseteq X$ is associated with a vector $\boldsymbol{e} = (e_0, e_1, \ldots, e_d)$ (known as the inner distribution of Y) satisfying

$$e_0 = 1, \quad e_1 \geqslant 0, \ldots, e_{d-t} \geqslant 0, \quad e_{d-t+1} = \cdots = e_d = 0,$$
$$|Y| = (\boldsymbol{e}Q)_0, \quad (\boldsymbol{e}Q)_1 \geqslant 0, \ldots, (\boldsymbol{e}Q)_d \geqslant 0.$$

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[9] See, e.g., [5, 14] for more applications as well as extensions of this method.

[10] We note that Φ is independent of $x \in X$ up to isomorphism.

[11] The matrix Q is denoted P^* in [26, p. 264].

Viewing these as forming a linear programming maximization problem, we are then to construct a vector $\boldsymbol{f} = (f_0, f_1, \dots, f_d)$ such that

$$f_0 = 1,\ f_1 = \cdots = f_t = 0,\ (\boldsymbol{f}Q^{\mathsf{T}})_1 = \cdots = (\boldsymbol{f}Q^{\mathsf{T}})_{d-t} = 0, \tag{6}$$

which turns out to give a feasible solution to the dual problem (provided that $f_{t+1} \geqslant 0, \dots, f_d \geqslant 0$).

Set $\boldsymbol{w} = \sum_{j=0}^{d} f_j E_j \boldsymbol{v}^*$. Then

$$\boldsymbol{w} = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle}{\|\boldsymbol{v}\|^2} \sum_{j=0}^{d} f_j \sum_{i=0}^{d} Q_{ij} E_i^* \boldsymbol{v} = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle}{\|\boldsymbol{v}\|^2} \sum_{i=0}^{d} (\boldsymbol{f}Q^{\mathsf{T}})_i E_i^* \boldsymbol{v}.$$

Hence it follows that $\boldsymbol{f}$ satisfies (6) if and only if $\boldsymbol{w} = \boldsymbol{w}_t$. In particular, such a vector $\boldsymbol{f}$ is unique and is given by Theorem 3.8 (ii).

We now give three examples. First, suppose Φ is of "dual Hahn" type [27, Example 5.12], i.e.,

$$\theta_i = \theta_0 + hi(i+1+s), \quad \theta_i^* = \theta_0^* + s^* i$$

for $0 \leqslant i \leqslant d$, and

$$\varphi_i = hs^* i(i-d-1)(i+r), \quad \phi_i = hs^* i(i-d-1)(i+r-s-d-1)$$

for $1 \leqslant i \leqslant d$, where h, s are nonzero. Then it follows that

$$f_j = \frac{(1-j)_t (j+s+2)_t (s-r+1)_j (-1)^{j-1}}{(t-r+s+1)(s+2)_t t! (r+2)_{j-1}} \cdot {}_3F_2\left(\begin{matrix} t-j+1, t+j+s+2, 1 \\ t+1, t-r+s+2 \end{matrix} \,\middle|\, 1 \right)$$

for $t+1 \leqslant j \leqslant d$. If Γ is the Johnson graph $J(v,d)$ [3, Section 9.1], then Φ is of dual Hahn type with $r = d-v-1$, $s = -v-2$ and $s^* = -v(v-1)/d(v-d)$; cf. [22, pp. 191–192]. In this case, the vector $\boldsymbol{f}$ was essentially constructed by Wilson [29] and was used to prove the original Erdős–Ko–Rado theorem [6] in full generality.

Suppose Φ is of "Krawtchouk" type [27, Example 5.13], i.e.,

$$\theta_i = \theta_0 + si, \quad \theta_i^* = \theta_0^* + s^* i$$

for $0 \leqslant i \leqslant d$, and

$$\varphi_i = ri(i-d-1), \quad \phi_i = (r - ss^*) i (i-d-1)$$

for $1 \leqslant i \leqslant d$, where r, s, s^* are nonzero. Then it follows that

$$f_j = \frac{(1-j)_t}{t!} \left(\frac{r-ss^*}{r} \right)^{j-1} \cdot {}_2F_1\left(\begin{matrix} t-j+1, 1 \\ t+1 \end{matrix} \,\middle|\, -\frac{ss^*}{r-ss^*} \right)$$

for $t+1 \leqslant j \leqslant d$. If Γ is the Hamming graph $H(d,n)$ [3, Section 9.2], then Φ is of Krawtchouk type with $r = n(n-1)$ and $s = s^* = -n$; cf. [22, p. 195]. In this case, the vector $\boldsymbol{f}$ coincides (up to normalization) with the weight distribution of an *MDS* code [13, Chapter 11], i.e., a code attaining the Singleton bound.[12]

Finally, suppose Φ is of the most general "q-Racah" type [27, Example 5.3], i.e.,

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i}, \quad \theta_i^* = \theta_0^* + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}$$

for $0 \leqslant i \leqslant d$, and

$$\begin{aligned} \varphi_i &= hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-r_1q^i)(1-r_2q^i), \\ \phi_i &= hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(r_1 - s^*q^i)(r_2 - s^*q^i)/s^* \end{aligned}$$

for $1 \leqslant i \leqslant d$, where r_1, r_2, s, s^*, q are nonzero and $r_1 r_2 = ss^* q^{d+1}$. Then it follows that the f_j are expressed as balanced ${}_4\phi_3$ series:

$$\begin{aligned} f_j = &\frac{s^{*j-1} q^{(d+1)(j-1)+t} (q^{1-j};q)_t (sq^{j+2};q)_t (r_1q^{-d}/s^*;q)_j (r_2q^{-d}/s^*;q)_j}{(1-r_1q^{t-d}/s^*)(1-r_2q^{t-d}/s^*)(q;q)_t (sq^2;q)_t (r_1q^2;q)_{j-1} (r_2q^2;q)_{j-1}} \\ &\times {}_4\phi_3\left(\begin{matrix} q^{t-j+1}, sq^{t+j+2}, q^{t-d-1}/s^*, q \\ q^{t+1}, r_1q^{t-d+1}/s^*, r_2q^{t-d+1}/s^* \end{matrix} \,\middle|\, q;q \right) \end{aligned}$$

for $t+1 \leqslant j \leqslant d$.

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[12] In this regard, one may also wish to call $\{\boldsymbol{w}_t\}_{t=0}^{d}$ an *MDS* basis or a *Singleton* basis.

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