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Omae's knot and 12_{a990} are ribbon

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ABSTRACT. The purpose of this note is twofold: First, we prove that Omae's knot is ribbon, which was known to be homotopically slice. Second, we give a sufficient condition for a given knot to be ribbon. As a corollary, we show that the knot 12_{a990} is ribbon, which was known to be slice.

1. OMAE'S KNOT IS RIBBON

A knot K in the 3-sphere $S^3 = \partial D^4$ is *slice* if there exists a smoothly embedded disk $D^2 \subset D^4$ such that $\partial D^2 = K$. A knot K is *ribbon* if there exists a smoothly immersed disk $D^2 \subset S^3$ with only ribbon singularities such that $\partial D^2 = K$. It is easy to see that every ribbon knot is slice. The slice-ribbon conjecture due to Fox [5] states that every slice knot is ribbon, which has been a long-standing unsolved problem in knot theory.

In the positive direction, the slice-ribbon conjecture was conformed for two-bridge knots [19, Lisca], certain pretzel knots [11, Greene-Jabuka], certain Montesinos knots [17, Lecuona] and simple slice knots [23, Shibuya]. On the other hand, potential counterexamples to the slice-ribbon conjecture are demonstrated through the study of the 4-dimensional smooth Poincaré conjecture [2, 6, 7, 9].

Omae [22] studied the knot depicted in the left of Figure 1. The first author and Jong [1] observed that Omae's knot bounds a smoothly embedded disk in a homotopy 4-ball W which is represented by the handle diagram as in the right of Figure 1 (see also Section 4). In this note, we prove the following.

Theorem 1.1. *The* 4*-manifold* W *is diffeomorphic to the standard* 4*-ball.*

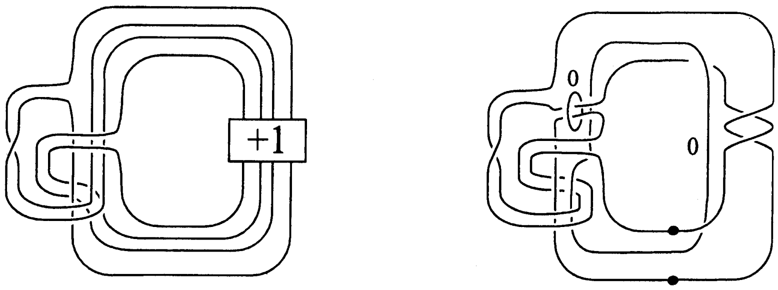

FIGURE 1. Omae's knot and a homotopy 4-ball W.

Proof. Handle calculus in Figure 2 implies that W is diffeomorphic to the standard 4-ball. □

Corollary 1.2. *Omae's knot is slice. Furthermore, it is ribbon.*

Proof. Theorem 1.1 implies that Omae's knot is slice. Recall that Omae's knot is isotopic to the boundary of cocore disk of the 2-handle (colored Grey) of the top left handle diagram in Figure 2. By chasing Omae's knot in handle diagrams in Figure 2, we obtain a ribbon presentation of Omae's knot as in Figure 3. □

Remark 1.3. Another potential counterexample to the slice-ribbon conjecture is the $(2, 1)$-cable of the figure eight knot. Livingston and Melvin [18] and Kawauchi [14] proved that it is algebraically slice. Furthermore Kawauchi [15] showed that it is rationally slice. On the other hand, by the theorem of Casson-Gordon [4], Miyazaki [21] proved that it is not ribbon. Untill now, it is not known whether the $(2, 1)$-cable of the figure eight knot is slice or not. See also Gomp-Miyazaki [8].

2. THE KNOT 12_{a990} IS RIBBON

The simplest slice knot which might not be ribbon is 12_{a990}. Indeed, Herald, Kirk and Livingston [12] showed that the connected sum of 12_{a990} and right- and left-handed trefoils is ribbon, implying that 12_{a990} is slice. However it was unknown whether 12_{a990} is ribbon [1].

A t_n*-move* is a tangle replacement as in Figure 4. In this section, we show the following.

[1]C. Livingston (e-mail communication) informed us that they knew that 12_{a990} is ribbon, however they did not write that 12_{a990} is ribbon in [12].

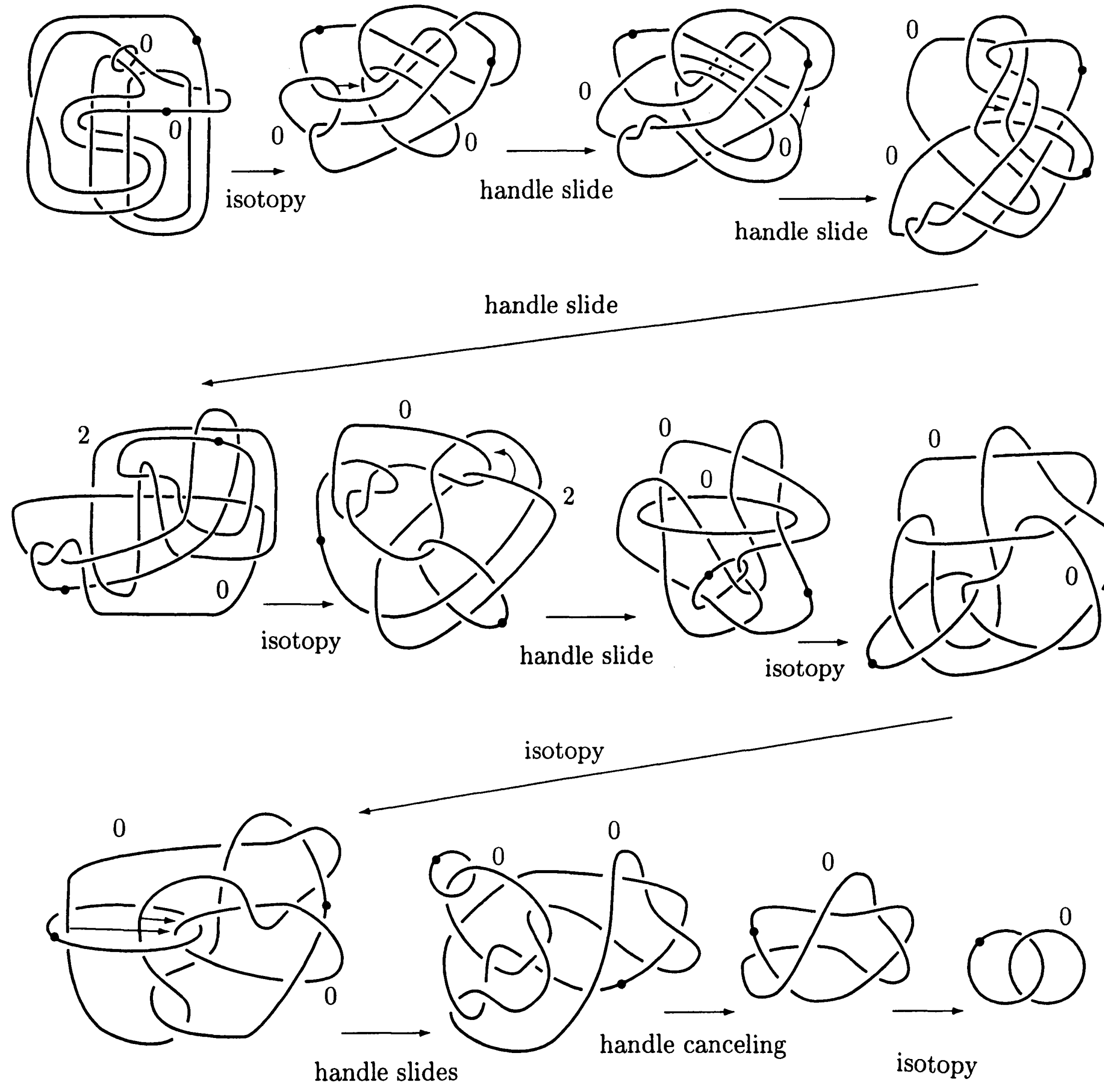

FIGURE 2. Handle diagrams which represent W.

Theorem 2.1. *Let K be a knot. If we obtain the 3-component unlink from K by applying a t_{2n+1}- and $t_{-(2n+1)}$-move, then K is ribbon.*

We denote by $T(p,q)$ the torus knot of type (p,q). First, we show the following.

Lemma 2.2. *Let K be a knot. If we obtain the 3-component unlink from K by applying a t_{2n+1}- and $t_{-(2n+1)}$-move, then $K\#T(2,2n+1)\#T(2,-(2n+1))$ is ribbon, where $\#$ denotes the connected sum.*

Proof. We may assume that a t_{2n+1}-move and a $t_{-(2n+1)}$-move are done simultaneously. In other words, there exist two trivial tangles (B_+, T_+) and

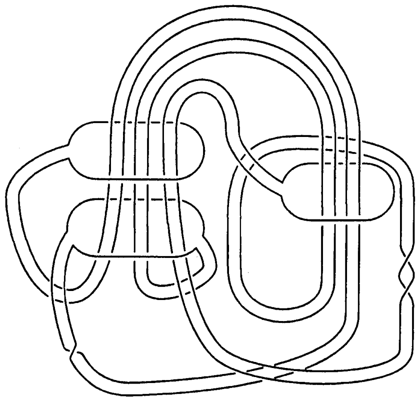

FIGURE 3. A ribbon presentation of Omae's knot.

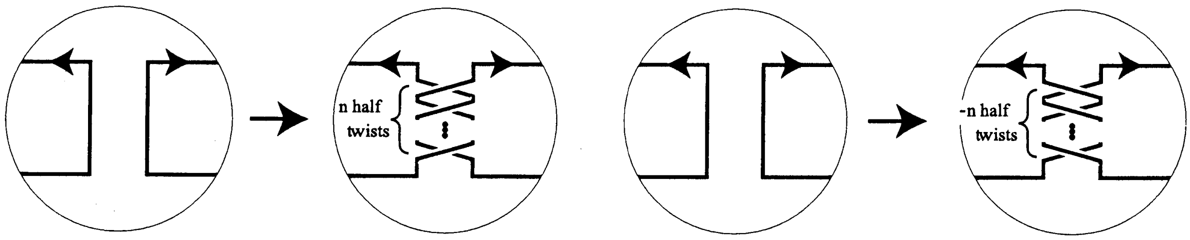

FIGURE 4. The definition of a t_n-move for $n > 0$ (left) and for $n < 0$ (right).

(B_-, T_-) with $B_+ \cap B_- = \emptyset$ such that if we apply a t_{2n+1}-move for (B_+, T_+) and a $t_{-(2n+1)}$-move for (B_-, T_-), then we obtain the 3-component unlink. Now we consider $K\#T(2, 2n+1)\#T(2, -(2n+1))$ as in Figure 5. If we add

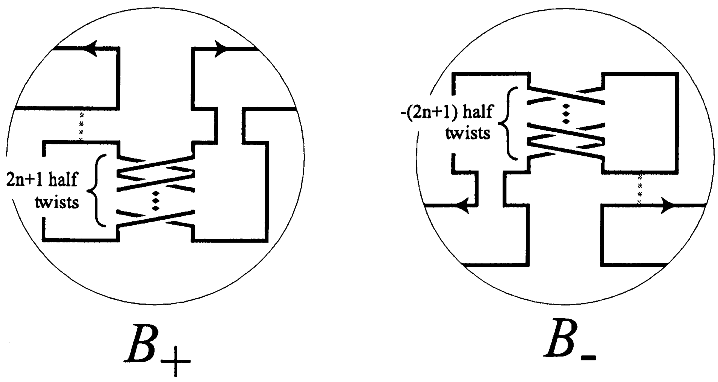

FIGURE 5. The knot $K\#T(2, 2n+1)\#T(2, -(2n+1))$.

two bands along dotted arcs in Figure 5, then the resulting 3-component link is trivial by the assumption. Therefore $K\#T(2, 2n+1)\#T(2, -(2n+1))$ is ribbon. $\square$

Now we prove Theorem 2.1.

Proof of Theorem 2.1. By the assumption, there exist two trivial tangles (B_+, T_+) and (B_-, T_-) with $B_+ \cap B_- = \emptyset$ such that if we apply a t_{2n+1}-move for (B_+, T_+) and a $t_{-(2n+1)}$-move for (B_-, T_-), then we obtain the 3-component unlink. If we need, by choosing another 3-balls, we may assume that two trivial tangles (B_+, T_+) and (B_-, T_-) are connected as in Figure 6. Now we consider again $K\#T(2, 2n+1)\#T(2, -(2n+1))$ as in Figure 5 with

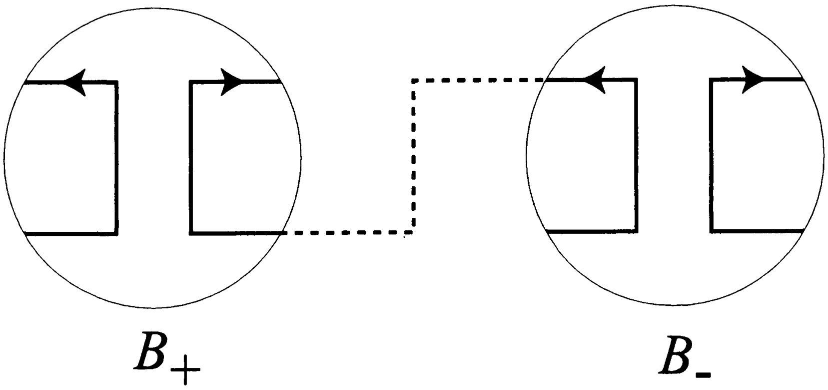

FIGURE 6. Connectivity of two trivial tangles (B_+, T_+) and (B_-, T_-).

two bands attached along dotted arcs. Then we deform $T(2, -(2n+1))$ as in Figure 7 with the band. We can see the knot $T(2, 2n+1)\#T(2, -(2n+1))$

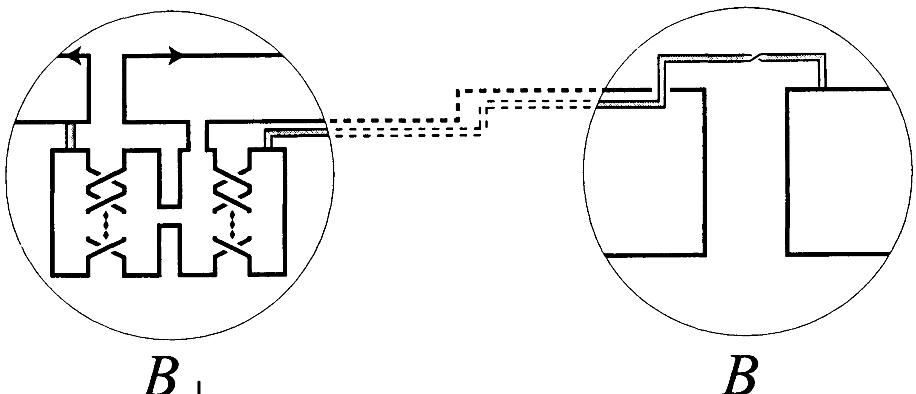

FIGURE 7. A deformation of $T(2, -(2n+1))$.

in B_+ which is known to be ribbon. We concentrate on B_+ and deform the tangle (in B_+) as in Figure 8. Then we obtain a ribbon presentation of K. □

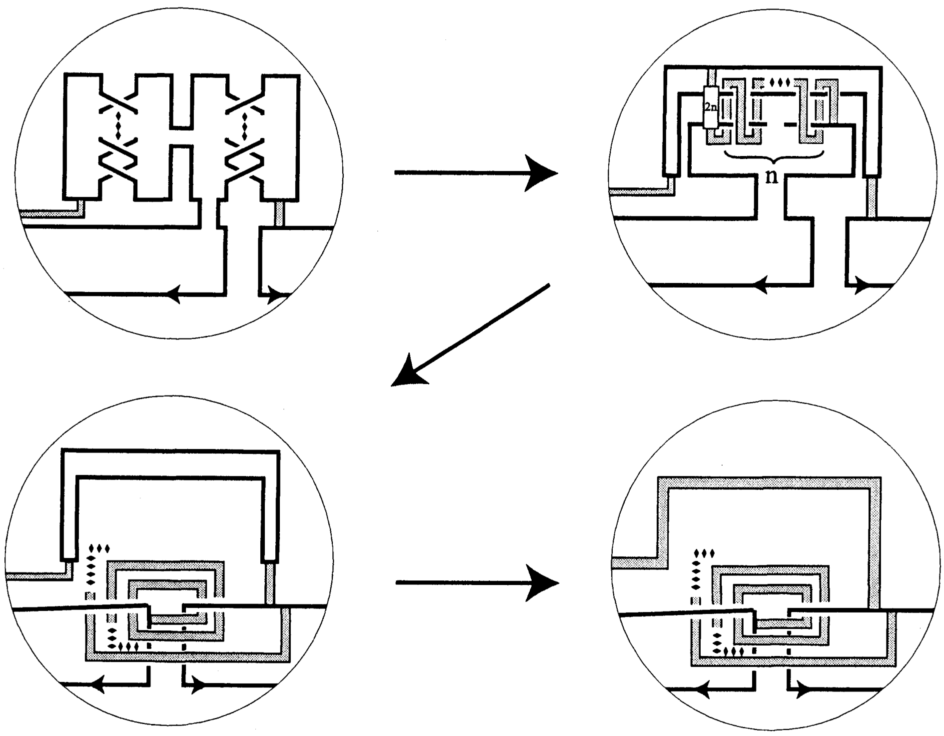

FIGURE 8. Deformations in B_+.

As a corollary of Theorem 2.1, we obtain the following.

Corollary 2.3. *The knot K_n in the left of Figure 9 is ribbon. In particular, $K_1 = 12_{a990}$ is ribbon.*

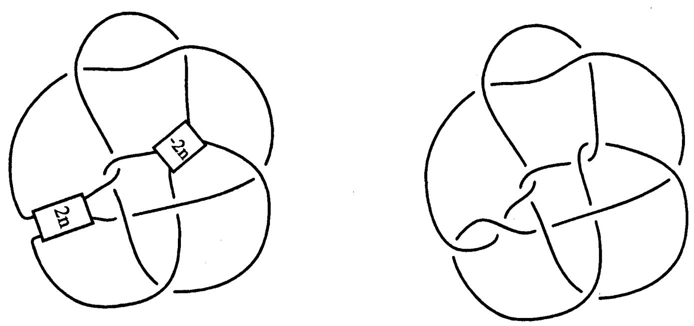

FIGURE 9. Left: the knot K_n, Right: the knot 12_{a990}.

Proof. We choose two 3-balls B_+ and B_- as in the left of Figure 10. We apply a t_{2n+1}-move for $(B_+, K_n \cap B_+)$ and a $t_{-(2n+1)}$-move for $(B_-, K_n \cap B_-)$. Then we obtain the 3-component link as in the right of Figure 10 which is trivial. Therefore K_n is ribbon by Theorem 2.1. □

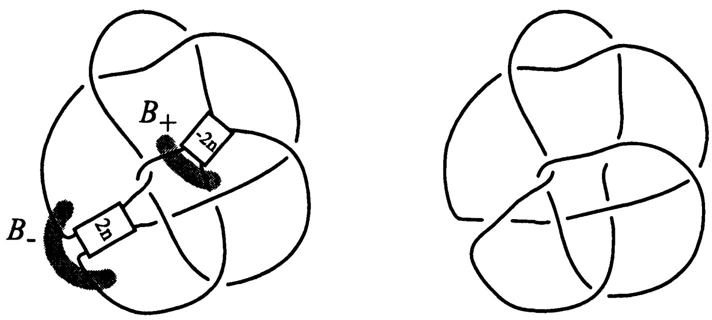

FIGURE 10. Left: 3-balls B_+ and B_-, Right: the 3-component unlink.

3. On the ribbon fusion number

A ribbon knot K is *of m-fusions* if K is isotopic to

$$\bigcup_{i=0}^{m} S_i^1 - \mathrm{int}(\bigcup_{j=1}^{m} b_j(\partial I \times I) \cup \bigcup_{j=1}^{m} b_j(I \times \partial I)$$

where $\bigcup_{i=0}^{m} S_i^1$ is the $(m+1)$-component unlink and $b_j : I \times I \longrightarrow S^3$ $(j = 1, 2, ..., m)$ are disjoint embeddings such that

$$S_i^1 \cap b_j = \begin{cases} b_j(\{0\} \times I) & \text{if } i = 0, \\ b_j(\{1\} \times I) & \text{if } i = j, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is known that a ribbon knot is of m-fusions for some m [20, 25]. The *ribbon fusion number* of a ribbon knot is defined to be the minimal number of such m. For the study of the ribbon fusion number, see [3, 13, 24].

Question 1. *Is the ribbon fusion number of Omae's knot two ?*

Question 2. *Is the ribbon fusion number of the knot* 12_{a990} *two ?*

4. Homotopy 4-spheres associated to unknotting number one ribbon knots

In the conference, Intelligence of Low-dimensional Topology, the first author talked on annulus twist, diffeomorphic 4-manifolds, and slice knots. In this section, we assume some terminologies in [1]. The first author and Jong showed the following.

Proposition 4.1 ([1]). *Let K be an unknotting number one knot, (A, b, c, ε) the associated band presentation and K_n the knot obtained from K by applying an annulus twist n times. If K is ribbon, then there exists a homotopy 4-ball W_n with $\partial W_n = S^3$ such that K_n bounds a smoothly embedded disk in W_n. In particular, we can associate a homotopy 4-sphere for each n.*

Let K be the knot 8_{20}. Note that the unknotting number of 8_{20} is one and the associated band presentation of K is depicted in Figure 11. Let K_n the knot obtained from K by applying an annulus twist n times. Then K_1 is Omae's knot. Since 8_{20} is ribbon, we can associate a homotopy 4-sphere Σ_n for each n by Proposition 4.1. Theorem 1.1 implies that Σ_1 is standard.

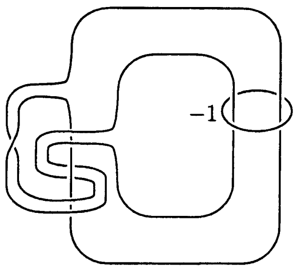

FIGURE 11. The associated band presentation for 8_{20}.

Conjecture 4.2. *The homotopy 4-sphere Σ_n is standard for each n.*

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