Description of a mean curvature sphere of a surface by quaternionic holomorphic geometry

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1 Introduction

In this paper, we collect definitions and propositions from the surface theory in terms of quaternions. These are selected so that they complement the paper [7]. Proofs are omitted. The details are described in [2], [3] and [5].

2 Mean curvature spheres

We explain the notion of a mean curvature sphere of a conformal map.

2.1 Sphere congruences

We model $S^4$ on the quaternionic projective line $\mathbb{H}P^1$. Set

$$Z := \{ C \in \text{End}(\mathbb{H}^2) \mid C^2 = -\text{Id} \}.$$ 

This is the set of all quaternionic linear complex structures of $\mathbb{H}^2$. Then two-spheres are parametrized by $Z$:

Lemma 1 ([2], Proposition 2).

$$\{\text{oriented two-spheres in } \mathbb{H}P^1\} = Z.$$ 

In a classical terminology, a sphere congruence is a smooth family of two-spheres. Hence a map from a Riemann surface $M$ to $Z$ is a sphere congruence in $\mathbb{H}P^1$ parametrized by $M$.

2.2 Mean curvature spheres

Let $M$ be a Riemann surface with complex structure $J$ and $f : M \to \mathbb{R}^4$ a conformal map.

Definition 1. At a point $p \in M$, a two-sphere in $M$ is called the mean curvature sphere of $f$ at $p$ if
• the sphere is tangent to \( f(M) \) at \( p \),
• the sphere is centered in the direction of the mean curvature vector at \( p \), and
• the radius of the sphere is equal to the reciprocal of the norm of the mean curvature vector at \( p \).

A sphere congruence parametrized by \( M \) which consists of the mean curvature spheres of \( f \) is called the mean curvature sphere of \( f \).

We see that \( f \) is the envelop of the mean curvature sphere of \( f \). The mean curvature of \( f \) at \( p \in M \) is equal to the mean curvature of the mean curvature sphere of \( f \) at \( p \).

Let \( S \) be the mean curvature sphere of \( f \) and \( \tau \) a conformal transformation of \( \mathbb{R}^4 \). Then \( \tau \circ S \) is the mean curvature sphere of \( \tau \circ f \). Hence the mean curvature sphere is a concept for conformal geometry of surfaces in \( S^4 \). For a conformal map \( f: M \to S^4 \cong \mathbb{H}P^1 \), the mean curvature sphere is a map from \( M \) to \( Z \).

### 2.3 Conformal Gauss maps

A mean curvature sphere is called a conformal Gauss map in [1]. This terminology is valid as follows. For \( C \in \text{End}(\mathbb{H}^2) \), we set \( \langle C \rangle := \frac{1}{8} \text{tr}_\mathbb{R} C \). Then an indefinite scalar product \( \langle \ , \ \rangle \) of \( \text{End}(\mathbb{H}^2) \) is defined by setting \( \langle C_1, C_2 \rangle := \langle C_1 C_2 \rangle \) for \( C_1, C_2 \in \text{End}(\mathbb{H}^2) \).

**Lemma 2** ([1], [2], Proposition 4). The mean curvature sphere \( S \) of a conformal map \( f: M \to S^4 \) is conformal with respect to \( \langle \ , \ \rangle \).

### 2.4 Energy of a sphere congruence

Let \( C: M \to Z \) be a sphere congruence. For a one-form \( \omega \) on \( M \), we set \( \ast \omega := \omega \circ J \).

**Definition 2** ([2], Definition 7).

\[
E(C) := \int_M \langle dC \wedge \ast dC \rangle
\]

is called the energy of a sphere congruence.

Because \( \langle \ , \ \rangle \) is indefinite, the functional \( E \) might take negative values. Set \( A_C := \frac{1}{4}(\ast dC + C dC) \). The Euler-Lagrange equation of \( E(C) \) is written by the one-form \( A_C \).

**Proposition 1** ([2], Proposition 5). A sphere congruence \( C \) is harmonic if and only if \( d \ast A_C = 0 \).

### 3 Associated vector bundles

We explain a conformal map in terms of vector bundles.
3.1 Conformal maps

Let $\mathbb{H}^2$ be the trivial right quaternionic vector bundle over $M$ of rank two. We consider a standard basis $e_1, e_2$ of $\mathbb{H}^2$ as a section of $\mathbb{H}^3$. Then $de_1 = de_2 = 0$. A conformal map $f: M \to \mathbb{H}P^1$ with mean curvature sphere $S$ is translated in terms of vector bundles as Table 1 (See [2], Section 4, Section 5).

<table>
<thead>
<tr>
<th>map</th>
<th>vector bundle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f: M \to \mathbb{H}P^1$: map</td>
<td>$L \subset \mathbb{H}^2$: quaternionic line subbundle $L_p = f(p)$</td>
</tr>
<tr>
<td>$df: TM \to T\mathbb{H}P^1$</td>
<td>$\pi: \mathbb{H}^2 \to \mathbb{H}^2/L$: projection $\delta := \pi d\Gamma(L)$</td>
</tr>
<tr>
<td>$f$: conformal</td>
<td>$S\delta = L$</td>
</tr>
<tr>
<td>$S$: the mean curvature sphere</td>
<td>$*\delta = S\delta = \delta S</td>
</tr>
</tbody>
</table>

Table 1: Vector bundles

3.2 The Willmore functional

Let $L$ be a conformal map with mean curvature sphere $S$.

Definition 3 ([2], Definition 8).

$$W(L) := \frac{1}{\pi} \int_M (A_S \wedge *A_S)$$

is called the Willmore energy of $L$.

Lemma 3 ([2], Lemma 8). For any conformal map $L$, the functional $W$ takes non-negative values.

A critical conformal map of the Willmore functional is called a Willmore conformal map.

Theorem 1 ([4], [8], [2]). A conformal map with mean curvature sphere $S$ is Willmore if and only if $S$ is harmonic.

By Proposition 1, the mean curvature sphere $S$ is harmonic if and only if $d * A_S = 0$.

We connect the above discussion with the classical terminology. Let $L$ be a conformal map and $f: M \to \mathbb{H}$ a stereographic projection of $S^4$ followed by $L$. We induce a (singular) metric on $M$ by a conformal map $f: M \to \mathbb{H}$. Let $K$ be the Gauss curvature, $K^\perp$ the normal curvature, and $\mathcal{H}$ the mean curvature vector of $f$.

Lemma 4 ([2], Example 19).

$$W(L) = \frac{1}{4\pi} \int_M (|\mathcal{H}|^2 - K - K^\perp)|df|^2.$$
4 Transforms

We explain transforms of conformal maps and sphere congruences.

4.1 Darboux transforms

Let $L$ be a conformal map with mean curvature sphere $S$. For $\phi \in \Gamma(\mathbb{H}^2/L)$, we denote by $\tilde{\phi} \in \Gamma(\mathbb{H}^2)$ a lift of $\phi$, that is $\pi \tilde{\phi} = \phi$. Set

$$D(\phi) := \frac{1}{2}(\pi d\tilde{\phi} + S \ast \pi d\tilde{\phi}).$$

We denote by $\widetilde{M}$ the universal covering of $M$. Similarly, for an object $B$ defined on $M$, we denote by $\tilde{B}$ for the object induced from $B$ by the universal covering map of $M$.

Theorem 2 ([3], Lemma 2.1). Let $\phi \in \Gamma(\mathbb{H}^2/L)$. If $D(\phi) = 0$, then there exists $\tilde{\phi} \in \Gamma(\mathbb{H}^2)$ uniquely such that $\pi \tilde{\phi} = 0$. The line bundle $\tilde{L} := \tilde{\phi} \mathbb{H}$ is conformal.

Definition 4 ([3], Definition 2.2). The line bundle $\tilde{L}$ in the above theorem is called the Darboux transform of $L$.

4.2 $\mu$-Darboux transforms

Let $C : M \to Z$. We set $I \phi := \phi i$. We identify $\mathbb{H}^2$ with $\mathbb{C}^4$ by taking $I$ as a complex structure.

Theorem 3 ([5], Theorem 4.1). The sphere congruence $C$ is harmonic if and only if $d_{\lambda} := d + (\lambda - 1)A_{C}^{(1,0)} + (\lambda^{-1} - 1)A_{C}^{(0,1)}$ is flat for all $\lambda \in \mathbb{C} \setminus \{0\}$.

Definition 5. We call $d_{\lambda}$ the associated family of $d$.

Theorem 4 ([5], Theorem 4.2). We assume that $C : M \to Z$ is harmonic, $A_{C} \neq 0$, $\mu \in \mathbb{C} \setminus \{0\}$, $\psi_1, \psi_2 \in \Gamma(\mathbb{H}^2)$ are linearly independent over $\mathbb{C}$, $d_{\mu}\psi_1 = d_{\mu}\psi_2 = 0$, $W_{\mu} := \text{span}\{\psi_1, \psi_2\}$, and $\Gamma(\mathbb{H}^2) = W_{\mu} \oplus jW_{\mu}$. Then for $G := (\psi_1, \psi_2) : M \to \text{GL}(2, \mathbb{H})$, $a = G \left(\frac{\mu + \mu^{-1}}{2}E_2\right)G^{-1}$, $b = G \left(I\left(\frac{\mu^{-1}-\mu}{2}E_2\right)\right)G^{-1}$, and $T := C(a - 1) + b$, the sphere congruence $\hat{C} := T^{-1}CT : M \to Z$ is harmonic.

Definition 6 ([5]). The sphere congruence $\hat{C}$ is called the $\mu$-Darboux transform of $C$.

It is known that a $\mu$-Darboux transform is a Darboux transform.

Let $S$ be a mean curvature sphere of a Willmore conformal map $L$. Then $S$ is harmonic by Theorem 1. Hence a harmonic sphere congruence $\hat{S}$ is defined.

Theorem 5 ([5], Theorem 4.4). Let $L$ be a Willmore conformal map with harmonic mean curvature sphere $S$ such that $A_S \neq 0$. Then, $\hat{L} := T(a - 1)^{-1}L$ is a Willmore conformal map and $\hat{S}$ is the mean curvature sphere of $\hat{L}$.

Hence a $\mu$-Darboux transform of a mean curvature sphere induces a transform of a Willmore conformal map.
4.3 Simple factor dressing

Let $L$ be a conformal map with the mean curvature sphere $S$. Because $S$ is a harmonic sphere congruence, the associated family $d_\lambda$ is defined. We assume that $r_\lambda: M \to \text{GL}(4, \mathbb{C})$ is a map parametrized by $\lambda \in \mathbb{C} \setminus \{0\}$ such that, with respect to $\lambda$, it is meromorphic with the only simple pole on $\mathbb{C} \setminus \{0\}$ and holomorphic at 0 and $\infty$.

**Definition 7** ([6]). If $\hat{d}_\lambda := r_\lambda \circ d_\mu \circ r_\lambda^{-1}$ is an associated family of a harmonic map $\hat{C}$, then $\hat{C}$ is called a simple factor dressing of $C$.

A simple factor dressing is a harmonic map.

**References**


