THE REAL REPRESENTATION ASSOCIATED WITH COPRIME NORMAL SUBGROUPS

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Dedicated to Professor K. Shimakawa on his 60th birthday

Abstract. Let $G$ be a finite group. In this article, we introduce a mutually coprime family of normal subgroups of $G$ and the real $G$-module associated with the family, and we report interesting results on the real $G$-module.

1. PRELIMINARY

Throughout this paper, $G$ is a finite group. We mean by a real $G$-module a real $G$-representation space of finite dimension. Let $S(G)$ denote the set of all subgroups of $G$.

In the study of smooth $G$-actions on disks and spheres, there are important families of normal subgroups of $G$: for examples, $\{G\}$, $\{G^{(2)}\}$, $\{G^{\text{nil}}\}$,

$$\mathcal{K}(G) = \{G^{(p)} \mid p \text{ is a prime}\},$$

and

$$\mathcal{N}_p(G) = \{H \triangleleft G \mid |G/H| = 1 \text{ or } p\},$$

where $G^{(p)}$ is the smallest normal subgroup $H$ such that $G/H$ has order of $p$-power (possibly $|G/H| = 1$), and $G^{\text{nil}}$ is the smallest normal subgroup $N$ such that $G/N$ is nilpotent.

Let $\mathcal{L}$ be a set of subgroups of $G$ such that each minimal element of $\mathcal{L}$ is a normal subgroup of $G$. Let $\mathbb{R}[G]$ denote the regular representation of $G$ and let $\mathbb{R}[G]^\mathcal{L}$ denote the smallest $G$-submodule of $\mathbb{R}[G]$ containing all $\mathbb{R}[G]^L$ with $L \in \mathcal{L}$. Let $\mathbb{R}[G]_\mathcal{L}$ be

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the orthogonal complement of $\mathbb{R}[G]^{\mathcal{L}}$ in $\mathbb{R}[G]$ with respect to some $G$-invariant inner product on $\mathbb{R}[G]$, i.e.
$$\mathbb{R}[G]_{\mathcal{L}} = \mathbb{R}[G] - \mathbb{R}[G]^{\mathcal{L}}.$$ In this paper we call $\mathbb{R}[G]_{\mathcal{L}}$ the real $G$-module associated with $\mathcal{L}$.

**Definition 1.1.** A nonempty family $\mathcal{K}$ of normal subgroups of $G$ is called *mutually coprime* if either
1. $\mathcal{K} = \{G\}$, or
2. $G \notin \mathcal{K}$ and $|G/K|$'s are mutually prime integers, i.e.
$$(|G/K|, |G/K'|) = 1 \text{ for all } K, K' \in \mathcal{K} \text{ such that } K \neq K'.$$

If $\mathcal{K}$ is a mutually coprime family of normal subgroups of $G$, then the equality

$$(1.1) \quad \mathbb{R}[G]_{\mathcal{K}} = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{K \in \mathcal{K}} (\mathbb{R}[G/K] - \mathbb{R})$$

holds, where $\mathbb{R}$ is the 1-dimensional trivial real $G$-module.

**Definition 1.2.** Let $\mathcal{L}$ be a set of subgroups of $G$. Then we define the *upper closure* $\overline{\mathcal{L}}$ of $\mathcal{L}$ by

$$(1.2) \quad \overline{\mathcal{L}} = \{H \in S(G) \mid H \supset L \text{ for some } L \in \mathcal{L}\},$$

and the *exterior* $\underline{\mathcal{L}}$ of $\mathcal{L}$ by

$$(1.3) \quad \underline{\mathcal{L}} = S(G) \setminus \overline{\mathcal{L}}.$$ 

With this notation, we have $\mathcal{L}(G) = \overline{\mathcal{K}(G)}$ and $\mathcal{M}(G) = \mathcal{K}(G)$, cf. E. Laitinen-M. Morimoto [1].

**Definition 1.3.** Let $V$ be a real $G$-module and $\mathcal{H}$ a family of subgroups of $G$. We say that $V$ is *$\mathcal{H}$-complete* if for each $H \in \mathcal{H}$, any irreducible real $H$-module is isomorphic to a submodule of $\text{res}^{G}_{H}V$.

The main results which will be reported in this article are Theorems 2.1, 2.2 and 3.2. The proofs will appear somewhere else.
2. Completeness and Gap Property

Let $\mathcal{K}$ be a mutually coprime family of normal subgroups of $G$. We introduce two practically important properties of $\mathbb{R}[G]_{\mathcal{K}}$ as the theorems below.

**Theorem 2.1.** Let $G$ be a finite group and let $\mathcal{K}$ be a mutually coprime family of normal subgroups of $G$. Then for any $H \in \mathcal{K}$, $\text{res}^G_H \mathbb{R}[G]_{\mathcal{K}}$ contains a real $H$-submodule isomorphic to $\mathbb{R}[H]$. Hence the real $G$-module $\mathbb{R}[G]_{\mathcal{K}}$ is $\mathcal{K}$-complete.

**Theorem 2.2.** Let $G$ be a finite group and let $\mathcal{K}$ be a mutually coprime family of normal subgroups of $G$. Then the real $G$-module $\mathbb{R}[G]_{\mathcal{K}}$ possesses the following properties.

1. $\mathbb{R}[G]_{\mathcal{K}}^H \neq 0$ if and only if $H \in \mathcal{K}$.
2. Let $p$ be a prime and $H < K \leq G$ with $|K : H| = p$. Then
   \[
   \dim \mathbb{R}[G]_{\mathcal{K}}^H \geq p \dim \mathbb{R}[G]_{\mathcal{K}}^K
   \]
   holds; the equality holds if and only if there exists $K_k \in \mathcal{K}$ such that $p ||G : K_k|$, $|KK_k : HK_k| = p$, and $HK_i = G$ for all $K_i \in \mathcal{K} \smallsetminus \{K_k\}$.
3. Let $H < K \leq G$. Then
   \[
   \dim \mathbb{R}[G]_{\mathcal{K}}^H \geq 2 \dim \mathbb{R}[G]_{\mathcal{K}}^K
   \]
   holds; the equality holds if and only if
   (a) $H \in \overline{\mathcal{K}}$, or
   (b) $K \in \mathcal{K}$, $|K : H| = 2$, there exists $K_k \in \mathcal{K}$ such that $2 ||G : K_k|$, $|KK_k : HK_k| = 2$ and $HK_i = G$ for all $K_i \in \mathcal{K} \smallsetminus \{K_k\}$.

The next proposition has been used in the induction argument of the equivariant surgery theory, cf. [1, 4, 5].

**Proposition 2.3.** Let $G$ be an Oliver group, and let $P$, $H_1$, $H_2$ be subgroups of $G$ such that $P \in \mathcal{P}(G)$, $P < H_1$, and $P < H_2$. If the equality

\[
2 \dim \mathbb{R}[G]_{\mathcal{L}(G)}^{H_1} = \dim \mathbb{R}[G]_{\mathcal{L}(G)}^P
\]

holds for each $i = 1$ and 2, then the smallest subgroup $K$ containing $H_1$ and $H_2$ belongs to $\mathcal{M}(G) = S(G) \smallsetminus \mathcal{L}(G)$. 

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3. Canonical line bundle of real projective space

Let $V$ be a real $G$-module (of finite dimension). The real projective space $P(V)$ is the space of all 1-dimensional real vector subspaces of $V$, and $P(V)$ has the canonically induced $G$-action. Let $\gamma_M$, where $M = P(V)$, denote the canonical line bundle of $M$.

**Lemma 3.1.** Let $V$ be a real $G$-module and $M = P(V)$. Then the following equalities hold as real $G$-vector bundles via canonical isomorphisms.

1. $\text{Hom}(\gamma_M, \gamma_M) = \epsilon_M(\mathbb{R})$.
2. $\text{Hom}(\gamma_M, \epsilon_M(\mathbb{R})) = \gamma_M$.
3. $T(M) = \text{Hom}(\gamma_M, \gamma_M^\perp)$.
4. $T(M) \oplus \epsilon_M(\mathbb{R}) = \text{Hom}(\gamma_M, \epsilon_M(V))$.
5. $\text{Hom}(\gamma_M, \epsilon_M(V)) = \gamma_M \otimes V$.

The equalities (1)–(4) above follow from the proof of [3, Lemma 4.4]. The equality (5) holds because

$$\text{Hom}(\gamma_M, \epsilon_M(V)) = \text{Hom}(\gamma_M, \epsilon_M(\mathbb{R})) \otimes_{\mathbb{R}} V = \gamma_M \otimes_{\mathbb{R}} V.$$

**Theorem 3.2.** Let $\mathcal{K}$ be a mutually coprime family of normal subgroups of $G$ and let $V$ be a real $G$-module such that $V = V^\mathcal{K}$. Then for $K_i \in \mathcal{K},$

1. $P(V)^{K_i} = \begin{cases} P(V^{K_i}) & \text{if } 2 \mid |G : K_i| \\ P(V^{K_i}) \prod_{L \in \mathcal{A}_i} P(V_{G/L}^{L}) & \text{if } 2 \nmid |G : K_i| \end{cases}$

and

2. $\gamma_{P(V)}|_{P(V)^{K_i}} = \begin{cases} \gamma_{P(V^{K_i})} & \text{if } 2 \mid |G : K_i| \\ \gamma_{P(V^{K_i})} \prod_{L \in \mathcal{A}_i} \gamma_{P(V_{G/L}^{L})} & \text{if } 2 \nmid |G : K_i|, \end{cases}$

where $\mathcal{A}_i$ is the set of all subgroups $L$ such that $|G : L| = 2$ and $|K_i : K_i \cap L| = 2$.

In addition

$$\left(\gamma_{P(V)} \otimes_{\mathbb{R}} V\right)^{K_i} = \begin{cases} \gamma_{P(V^{K_i})} \otimes_{\mathbb{R}} V^{K_i} & \text{if } 2 \mid |G : K_i| \\ \gamma_{P(V^{K_i})} \otimes_{\mathbb{R}} V^{K_i} \prod_{L \in \mathcal{A}_i} \gamma_{P(V_{G/L}^{L})} \otimes_{\mathbb{R}} V_{G/L}^{L} & \text{if } 2 \nmid |G : K_i| \end{cases}$$
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