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Complete self-shrinkers in Euclidean space

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1. Introduction

1.1. Self-shrinkers.

Let $X : M^n \to \mathbb{R}^{n+p}$ be an n-dimensional submanifold in the $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. If the position vector X evolves in the direction of the mean curvature H, then it gives rise to a solution to the mean curvature flow:

$$F(\cdot, t) : M^n \to \mathbb{R}^{n+p}$$

satisfying $F(\cdot, 0) = X(\cdot)$ and

$$\left(\frac{\partial F(p,t)}{\partial t}\right)^N = H(p,t), \quad p \in M, \tag{1.1}$$

where $H(p,t)$ denotes the mean curvature vector of submanifold $M_t = F(M^n, t)$ at point $F(p,t)$.

One of the most important problems in mean curvature flow is to understand the possible singularities that the flow goes through. Singularities are unavoidable as the flow contracts any closed embedded submanifold in Euclidean space eventually leading to extinction of the evolving submanifolds. A key starting point for singularity analysis is Huisken's monotonicity formula.

Let $\rho(F,t) = (4\pi(t_0 - t))^{-n/2} e^{-\frac{|F-p_0|^2}{4(t_0-t)}}$, $t < t_0$ at (p_0, t_0). Huisken proved the following monotonicity formula for the mean curvature flow:

$$\frac{d}{dt}\int_{M_t} \rho(F,t)dv_t = -\int_{M_t} \rho(F,t)\left|H + \frac{(F(p,t)-p_0)^N}{2(t_0-t)}\right|^2 dv_t$$

where dv_t is the measure on M_t.

A solution of (1.1) is called *self-shrinking* about (p_0, t_0) if it satisfies

$$H = -\frac{(F(p,t)-p_0)^N}{2(t_0-t)}.$$

A submanifold is said to be a *self-shrinker* if it is the time $t = 0$ slice of solution which is self-shrinking about (p_0, t_0). That is, we call the submanifold $X : M^n \to \mathbb{R}^{n+p}$ satisfying

$$H = -\frac{(X - p_0)^N}{2t_0} \tag{1.2}$$

a self-shrinker.

Define the functional

$$\mathcal{F} = \frac{1}{(4\pi t_0)^{n/2}} \int_M e^{-\frac{|X-p_0|^2}{4t_0}} dv$$

Obviously, we know that $\mathcal{F} = \int_M \rho(X, 0) dv$.

By computing the first variation formula, we can prove that M^n is a critical point of $\mathcal{F}$ if and only if

$$H = -\frac{(X - p_0)^N}{2t_0}.$$

Without loss of generality, we only consider the case $p_0 = 0$ and $t_0 = \frac{1}{2}$. The self-shrinker equation (1.2) is equivalent to

$$H = -X^N. \tag{1.3}$$

Notice that M^n is a critical point of $\mathcal{F} = (2\pi)^{-n/2} \int_M e^{-\frac{|X|^2}{2}} dv$. So we can use it to characterize self-shrinkers.

1.2. Some typical examples.

Example 1.1. $X : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ *is a complete self-shrinker with* $|A|^2 = 0$.

Example 1.2. *For any positive integers* $m_1, \cdots, m_p$ *such that* $m_1 + \cdots + m_p = n$, *the submanifold*

$$\mathbb{S}^{m_1}(\sqrt{m_1}) \times \cdots \times \mathbb{S}^{m_p}(\sqrt{m_p}) \subset \mathbb{R}^{n+p}$$

is an n-dimensional compact self-shrinker in $\mathbb{R}^{n+p}$ *with*

$$H = -X, \quad |A|^2 = p.$$

Here, $\mathbb{S}^{m_i}(r_i) = \{X_i \in \mathbb{R}^{m_i+1} : |X_i|^2 = r_i^2, i = 1, \cdots, p\}$ *is a* m_i-*dimensional round sphere with radius* r_i.

In particular, we consider the product submanifold $\mathbb{S}^{n_1}(\sqrt{n_1}) \times \mathbb{S}^{n_2}(\sqrt{n_2})$ of two submanifolds $\mathbb{S}^{n_1}(\sqrt{n_1})$ and $\mathbb{S}^{n_2}(\sqrt{n_2})$, and give a simple computation.

Example 1.3. $X : \mathbb{S}^{n_1}(\sqrt{n_1}) \times \mathbb{S}^{n_2}(\sqrt{n_2}) \hookrightarrow \mathbb{R}^{n_1+n_2+2}$.

Choose local orthonormal vectors $\{e_1, \cdots, e_{n_1}\}$ and $\{\delta_1, \cdots, \delta_{n_2}\}$ in the tangent spaces of $\mathbb{S}^{n_1}(\sqrt{n_1})$ and $\mathbb{S}^{n_2}(\sqrt{n_2})$, respectively. Put

$$E_1 = (e_1, 0), \cdots, E_{n_1} = (e_{n_1}, 0), E_{n_1+1} = (0, \delta_1), \cdots, E_{n_1+n_2} = (0, \delta_{n_2}).$$

Suppose that $\nu_1 = \frac{1}{r_1}X_1, \quad \nu_2 = \frac{1}{r_2}X_2$ are the unit normal vectors.
Write

$$V_1 = (\nu_1, 0), \quad V_2 = (0, \nu_2).$$

Obviously, V_1, V_2 are the unit normal vectors in $\mathbb{R}^{n_1+n_2+2}$. It is easy to see that $\{E_1, \cdots, E_{n_1}, E_{n_1+1}, \cdots, E_{n_1+n_2}, V_1, V_2\}$ forms an orthonormal basis in $\mathbb{R}^{n_1+n_2+2}$.

Denote by h^1_{ij} and h^2_{ij} the second fundamental form for V_1, V_2, respectively.

By a direct calculation, we obtain that

$$h^1_{ij} = \begin{cases} -\frac{1}{r_1}\delta_{ij}, & 1 \leq i, j \leq n_1, \\ 0, & \text{othercases,} \end{cases}$$

$$h^2_{ij} = \begin{cases} -\frac{1}{r_2}\delta_{ij}, & n_1 + 1 \leq i, j \leq n_1 + n_2, \\ 0, & \text{othercases.} \end{cases}$$

Hence, the mean curvature vector $H = -X$, and $|A|^2 = 2$. Therefore, $H = -X^N$ and $X : S^{n_1}(\sqrt{n_1}) \times S^{n_2}(\sqrt{n_2}) \hookrightarrow \mathbb{R}^{n_1+n_2+2}$ is an $(n_1 + n_2)$-dimensional compact self-shrinker in $\mathbb{R}^{n_1+n_2+2}$.

Example 1.4. *For positive integers $m_1, \cdots, m_p$, $q \geq 1$, with $m_1 + \cdots + m_p + q = n$, the submanifold*

$$M^n = \mathbb{S}^{m_1}(\sqrt{m_1}) \times \cdots \times \mathbb{S}^{m_p}(\sqrt{m_p}) \times \mathbb{R}^q \subset \mathbb{R}^{n+p}$$

is an n-dimensional complete noncompact self-shrinker in $\mathbb{R}^{n+p}$ with polynomial volume growth which satisfies

$$H = -X^N, \quad |A|^2 = p.$$

Example 1.5. *Let $X : \mathbb{S}^2(\sqrt{m(m+1)}) \hookrightarrow \mathbb{S}^{2m}(\sqrt{2}) \subset \mathbb{R}^{2m+1}$ be a minimal surface in $\mathbb{S}^{2m}(\sqrt{2})$. Consider it as a surface in $\mathbb{R}^{2m+1}$, it is a self-shrinker with*

$$H = -X, \quad |A|^2 = 2 - \frac{2}{m(m+1)}.$$

2. Some known results

2.1. Complete self-shrinkers in $\mathbb{R}^{n+1}$.

In this section, I would like to introduce some known results. The classification of mean convex (i.e. $H \geq 0$) self-shrinkers began with Huisken's classification which can be stated as follows:

Theorem 2.1. *([8]) If $X : M^n \to \mathbb{R}^{n+1}(n \geq 2)$ is a smooth compact self-shrinker with nonnegative mean curvature H in $\mathbb{R}^{n+1}$. Then $X(M^n) = S^n(\sqrt{n})$.*

When $n = 1$, Abresch and Langer [1] had already classified all smooth closed self-shrinker curves in $\mathbb{R}^2$ and showed that the embedded ones are round circles.

In a second paper, Huisken dealt with the complete case.

Theorem 2.2. *([8]) Let $X : M^n \to \mathbb{R}^{n+1}$ be a smooth complete embedded self-shrinker in $\mathbb{R}^{n+1}$ with $H \geq 0$, polynomial volume growth. Suppose that the second fundamental form $|A|$ is bounded. Then M^n is one of the following:*
(1) *a round sphere $S^n(\sqrt{n})$ in $\mathbb{R}^{n+1}$,*
(2) *a cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$ in $\mathbb{R}^{n+1}$,*
(3) *a hyperplane in $\mathbb{R}^{n+1}$.*

Then, Colding-Minicozzi showed that Huisken's classification holds even without the assumption that $|A|$ *is bounded.*

Theorem 2.3. *([3]) Let $X : M^n \to \mathbb{R}^{n+1}$ be a smooth complete embedded self-shrinker in $\mathbb{R}^{n+1}$ with $H \geq 0$, polynomial volume growth. Then M^n is one of the following:*
(1) *a round sphere $S^n(\sqrt{n})$ in $\mathbb{R}^{n+1}$,*
(2) *a cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$ in $\mathbb{R}^{n+1}$,*
(3) *a hyperplane in $\mathbb{R}^{n+1}$.*

In Huisken and Colding-Minicozzi's theorems, they all assumed the condition *polynomial volume growth.* In fact, in order to prove their theorems, they need to use integral formulas which are similar to Stokes formula for compact manifolds. To guarantee that integral formulas hold, the condition of *polynomial volume growth* plays a very important role.

2.2. Polynomial volume growth.

Definition 2.1. *We say that a submanifold M^n in $\mathbb{R}^{n+p}$ has polynomial volume growth if there exist constants C and d such that for all $r \geq 1$,* *there holds*

$$Vol(B_r(0)\cap M)\le Cr^d,$$

where $B_r(0)$ is an Euclidean ball in $\mathbb{R}^{n+p}$ with radius r and centered at the origin.

In 2011, Ding-Xin, Cheng-Zhou proved the following theorem:

Theorem 2.4. *([6],[5]) Any complete non-compact properly immersed self-shrinker M^n in $\mathbb{R}^{n+p}$ has Euclidean volume growth at most. Precisely,*

$$Vol(B_r(0)\cap M)\le Cr^n \quad for \quad r\ge 1.$$

Proof. From $H=-X^N$, we have

$$\Delta|X|^2=2n-2|X^N|^2,\quad \nabla|X|^2=2X^T.$$

Since X is a proper immersion, it is well defined for

$$I(t)=t^{-\frac{n}{2}}\int_{D_r}e^{-\frac{|X|^2}{2t}}dv,$$

where $D_r=M\cap B_r(0)$ for the submanifold in $\mathbb{R}^{n+p}$, $B_r(0)$ is a standard ball in $\mathbb{R}^{n+p}$ with radius r and centered at the origin. Then

$$I'(t)=t^{-\frac{n}{2}-1}\int_{D_r}(-\frac{n}{2}+\frac{|X|^2}{2t})e^{-\frac{|X|^2}{2t}}dv.$$

On the other hand,

$$\begin{aligned}
div(e^{-\frac{|X|^2}{2t}}\nabla(\frac{|X|^2}{2}))&=e^{-\frac{|X|^2}{2t}}(\Delta\frac{|X|^2}{2}-\frac{1}{4t}|\nabla|X|^2|^2)\\
&=e^{-\frac{|X|^2}{2t}}(n-|X^N|^2-\frac{1}{t}|X^T|^2)\\
&=e^{-\frac{|X|^2}{2t}}(n-|X^N|^2-\frac{1}{t}|X|^2+\frac{|X^N|^2}{t})\\
&=e^{-\frac{|X|^2}{2t}}(n-\frac{|X|^2}{t}+(\frac{1}{t}-1)|X^N|^2)\\
&\le e^{-\frac{|X|^2}{2t}}(n-\frac{|X|^2}{t}),\quad (\text{for}\quad t\ge 1).
\end{aligned}$$

For $t \geq 1$,

$$\begin{aligned} I'(t) &= t^{-\frac{n}{2}-1}\int_{D_r}(-\frac{n}{2}+\frac{|X|^2}{2t})e^{-\frac{|X|^2}{2t}}dv \\ &\leq -\frac{1}{2}t^{-\frac{n}{2}-1}\int_{D_r} div(e^{-\frac{|X|^2}{2t}}\nabla(\frac{|X|^2}{2}))dv \\ &= -\frac{1}{2}t^{-\frac{n}{2}-1}\int_{\partial D_r}\langle e^{-\frac{|X|^2}{2t}}\nabla\frac{|X|^2}{2}, \frac{\nabla\frac{|X|^2}{2}}{|\nabla\frac{|X|^2}{2}|}\rangle dv \\ &= -\frac{1}{2}t^{-\frac{n}{2}-1}\int_{\partial D_r} e^{-\frac{|X|^2}{2t}}|\nabla\frac{|X|^2}{2}|dv \leq 0, \end{aligned}$$

then we get $I(r^2) \leq I(1)$ for $r \geq 1$, i.e.

$$r^{-n}\int_{D_r} e^{-\frac{|X|^2}{2r^2}}dv \leq \int_{D_r} e^{-\frac{|X|^2}{2}}dv,$$

but on the other hand, $\frac{|X|^2}{r^2} \leq 1$ holds in D_r. Therefore,

$$e^{-\frac{1}{2}}r^{-n}\int_{D_r} dv \leq r^{-n}\int_{D_r} e^{-\frac{|X|^2}{2r^2}}dv \leq \int_{D_r} e^{-\frac{|X|^2}{2}}dv. \tag{2.1}$$

Note that

$$\int_{D_r\backslash D_{r-1}} e^{-\frac{|X|^2}{2}}dv \leq e^{-\frac{(r-1)^2}{2}}e^{\frac{1}{2}}r^n\int_{D_r} e^{-\frac{|X|^2}{2}}dv \leq e^{-r}\int_{D_r} e^{-\frac{|X|^2}{2}}dv,$$

the last inequality holds for $r \geq r_0$ with r_0 sufficiently large. Then the above inequality implies

$$\int_{D_r} e^{-\frac{|X|^2}{2}}dv \leq \frac{1}{1-e^{-r}}\int_{D_{r-1}} e^{-\frac{|X|^2}{2}}dv.$$

Then for any N,

$$\int_{D_{r_0+N}} e^{-\frac{|X|^2}{2}}dv \leq \prod_{i=0}^{N}\frac{1}{1-e^{-(r_0+i)}}\int_{D_{r_0-1}} e^{-\frac{|X|^2}{2}}dv.$$

This implies that

$$\int_M e^{-\frac{|X|^2}{2}}dv \leq C_1\int_{D_{r_0-1}} e^{-\frac{|X|^2}{2}}dv < C_2.$$

From (2.1), we have

$$\int_{D_r} dv \leq e^{\frac{1}{2}}r^n\int_M e^{-\frac{|X|^2}{2}}dv \leq Cr^n.$$

□

Remark 2.1. *Take self-shrinker*

$$\mathbb{S}^k(\sqrt{k}) \times \mathbb{R}^{n-k}, \quad 0 \leq k \leq n, \quad |H| = \sqrt{k},$$

then the above estimate implies that

$$Vol(B_r(0) \cap M) = Cr^{n-k},$$

which is sharp.

By Theorem 2.4, it's not hard to prove the following lemma:

Lemma 2.1. *If self-shrinker $M^n \to \mathbb{R}^{n+p}$ has polynomial volume growth, then for any $m \geq 0$, we have*

$$\int_M |X|^m e^{-\frac{|X|^2}{2}} dv < \infty.$$

Proof. We can prove our lemma by the following

$$\begin{aligned}
\int_M |X|^m e^{-\frac{|X|^2}{2}} dv &= \sum_{j=0}^{+\infty} \int_{D_{j+1} \setminus D_j} |X|^m e^{-\frac{|X|^2}{2}} dv \\
&\leq \sum_{j=0}^{+\infty} (j+1)^m \int_{D_{j+1}} e^{-\frac{j^2}{2}} dv \\
&= \sum_{j=0}^{+\infty} (j+1)^m e^{-\frac{j^2}{2}} \int_{D_{j+1}} dv \\
&\leq C \sum_{j=0}^{+\infty} (j+1)^{m+n} e^{-\frac{j^2}{2}}
\end{aligned}$$

□

2.3. Outlines of proofs.

The following linear operator which was introduced and studied firstly on self-shrinker by Colding-Minicozzi (see (3.7) in [3]):

$$\mathcal{L} = \Delta - \langle X, \nabla(\cdot) \rangle = e^{\frac{|X|^2}{2}} div(e^{-\frac{|X|^2}{2}} \nabla \cdot) \tag{2.2}$$

where Δ and ∇ denote the Laplacian and the gradient operator on the self-shrinker, respectively and $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^{n+p}$.

The operator $\mathcal{L}$ is self-adjoint in a weighted L^2-space:

Lemma 2.2. *If $X : M^n \to \mathbb{R}^{n+p}$ is a submanifold, u is a C^1-function with compact support, and v is a C^2-function, then*

$$\int_M u(\mathcal{L}v)e^{-\frac{|X|^2}{2}} = -\int_M \langle \nabla v, \nabla u\rangle e^{-\frac{|X|^2}{2}}. \tag{2.3}$$

Outline of proof of Theorem 2.1

Step 1. From $\mathcal{L}H = H - |A|^2H$, and by maximum principle, we know that H satisfies the strict inequality $H > 0$.

Step 2. By lemma 2.2, we can get a key inequality:

$$\int_M |\nabla|A| - |A|\nabla logH|^2 e^{-\frac{|X|^2}{2}} \leq 0.$$

equivalently, $|A| = \beta H$, where β is some positive constant.

Step 3. By $|A| = \beta H$, we can prove theorem 2.1.

The following corollary is an extension of lemma 2.1, used later to justify computations when M is not closed.

Corollary 2.1. *Suppose that $X : M^n \to \mathbb{R}^{n+p}$ is a complete submanifold, if u, v are C^2-functions satisfying*

$$\int_M (|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|)e^{-\frac{|X|^2}{2}} < +\infty$$

then we get

$$\int_M u(\mathcal{L}v)e^{-\frac{|X|^2}{2}} = -\int_M \langle \nabla v, \nabla u\rangle e^{-\frac{|X|^2}{2}}. \tag{2.4}$$

Proof. Let η be a cut-off function with compact support on M. Then

$$\begin{aligned}\int_M \eta u(\mathcal{L}v)e^{-\frac{|X|^2}{2}} &= \int_M \eta u div(e^{-\frac{|X|^2}{2}}\nabla v)\\ &= -\int_M \langle \nabla(\eta u), \nabla v\rangle e^{-\frac{|X|^2}{2}}\\ &= -\int_M u\langle \nabla \eta, \nabla v\rangle e^{-\frac{|X|^2}{2}} - \int_M \eta\langle \nabla u, \nabla v\rangle e^{-\frac{|X|^2}{2}}\end{aligned}$$

Let $\eta = \eta_j$ be a cut-off fuction linearly to zero from B_j to B_{j+1}, where $B_j = M \cap B_j(0)$ with $B_j(0)$ is the Euclidean ball of radius j centered at at the origin. Since $|\eta_j|$ and $|\nabla \eta_j|$ are bounded by one, $\eta_j \to 1$ and $|\nabla \eta_j| \to 0$.

$$\int_M \eta_j u(\mathcal{L}v)e^{-\frac{|X|^2}{2}} = -\int_M u\langle \nabla \eta_j, \nabla v\rangle e^{-\frac{|X|^2}{2}} - \int_M \eta_j\langle \nabla u, \nabla v\rangle e^{-\frac{|X|^2}{2}}$$

By dominated convergence theorem, we can complete the proof of Corollary 2.1. □

Outline of proofs of Theorem 2.2 and Theorem 2.3

Step 1. From $\mathcal{L}H = H - |A|^2 H$, and by maximum principle, we know that H satisfies the strict inequality $H > 0$.

Step 2. We show that $|A| = \beta H$ for a constant $\beta > 0$. This geometric identity is a key for proving the classification.

$$\int_M |\nabla|A| - |A|\nabla logH|^2 e^{-\frac{|X|^2}{2}}$$
$$= \int_M |\nabla|A||^2 e^{-\frac{|X|^2}{2}} - \int_M \langle\nabla|A|^2, \nabla logH\rangle e^{-\frac{|X|^2}{2}} + \int_M |A|^2|\nabla logH|^2 e^{-\frac{|X|^2}{2}}.$$

In order to insure the above integrations to make sense, the following lemma is needed.

Proposition 2.1. *Let $X : M^n \to \mathbb{R}^{n+p}$ be an n-dimensional complete self-shrinker with $H > 0$. If M^n has polynomial volume growth, then*

$$\int_M (|A|^2|\nabla logH| + |\nabla|A|^2||\nabla logH| + |A|^2|\mathcal{L}logH|)e^{-\frac{|X|^2}{2}} < \infty,$$

$$\int_M (|A||\nabla|A|| + |\nabla|A||^2 + |A||\mathcal{L}|A||)e^{-\frac{|X|^2}{2}} < \infty.$$

Let $u = |A|, v = |A|$ in corollary 2.1, then we obtain that by proposition 2.1

$$\int_M (|A||\nabla|A|| + |\nabla|A||^2 + |A||\mathcal{L}|A||)e^{-\frac{|X|^2}{2}} < \infty,$$

which satisfies the condition of corollary 2.1, so we have

$$\int_M |\nabla|A||^2 e^{-\frac{|X|^2}{2}} = -\int_M |A|(\mathcal{L}|A|)e^{-\frac{|X|^2}{2}}.$$

Let $u = |A|^2, v = logH$ in corollary 2.1, then we obtain that by proposition 2.1

$$\int_M (|A|^2|\nabla logH| + |\nabla|A|^2||\nabla logH| + |A|^2|\mathcal{L}logH|)e^{-\frac{|X|^2}{2}} < \infty,$$

which satisfies the condition of corollary 2.1, so we have

$$\int_M \langle\nabla|A|^2, \nabla logH\rangle e^{-\frac{|X|^2}{2}} = -\int_M |A|^2(\mathcal{L}logH)e^{-\frac{|X|^2}{2}}$$

Step 3. From $|A| = \beta H$, we can get theorem 2.2 and theorem 2.3 by a series of discussions.

2.4. Complete self-shrinkers in $\mathbb{R}^{n+p}$.

Recently, based on the differential operator $\mathcal{L}$ defined by (2.2), N. Q. Le and N. Sesum [10] proved a gap theorem for self-shrinkers of codimension one: if a hypersurface $M^n \subset \mathbb{R}^{n+1}$ is a smooth complete embedded self-shrinker and with polynomial volume growth, and satisfies $|A|^2 < 1$, then M^n is a hyperplane.

In 2012, Cao-Li extended Le-Sesum's result to any codimension:

Theorem 2.5. *([2]) Let $X : M^n \to \mathbb{R}^{n+p}(p \geq 1)$ be a complete self-shrinker with polynomimal volume growth in $\mathbb{R}^{n+p}$. If the squared norm of the second fundamental form satisfies*

$$|A|^2 \leq 1.$$

Then M is one of the following:
(1) *a round sphere $S^n(\sqrt{n})$ in $\mathbb{R}^{n+1}$,*
(2) *a cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$ in $\mathbb{R}^{n+1}$,*
(3) *a hyperplane in $\mathbb{R}^{n+1}$.*

Question 2.1: Is it possible to remove the assumption on *polynomial volume growth* in Theorem 2.5?

3. Classifications of self-shrinkers in $\mathbb{R}^{n+p}$

We study complete self-shrinkers without the assumption on *polynomial volume growth* and obtain the following results by extending the generalized maximum principle of Yau to $\mathcal{L}$-operator. Our main results can be stated as follows:

Theorem 3.1. *([4]) Let $X : M^n \to \mathbb{R}^{n+p}$ $(p \geq 1)$ be an n-dimensional complete self-shrinker in $\mathbb{R}^{n+p}$, then one of the following holds:*
(1) $\sup |A| \geq 1$,
(2) $|A| \equiv 0$, *i.e. M^n is a hyperplane in $\mathbb{R}^{n+1}$.*

Corollary 3.1. *Let $X : M^n \to \mathbb{R}^{n+p}$ $(p \geq 1)$ be a complete self-shrinker, and satisfy*

$$\sup |A|^2 < 1.$$

Then M is a hyperplane in $\mathbb{R}^{n+1}$.

Remark 3.1. *The round sphere $S^n(\sqrt{n})$ and the cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$ are complete self-shrinkers in $\mathbb{R}^{n+1}$ with $|A| = 1$. Thus, our result is sharp.*

Remark 3.2. *From corollary 3.1 and remark 3.1, we can easily know that theorem 2.5 holds without assumption on* polynomial volume growth, *which gives an affirmative answer to Question 2.1.*

In order to prove our results, first of all, we extend the generalized maximum principle of Yau to $\mathcal{L}$-operator on complete self-shrinkers. We call the following theorem as *generalized maximum principle for $\mathcal{L}$-operator* which can be stated as follows:

Theorem 3.2. ([4]) *Let $X : M^n \to \mathbb{R}^{n+p}$ be a complete self-shrinker with Ricci curvature bounded from below. Let f be any C^2-function bounded from above on this self-shrinker. Then, there exists a sequence of points $\{p_k\} \subset M^n$, such that*

$$\lim_{k\to\infty} f(X(p_k)) = supf, \lim_{k\to\infty} |\nabla f|(X(p_k)) = 0, \quad \limsup_{k\to\infty} \mathcal{L}f(X(p_k)) \leq 0.$$

Since M^n is a complete self-shrinker, the self-shrinker equation (1.3) is equivalent to

$$H^\alpha = -\langle X, e_\alpha\rangle, \quad n+1 \leq \alpha \leq n+p. \tag{3.1}$$

Taking covariant derivative of (3.1) with respect to e_i, we have

$$H^\alpha_{,i} = \sum_k h^\alpha_{ik}\langle X, e_k\rangle, \quad 1 \leq i \leq n, \quad n+1 \leq \alpha \leq n+p. \tag{3.2}$$

Furthermore, by taking covariant derivative of (3.2) with respect to e_j, we have

$$\begin{aligned} H^\alpha_{,ij} &= \sum_k h^\alpha_{ikj}\langle X, e_k\rangle + h^\alpha_{ij} + \sum_{\beta,k} h^\alpha_{ik}h^\beta_{kj}\langle X, e_\beta\rangle \\ &= \sum_k h^\alpha_{ikj}\langle X, e_k\rangle + h^\alpha_{ij} - \sum_{\beta,k} H^\beta h^\alpha_{ik}h^\beta_{kj}, \end{aligned} \tag{3.3}$$

According to (3.3), we obtain

$$\mathcal{L}|H|^2 = 2|\nabla H|^2 + 2|H|^2 - 2\sum_{\alpha,\beta,i,k} H^\alpha H^\beta h^\alpha_{ik}h^\beta_{ik}. \tag{3.4}$$

Outline of proof of Theorem 3.1

Step 1. By using Cauchy-Schwarz inequality, we can obtain that

$$\begin{aligned} \mathcal{L}|H|^2 &= 2|\nabla H|^2 + 2|H|^2 - 2\sum_{\alpha,\beta,i,k} H^\alpha H^\beta h^\alpha_{ik}h^\beta_{ik} \\ &\geq 2|\nabla H|^2 + 2(1-|A|^2)|H|^2. \end{aligned}$$

Step 2. If $\sup|A|^2 \geq 1$, there is nothing to do.

If $\sup|A|^2 < 1$, $\sum_{\alpha,i,j}(h^\alpha_{ij})^2 < 1$. We can know that the Ricci curvature is bounded from below and $|H|^2$ is bounded from above.

Step 3. By applying the generalized maximum principle for $\mathcal{L}$-operator to the function H^2, we have

$$0 \geq \limsup \mathcal{L}|H|^2 \geq 2(1 - \sup|A|^2) \sup|H|^2.$$

From $\sup|A|^2 < 1$, we have $\sup|H|^2 = 0$, that is $H \equiv 0$. From the self-shrinker equation (1.3), we can know that M^n is a smooth minimal cone. Hence, it follows that the only smooth cone through 0 is a hyperplane.

Theorem 3.3. *([4]) Let $X : M^n \to \mathbb{R}^{n+1}$ be a complete self-shrinker. If $\inf H^2 > 0$ and $|A|^2$ is bounded, then $\inf|A|^2 \leq 1$.*

Corollary 3.2. *([4]) Let $X : M^n \to \mathbb{R}^{n+1}$ be a complete self-shrinker. If $\inf H^2 > 0$ and $|A|^2$ is constant, then $|A|^2 \equiv 1$ and M^n is the round sphere $S^n(\sqrt{n})$ or the cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$.*

Remark 3.3. *Notice that condition $\inf H^2 > 0$ is necessary since Angenent has proved that there exist embedded self-shrinkers from $S^1 \times S^{n-1}$ into $\mathbb{R}^{n+1}$ with $\inf H^2 = 0$.*

Outline of proof of Theorem 3.3

Step 1. By a direct calculation, we can get

$$\mathcal{L}H = (1 - |A|^2)H.$$

Step 2. Since $|A|^2$ is bounded, we know that H is bounded and the Ricci curvature is bounded from below.

Since $\inf H^2 > 0$, we can divide it into two cases: $\inf H > 0$ or $\sup H < 0$.

1° If $\inf H > 0$, applying the generalized maximum principle for $\mathcal{L}$-operator to $-H$, we obtain

$$0 \leq (1 - \inf|A|^2) \inf H.$$

Hence, $\inf|A|^2 \leq 1$.

2° If $\sup H < 0$, applying the generalized maximum principle for $\mathcal{L}$-operator to H, we obtain

$$0 \geq \limsup \mathcal{L}H \geq (1 - \sup|A|^2) supH.$$

Then, $\inf|A|^2 \leq \sup|A|^2 \leq 1$. This finishes the proof of Theorem 3.3.

Outline of proof of Corollary 3.3

Step 1. $|A|^2 \equiv 1$.

In fact, from Theorem 3.3, we know that $\inf|A|^2 \leq 1$. Since $\inf|H|^2 > 0$, $H \neq 0$, that is M^n is not totally geodesic. According to Theorem

3.1, we have $\sup |A|^2 \geq 1$. From our assumption that $|A|^2$ is a constant, so we get $|A|^2 \equiv 1$.

Step 2. By a direct calculation, we can obtain that

$$\frac{1}{2}\mathcal{L}|A|^2 = |\nabla A|^2 + |A|^2(1 - |A|^2).$$

Step 3. Substituting $|A|^2 = 1$ into the above equation, we know $|\nabla A|^2 \equiv 0$. According to the theorem of Lawson [9], we know that M^n is isometric to the round sphere $S^n(\sqrt{n})$ or the cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$.

4. Complete proper self-shrinkers of dimension 2 and 3

4.1. Complete proper self-shrinkers of dimension 3.

First, we study complete proper self-shrinkers of dimension 3 in $\mathbb{R}^5$ with constant squared norm of the second fundamental form, and then, obtain a complete classification which can be stated as follows:

Theorem 4.1. ([4]) *Let $X : M^3 \to \mathbb{R}^5$ be a 3-dimensional complete proper self-shrinker with $H > 0$. If the principal normal $\nu = \frac{\mathbf{H}}{H}$ is parallel in the normal bundle of M^3 and the squared norm of the second fundamental form is constant, then M is one of the following:*

(1) $S^k(\sqrt{k}) \times \mathbb{R}^{3-k}$, $1 \leq k \leq 3$ *with* $|A|^2 = 1$,

(2) $S^1(1) \times S^1(1) \times \mathbb{R}$ *with* $|A|^2 = 2$,

(3) $S^1(1) \times S^2(\sqrt{2})$ *with* $|A|^2 = 2$,

(4) *the three dimensional minimal isoparametric Cartan hypersurface with* $|A|^2 = 3$.

Outline of proof of Theorem 4.1

Step 1. Since M^3 is a complete proper self-shrinker, we know that M^3 has polynomial volume growth. By a result of Li-Wei [11], we know that M^3 is isometric to $\Gamma \times \mathbb{R}^2$ or $\tilde{M}^r \times \mathbb{R}^{3-r}$, where Γ is an Abresch-Langer curve and $\tilde{M}$ is a minimal hypersurface in sphere $S^{r+1}(\sqrt{r})$.

Step 2. Since $|A|^2$ is constant, then the Abresch-Langer curve Γ must be a circle. In this case, M^3 is isometric to $S^1(1) \times \mathbb{R}^2$.

Step 3. If $|A|^2 \leq 1$, by Cao-Li's results (see Theorem 2.5), we have $|A|^2 = 1$ and M^3 is $S^k(\sqrt{k}) \times \mathbb{R}^{3-k}$, $1 \leq k \leq 3$.

Step 4. We consider the case of $|A|^2 > 1$.

When $r = 2$, $\tilde{M}$ is the Clifford torus $S^1(1) \times S^1(1)$ in $S^3(\sqrt{2})$;

When $r = 3$, $\tilde{M}$ is the Clifford torus $S^1(1) \times S^2(\sqrt{2})$ in $S^4(\sqrt{3})$ with $|A|^2 = 2$ or the three dimensional minimal isoparametric Cartan hypersurface in $S^4(\sqrt{3})$ with $|A|^2 = 3$.

4.2. Complete proper self-shrinkers of dimension 2.

Furthermore, we study complete proper self-shrinker of dimension 2, and give a complete classification theorem for arbitrary codimension.

Theorem 4.2. *([4]) Let $X : M^2 \to \mathbb{R}^{2+p}$ be a 2-dimensional complete proper self-shrinker with $H > 0$. If the principal normal $\nu = \frac{\mathbf{H}}{H}$ is parallel in the normal bundle of M^2 and the squared norm of the second fundamental form is constant, then M is one of the following:*

(1) $S^k(\sqrt{k}) \times \mathbb{R}^{2-k}$, $1 \le k \le 2$ *with* $|A|^2 = 1$,

(2) *the Boruvka sphere* $S^2(\sqrt{m(m+1)})$ *in* $S^{2m}(\sqrt{2})$ *with* $p = 2m-1$ *and* $|A|^2 = 2 - \frac{2}{m(m+1)}$,

(3) *a compact flat minimal surface in* $S^{2m+1}(\sqrt{2})$ *with* $p = 2m$ *and* $|A|^2 = 2$.

Outline of proof of Theorem 4.2

Step 1. Since M^2 is a complete proper self-shrinker, we know that M^2 has polynomial volume growth. By a result of Li-Wei [11], we know that M^2 is isometric to $\Gamma \times \mathbb{R}^1$ or $\tilde{M}^2$, where Γ is an Abresch-Langer curve and $\tilde{M}$ is a compact minimal hypersurface in sphere $S^{p+1}(\sqrt{2})$.

Step 2. Since $|A|^2$ is constant, then the Abresch-Langer curve Γ must be a circle. In this case, M^2 is isometric to $S^1(1) \times \mathbb{R}$.

Step 3. If $|A|^2 \le 1$, we have $|A|^2 = 1$ and M^2 is $S^1(1) \times \mathbb{R}$.

Step 4. Consider the case of $|A|^2 > 1$. We can prove that M^2 is isometric to a Boruvka sphere $S^2(\sqrt{m(m+1)})$ in $S^{2m}(\sqrt{2})$ with $p = 2m - 1$ and $|A|^2 = 2 - \frac{2}{m(m+1)}$ or a compact flat minimal surface in $S^{2m+1}(\sqrt{2})$ with $p = 2m$ and $|A|^2 = 2$.

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