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Groups of uniform homeomorphisms of covering spaces

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In this article we discuss topological properties of spaces of uniform embeddings and groups of uniform homeomorphisms in metric covering spaces over compact manifolds and metric spaces with Euclidean ends.

1. Spaces of uniformly continuous maps

First we recall some basic facts on uniformly continuous maps and the uniform topology on the space of uniformly continuous maps. In this article, maps between topological spaces are always assumed to be continuous. For a topological space X and a subset A of X, the symbols $\mathrm{Int}_X A$, $cl_X A$ and $\mathrm{Fr}_X A$ denote the topological interior, closure and frontier of A in X.

Suppose (X,d) and (Y,ϱ) are metric spaces. A map $h:(X,d)\to(Y,\varrho)$ is said to be uniformly continuous if for each $\varepsilon>0$ there is a $\delta>0$ such that if $x,x'\in X$ and $d(x,x')<\delta$ then $\varrho(f(x),f(x'))<\varepsilon$. The map h is called a uniform homeomorphism if h is bijective and both h and h^{-1} are uniformly continuous. A uniform embedding is a uniform homeomorphism onto its image.

Let $\mathcal{C}(X,Y)$ and $\mathcal{C}^u((X,d),(Y,\varrho))$ denote the space of maps $f:X\to Y$ and the subspace of uniformly continuous maps $f:(X,d)\to(Y,\varrho)$. The metric ϱ on Y induces the sup-metric on $\mathcal{C}(X,Y)$ defined by

$$\varrho(f,g)=\sup\{\varrho(f(x),g(x))\mid x\in X\}\in[0,\infty].$$

The topology on $\mathcal{C}(X,Y)$ induced by this sup-metric ϱ is called the uniform topology. Below the space $\mathcal{C}(X,Y)$ and its subspaces are endowed with the sup-metric ϱ and the uniform topology, otherwise specified. To emphasize this point, sometimes we use the symbol $\mathcal{C}(X,Y)_u$. On the other hand, when the space $\mathcal{C}(X,Y)$ is endowed with the compact-open topology, we use the symbol $\mathcal{C}(X,Y)_{co}$. When X is compact, we have $\mathcal{C}^u((X,d),(Y,\varrho))_u=\mathcal{C}(X,Y)_{co}$. It is important to notice that the composition map

$$\mathcal{C}^u((X,d),(Y,\varrho))_u\times\mathcal{C}^u((Y,\varrho),(Z,\eta))_u\longrightarrow\mathcal{C}^u((X,d),(Z,\eta))_u.$$

is continuous, while the composition map $\mathcal{C}(X,Y)_u\times\mathcal{C}(Y,Z)_u\longrightarrow\mathcal{C}(X,Z)_u$ is not necessarily continuous.

Let $\mathcal{E}(X,Y)$ and $\mathcal{E}^u((X,d),(Y,\varrho))$ denote the space of embeddings $f : X \to Y$ and the subspace of uniform embeddings $f : (X,d) \to (Y,\varrho)$ (both with the sup-metric and the uniform topology). For a subset A of X let $\mathcal{E}_A(X,Y) = \{f \in \mathcal{E}(X,Y) \mid f|_A = \mathrm{id}_A\}$. When $X \subset Y \subset Z$, for a subset C of Z we also use the symbol $\mathcal{E}(X,Y;C)$ to denote $\mathcal{E}_{X\cap C}(X,Y)$ and for $\varepsilon > 0$ let $\mathcal{E}(i_X,\varepsilon;X,Y;C)$ denote the ε-neighborhood of the inclusion $i_X : X \subset Y$ in the space $\mathcal{E}(X,Y;C)$. The meaning of the symbols $\mathcal{E}^u_A((X,d),(Y,\varrho))$, $\mathcal{E}^u((X,d),(Y,\varrho);C)$, etc are obvious.

Similarly $\mathcal{H}_A(X)$ and $\mathcal{H}^u_A(X,d)$ denote the group of homeomorphisms h of X onto itself and the subgroup of uniform homeomorphisms h of (X,d) with $h|A = \mathrm{id}_A$ (both endowed with the uniform topology). We denote by $\mathcal{H}^u_A(X,d)_0$ the connected component of the identity map id_X of X in $\mathcal{H}^u_A(X,d)$ and define the subgroup

$$\mathcal{H}^u_A(X,d)_b = \{h \in \mathcal{H}^u_A(X,d) \mid d(h,\mathrm{id}_X) < \infty\}.$$

It follows that $\mathcal{H}^u_A(X,d)$ is a topological group and $\mathcal{H}^u_A(X,d)_b$ is an open (and closed) subgroup of $\mathcal{H}^u_A(X,d)$, so that $\mathcal{H}^u_A(X,d)_0 \subset \mathcal{H}^u_A(X,d)_b$. When $X - A$ is relatively compact in X, the group $\mathcal{H}^u_A(X,d)$ coincides with the whole group $\mathcal{H}_A(X)$. As usual, the symbol A is suppressed when it is an empty set.

Recall that a family $f_\lambda \in \mathcal{C}(X,Y)$ $(\lambda \in \Lambda)$ is said to be equi-continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, x' \in X$ and $d(x,x') < \delta$ then $\varrho(f_\lambda(x), f_\lambda(x')) < \varepsilon$ for any $\lambda \in \Lambda$. More generally, we say that a family of maps $\{f_\lambda : (X_\lambda, d_\lambda) \to (Y_\lambda, \varrho_\lambda)\}_{\lambda\in\Lambda}$ between metric spaces is equi-continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\lambda \in \Lambda$ if $x, x' \in X_\lambda$ and $d_\lambda(x,x') < \delta$ then $\varrho_\lambda(f_\lambda(x), f_\lambda(x')) < \varepsilon$. For embeddings, we also use the following terminology: a family of embeddings $\{h_\lambda : (X_\lambda, d_\lambda) \to (Y_\lambda, \varrho_\lambda)\}_{\lambda\in\Lambda}$ is equi-uniform if both of the families $\{h_\lambda : (X_\lambda, d_\lambda) \to (Y_\lambda, \varrho_\lambda)\}_{\lambda\in\Lambda}$ and $\{(h_\lambda)^{-1} : (h_\lambda(X_\lambda), \varrho_\lambda) \to (X_\lambda, d_\lambda)\}_{\lambda\in\Lambda}$ are equi-continuous.

The following lemmas are used in the proof of the main theorems. Let (X,d), (Y,ϱ) and (Z,η) be metric spaces. For a subset $\mathcal{C}$ of $\mathcal{C}(X,Y)$, the symbol $cl_u\,\mathcal{C}$ means the closure of $\mathcal{C}$ in $\mathcal{C}(X,Y)_u$.

Lemma 1.1. (1) $cl_u\,\mathcal{E}^u(X,Y) \subset \mathcal{C}^u(X,Y)$.

(2) Suppose $\mathcal{C} \subset \mathcal{E}^u(X,Y)$. If $\mathcal{C}' = \{f^{-1} : f(X) \to X \mid f \in \mathcal{C}\}$ is equi-continuous, then $cl_u\,\mathcal{C} \subset \mathcal{E}^u(X,Y)$.

The word "function" means a correspondence not assumed to be continuous.

Lemma 1.2. Suppose P is a topological space, $f : P \to \mathcal{C}(X,Y)_u$, $g : P \to \mathcal{C}(X,Z)_u$ are continuous maps and $h : P \to \mathcal{C}^u(Y,Z)_u$ is a function. If f_p is surjective and $h_p f_p = g_p$ for each $p \in P$, then h is continuous.

Lemma 1.3. Suppose S is a compact subset of X which has an open collar neighborhood $\theta : (S \times [0,4), S \times \{0\}) \approx (N, S)$ in X. Let $N_a = \theta(S \times [0,a])$ $(a \in [0,4))$. Then there exists a strong deformation retraction φ_t $(t \in [0,1])$ of $\mathcal{H}^u_{N_1}(X)_b$ onto $\mathcal{H}^u_{N_2}(X)_b$ such that

$$\varphi_t(h) = h \quad \text{on} \quad h^{-1}(X - N_3) - N_3 \quad \text{for any} \quad (h,t) \in \mathcal{H}^u_{N_1}(X)_b \times [0,1].$$

2. Spaces of uniform embeddings in metric covering spaces over compact manifolds

In [3] R. D. Edwards and R. C. Kirby obtained a fundamental local deformation theorem for embeddings of a compact subspace in a manifold (see §2.1). From this theorem and the Arzela-Ascoli theorem (cf. [2, Theorem 6.4]) we can deduce a local deformation lemma for uniform embeddings in a metric covering space over a compact manifold (Theorem 2.2).

2.1. Basic deformation theorem for topological embeddings in topological manifolds.

First we recall the basic deformation theorem on embeddings of a compact subset in topological manifold (R. D. Edwards and R. C. Kirby [3]). Suppose M is a topological n-manifold possibly with boundary and X is a subspace of M. An embedding $f : X \to M$ is said to be proper if $f^{-1}(\partial M) = X \cap \partial M$ (and quasi-proper if $f(X \cap \partial M) \subset \partial M$). For any subset $C \subset M$, let $\mathcal{E}_*(X, M; C)$ denote the subspaces of $\mathcal{E}(X, M; C)$ consisting of proper embeddings.

Theorem 2.1. ([3, Theorem 5.1]) Suppose M is a topological n-manifold possibly with boundary, C is a compact subset of M, U is a neighborhood of C in M and D and E are two closed subsets of M such that $D \subset \mathrm{Int}_M E$. Then, for any compact neighborhood K of C in U, there exists a neighborhood $\mathcal{U}$ of i_U in $\mathcal{E}_*(U, M; E)_{co}$ and a homotopy $\varphi : \mathcal{U} \times [0,1] \longrightarrow \mathcal{E}_*(U, M; D)_{co}$ such that

(1) for each $f \in \mathcal{U}$,
 (i) $\varphi_0(f) = f$, (ii) $\varphi_1(f)|_C = i_C$, (iii) $\varphi_t(f) = f$ on $U - K$ $(t \in [0,1])$,
 (iv) if $f = \mathrm{id}$ on $U \cap \partial M$, then $\varphi_t(f) = \mathrm{id}$ on $U \cap \partial M$ $(t \in [0,1])$,

(2) $\varphi_t(i_U) = i_U$ $(t \in [0,1])$.

Remark 2.1. Theorem 2.1 still holds if we replace the spaces of proper embeddings, $\mathcal{E}_*(U, M; D)$ and $\mathcal{E}_*(U, M; E)$, by the spaces of quasi-proper embeddings, $\mathcal{E}_\#(U, M; D)$ and $\mathcal{E}_\#(U, M; E)$. Note that $\mathcal{E}_\#(X, M; C)$ is closed in $\mathcal{E}(X, M; C)$, while $\mathcal{E}_*(U, M; D)$ is not necessarily closed.

2.2. **Metric covering projections.**

Since the notion of uniform continuity depends on the choice of metrics, it is necessary to select a reasonable class of metrics to obtain a suitable conclusion on spaces of uniform embeddings of a metric manifold (M, d). In [1] (cf, [5, Section 5.6]) A.V. Černavskiĭ considered the case where M is the interior of a compact manifold N and the metric d is a restriction of some metric on N. In this article we consider the case where M is a covering space over a compact manifold N and the metric d is the pull-back of some metric on N. The natural model is the class of Riemannian coverings in the smooth category. In order to remove the extra requirements in the smooth setting, here we introduce the notion of metric covering projection. For the basics on covering spaces, we refer to [6, Chapter 2, Section 1]. If $p : M \to N$ is a covering projection and N is a topological n-manifold possibly with boundary, then so is M and $\partial M = \pi^{-1}(\partial N)$.

Suppose (X, d) is a metric space. A neighborhood U of A in X is called a uniform neighborhood of A in (X, d) if U contains a δ-neighborhood of A for some $\delta > 0$. For $\varepsilon > 0$ a subset A of X is said to be ε-discrete if $d(x, y) \geq \varepsilon$ for any distinct points $x, y \in A$. We say that A is uniformly discrete if it is ε-discrete for some $\varepsilon > 0$.

Definition 2.1. A map $\pi : (X, d) \to (Y, \varrho)$ between metric spaces is called a metric covering projection if it satisfies the following conditions:

$(*)_1$ There exists an open cover $\mathcal{U}$ of Y such that for each $U \in \mathcal{U}$ the inverse $\pi^{-1}(U)$ is the disjoint union of open subsets of X each of which is mapped isometrically onto U by π.

$(*)_2$ For each $y \in Y$ the fiber $\pi^{-1}(y)$ is uniformly discrete in X.

$(*)_3$ $\varrho(\pi(x), \pi(x')) \leq d(x, x')$ for any $x, x' \in X$.

When the map π satisfies the condition $(*)_1$, we say that each $U \in \mathcal{U}$ is isometrically evenly covered by π. If an open subset U of Y is connected and isometrically evenly covered by π, then each connected component of $\pi^{-1}(U)$ is mapped isometrically onto U by π. If $\pi : (X, d) \to (Y, \varrho)$ is a metric covering projection and Y is compact, then there exists $\varepsilon > 0$ such that each fiber of π is ε-discrete. Riemannian covering projections are typical examples of metric covering projections.

2.3. **Deformation theorem for uniform embeddings.**

When (M, d) is a topological manifold possibly with boundary with a fixed metric d and X, C are subspaces of M, we denote by $\mathcal{E}^u_*(X, M; C)$ the space of uniform proper embeddings $f : (X, d|_X) \to (M, d)$ such that $f = \mathrm{id}$ on $X \cap C$. This space is endowed with the uniform topology induced from the sup-metric The following is our first main theorem.

Theorem 2.2. Suppose $\pi : (M, d) \to (N, \varrho)$ is a metric covering projection, N is a compact topological n-manifold possibly with boundary, X is a closed subset of M, $W' \subset W$ are uniform neighborhoods of X in (M, d) and Z, Y are closed subsets of M such that Y is a uniform neighborhood of Z. Then there exists a neighborhood $\mathcal{W}$ of the inclusion map $\iota_W : W \subset M$ in $\mathcal{E}^u_*(W, M; Y)$ and a homotopy $\varphi : \mathcal{W} \times [0,1] \longrightarrow \mathcal{E}^u_*(W, M; Z)$ such that

(1) for each $h \in \mathcal{W}$
  (i) $\varphi_0(h) = h$, (ii) $\varphi_1(h) = \mathrm{id}$ on X, (iii) $\varphi_t(h) = h$ on $W - W'$ $(t \in [0,1])$,
(2) $\varphi_t(\iota_W) = \iota_W$ $(t \in [0,1])$.

In [1] it is shown that $\mathcal{H}^u(M, d)$ is locally contractible in the case where M is the interior of a compact manifold N and the metric d is a restriction of some metric on N. The next corollary is a direct consequence of Theorem 2.2.

Corollary 2.1. Suppose $\pi : (M, d) \to (N, \varrho)$ is a metric covering projection onto a compact topological n-manifold N possibly with boundary. Then $\mathcal{H}^u(M, d)$ is locally contractible.

We conclude this section by indicating how to use the Arzela-Ascoli theorem in the proof of Theorem 2.2.

Idea of proof of Theorem 2.2.

We consider the special but essential case where $M \to N$ is the product metric covering projection $M = N \times \mathbb{N} \longrightarrow N$ and $X = \pi^{-1}(C)$ for some compact subset C of N (and $Z = Y = \emptyset$). For simplicity we pretend that $W = W' = X$. We apply Theorem 2.1 to the compact subset C of the topological manifold N (pretending that $U = K = C$), so to obtain a neighborhood $\mathcal{U}$ of the inclusion i_C in $\mathcal{E}_*(C, N)_{co}$ and a deformation $\psi : \mathcal{U} \times [0,1] \longrightarrow \mathcal{E}_*(C, M)_{co}$ as in Theorem 2.1.

Suppose a proper uniform embedding $f : X \to M$ is sufficiently close to the inclusion i_X. We have to construct the homotopy $\varphi_t(f)$ as in Theorem 2.2. On each sheet $N_i \equiv N \times \{i\}$ $(i \in \mathbb{N})$, the embedding f restricts to an embedding $f_i : X \cap N_i \to N_i$, which induces the embedding $\overline{f}_i : C \to N$. Then $\varphi_t(f)|_{N_i}$ is defined as the lift of $\psi_t(\overline{f}_i)$ by the isometry $\pi : N_i \to N$. Since f is a uniform embedding, the families $\{f_i\}_{i \in \mathbb{N}}$ and $\{\overline{f}_i\}_{i \in \mathbb{N}}$ are equi-uniform, so that $cl\{\overline{f}_i\}_{i \in \mathbb{N}}$ is compact by the Arzela-Ascoli theorem. This implies that $\psi(cl\{\overline{f}_i\}_{i \in \mathbb{N}} \times [0,1])$ is also compact and that $\{\psi_t(\overline{f}_i)\}_{i \in \mathbb{N}, t \in [0,1]}$ is equi-uniform. Hence we obtain the required homotopy $\varphi_t(f)$ in $\mathcal{E}^u_*(X, M)_u$.

3. Groups of uniform homeomorphisms of metric spaces with bi-Lipschitz Euclidean ends

In this section we discuss some global topological properties of groups of uniform homeomorphisms of metric spaces with bi-Lipschitz Euclidean ends.

3.1. **The Euclidean ends.**

The Euclidean space $\mathbb{R}^n$ with the standard Euclidean metric admits the canonical Riemannian covering projection $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n$ onto the flat torus. Therefore we can apply the local deformation theorem, Theorem 2.2, to uniform embeddings in $\mathbb{R}^n$. In this situation, the important feature of $\mathbb{R}^n$ is the existence of similarity transformations

$$k_\gamma : \mathbb{R}^n \approx \mathbb{R}^n : \ k_\gamma(x) = \gamma x \qquad (\gamma > 0).$$

This enables us to deduce a global deformation of uniform embeddings from a local one.

In a relation to other metric spaces we are especially concerned with the end of the Euclidean space $\mathbb{R}^n$. The model of Euclidean end is the complement $\mathbb{R}^n_r = \mathbb{R}^n - O(r)$ of the round open r-ball $O(r)$ centered at the origin. If we combine Theorem 2.2 with the similarity transformation k_γ for a sufficiently large $\gamma > 0$, then we have the following conclusion.

Lemma 3.1. For any $c, s_0 > 0$ and $\beta > \alpha > 1$ there exist $s > s_0$ and a homotopy

$$\psi : \mathcal{E}^u(\iota_s, c; \mathbb{R}^n_s, \mathbb{R}^n) \times [0,1] \longrightarrow \mathcal{E}^u(\iota_s, s; \mathbb{R}^n_s, \mathbb{R}^n)$$

such that

(1) for each $h \in \mathcal{E}^u(\iota_s, c; \mathbb{R}^n_s, \mathbb{R}^n)$
  (i) $\psi_0(h) = h$, (ii) $\psi_1(h) = \mathrm{id}$ on $\mathbb{R}^n_{\beta s}$, (iii) $\psi_t(h) = h$ on $\mathbb{R}^n_s - \mathbb{R}^n_{\alpha s}$ $(t \in [0,1])$,
(2) $\psi_t(\iota_s) = \iota_s$ $(t \in [0,1])$
(3) $\psi(\mathcal{E}^u(\iota_s, c; \mathbb{R}^n_s, \mathbb{R}^n_r) \times [0,1]) \subset \mathcal{E}^u(\iota_s, s; \mathbb{R}^n_s, \mathbb{R}^n_r)$ for any $r < s$.

3.2. **Bi-Lipschitz Euclidean ends.**

In order to transfer to more general metric spaces, we introduce the notion of bi-Lipschitz Euclidean ends. Recall that a map $h : (X, d) \to (Y, \varrho)$ between metric spaces is said to be Lipschitz if there exists a constant $C > 0$ such that $\varrho(f(x), f(x')) \leq C d_X(x, x')$ for any $x, x' \in X$. The map h is called a bi-Lipschitz homeomorphism if h is bijective and both h and h^{-1} are Lipschitz maps. The Euclidean ends $\mathbb{R}^n_r$ $(r > 0)$ are bi-Lipschitz homeomorphic to each other under similarity transformations.

Definition 3.1. A bi-Lipschitz n-dimensional Euclidean end of a metric space (X, d) is a closed subset L of X which admits a bi-Lipschitz homeomorphism of pairs, $\theta : (\mathbb{R}^n_1, \partial\mathbb{R}^n_1) \approx ((L, \mathrm{Fr}_X L), d|_L)$ and satisfies the condition $d(X - L, L_r) \to \infty$ as $r \to \infty$, where $L_r = \theta(\mathbb{R}^n_r)$ $(r \geq 1)$. We set $L' = \theta(\mathbb{R}^n_2)$ and $L'' = \theta(\mathbb{R}^n_3)$.

The following is our 2nd main theorem.

Theorem 3.1. Suppose X is a metric space and $L_1, \cdots, L_m$ are mutually disjoint bi-Lipschitz Euclidean ends of X. Let $L' = L'_1 \cup \cdots \cup L'_m$ and $L'' = L''_1 \cup \cdots \cup L''_m$. Then there exists a strong deformation retraction φ of $\mathcal{H}^u(X)_b$ onto $\mathcal{H}^u_{L''}(X)$ such that

$$\varphi_t(h) = h \quad \text{on} \quad h^{-1}(X - L') - L' \quad \text{for any} \quad (h,t) \in \mathcal{H}^u(X)_b \times [0,1].$$

The following lemmas are used in the proof of Theorem 3.1. We keep the notations in Definition 3.1. We set $\mathcal{H}^u(X;\lambda) = \{h \in \mathcal{H}^u(X,d) \mid d(h, \mathrm{id}_X) < \lambda\}$.

Lemma 3.2. For any $\lambda > 0$ and $r > r_0 \geq 1$ there exist $\lambda' > 0$ and a homotopy $\chi : \mathcal{H}^u(X;\lambda) \times [0,1] \longrightarrow \mathcal{H}^u(X;\lambda')$ such that for each $h \in \mathcal{H}^u(X;\lambda)$

(i) $\chi_0(h) = h$, (ii) $\chi_1(h) = \mathrm{id}$ on L_r, (iii) $\chi_t(h) = h$ on $h^{-1}(X - L_{r_0}) - L_{r_0}$ $(t \in [0,1])$,
(iv) if $h = \mathrm{id}$ on L_{r_0}, then $\chi_t(h) = h$ $(t \in [0,1])$.

Lemma 3.3. For any $r \in (1,2)$ there exists a homotopy $\psi : \mathcal{H}^u(X)_b \times [0,1] \longrightarrow \mathcal{H}^u(X)_b$ such that for each $h \in \mathcal{H}^u(X)_b$

(i) $\psi_0(h) = h$, (ii) $\psi_1(h) = \mathrm{id}$ on L_2, (iii) $\psi_t(h) = h$ on $h^{-1}(X - L_r) - L_r$ $(t \in [0,1])$,
(iv) if $h = \mathrm{id}$ on L_r, then $\psi_t(h) = h$ $(t \in [0,1])$,
(v) for any $\lambda > 0$ there exists $\mu > 0$ such that $\psi_t(\mathcal{H}^u(X;\lambda)) \subset \mathcal{H}^u(X;\mu)$ $(t \in [0,1])$.

Proposition 3.1. For any $1 < s < r < 2$ there exists a strong deformation retraction φ of $\mathcal{H}^u(X)_b$ onto $\mathcal{H}^u_{L_r}(X)_b$ such that

$$\varphi_t(h) = h \quad \text{on} \quad h^{-1}(X - L_s) - L_s \quad \text{for any} \quad (h,t) \in \mathcal{H}^u(X)_b \times [0,1].$$

3.3. **Some examples.**

Example 3.1. $\mathcal{H}^u(\mathbb{R}^n)_b$ is contractible for every $n \geq 0$. In fact, $\mathbb{R}^n$ has the model Euclidean end $\mathbb{R}^n_1$ and hence there exists a strong deformation retraction of $\mathcal{H}^u(\mathbb{R}^n)_b$ onto $\mathcal{H}^u_{\mathbb{R}^n_3}(\mathbb{R}^n)$. The latter is contractible by Alexander's trick.

Remark 3.1. Let $B(1)$ denote the closed unit ball in $\mathbb{R}^n$ centered at the origin. Using a suitable shrinking homeomorphism $\mathbb{R}^n \approx O(1)$ we can construct a natural continuous injection $\mathcal{H}^u(\mathbb{R}^n)_b \longrightarrow \mathcal{H}_\partial(B(1))$. The Alexander's trick yields a canonical contraction of $\mathcal{H}_\partial(B(1))$. However, the contraction of $\mathcal{H}^u(\mathbb{R}^n)_b$ induced by this injection is not continuous. In fact, it would mean that any $h \in \mathcal{H}^u(\mathbb{R}^n)_b$ could be approximated by compactly supported homeomorphisms in the sup-metric. But this does not hold, for example, for any translation $h(x) = x + a$ $(a \neq 0)$.

Example 3.2. The n-dimensional cylinder $M = \mathbb{S}^{n-1} \times \mathbb{R}$ is the product of the $(n-1)$-sphere $\mathbb{S}^{n-1}$ and the real line $\mathbb{R}$. If M is asigned a metric so that $\mathbb{S}^{n-1} \times (-\infty, -1]$ and $\mathbb{S}^{n-1} \times [1, \infty)$ are two bi-Lipschitz Euclidean ends of M, then $\mathcal{H}^u(M)_b$ includes the subgroup $\mathcal{H}_{\mathbb{S}^{n-1} \times \mathbb{R}_1}(M) \approx \mathcal{H}_\partial(\mathbb{S}^{n-1} \times [-1,1])$ as a strong deformation retract. This implies that $\mathcal{H}^u(M)_0$ admits a strong deformation retraction onto $\mathcal{H}_{\mathbb{S}^{n-1} \times \mathbb{R}_1}(M)_0 \approx \mathcal{H}_\partial(\mathbb{S}^{n-1} \times [-1,1])_0$.

Example 3.3. In dimension 2, we have a more explicit conclusion. Suppose N is a compact connected 2-manifold with a nonempty boundary and $C = \cup_{i=1}^m C_i$ is a nonempty union of some boundary circles of N. If the noncompact 2-manifold $M = N - C$ is assigned a metic d such that for each $i = 1, \cdots, m$ the end L_i of M corresponding to the boundary circle C_i is a bi-Lipschitz Euclidean end of (M, d), then it follows that $\mathcal{H}^u(M,d)_0 \simeq \mathcal{H}^u_{L''}(M)_0 \approx \mathcal{H}_C(N)_0 \simeq *$.

3.4. Conjecture.

In [4] we studied the topological type of $\mathcal{H}^u(\mathbb{R})_b$ as an infinite-dimensional manifold and showed that it is homeomorphic to ℓ_∞. Example 1.1 leads to the following conjecture.

Conjecture 3.1. $\mathcal{H}^u(\mathbb{R}^n)_b$ is homeomorphic to ℓ_∞ for any $n \geq 1$.

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